NORMS ON EARTHQUAKE MEASURES
AND ZYGMUND FUNCTIONS

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ABSTRACT. The infinitesimal earthquake theorem gives a one-to-one correspondence between Thurston bounded earthquake measures and normalized Zygmund bounded functions. In this paper, we provide an intrinsic proof of a theorem given in an earlier paper by the author; that is, we show that the cross-ratio norm of a Zygmund bounded function is equivalent to the Thurston norm of the earthquake measure in the correspondence.

1. INTRODUCTION

Consider the open unit disk $\mathbb{D}$ centered at the origin of the complex plane $\mathbb{C}$ as the hyperbolic plane. A geodesic lamination $\mathcal{L}$ in $\mathbb{D}$ is a collection of geodesics that foliate a closed subset $L$ of $\mathbb{D}$. Here $L$ is called the locus of $\mathcal{L}$, the geodesics are called the leaves of $\mathcal{L}$, the connected components of $\mathbb{D} \setminus L$ are called the gaps, and the gaps and the leaves of $\mathcal{L}$ are called the strata of $\mathcal{L}$. Let $\mathbb{S}^1$ denote the boundary circle of $\mathbb{D}$, and let $X$ be the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\text{the diagonal}\}$ factorized by the equivalence relation $(a, b) \sim (b, a)$. A Borel measure $\sigma$ defined on $X$ is called an earthquake measure if there is a lamination $\mathcal{L}$ such that $\sigma$ is supported on the pairs of the endpoints of the leaves in $\mathcal{L}$.

Let $\mathcal{L}$ be a lamination in $\mathbb{D}$ and $\beta$ a closed geodesic segment of hyperbolic length $\leq 1$. If $\beta$ is transversal to a leaf in $\mathcal{L}$, then $\mathcal{L}$ intersects $\beta$ in a parallel fashion in the sense that there are two geodesic lines $l_1$ and $l_2$ among the lines of $\mathcal{L}$ intersecting $\beta$ such that any other line in $\mathcal{L}$ intersecting $\beta$ separates $l_1$ from $l_2$. Suppose the strip $S$ bounded by $l_1$ and $l_2$ in the unit disk is of the form $[a, b] \times [c, d]$, where $a, d$ and $b, c$ are the endpoints of $l_1$ and $l_2$, respectively, and $a, b, c, d$ are arranged on $\mathbb{S}^1$ in counter-clockwise order. We denote by

$$\sigma(\beta) = \sigma([a, b] \times [c, d]).$$

In [5], Thurston defines the norm of $\sigma$ to be

$$\|\sigma\|_{TH} = \sup_{l(\beta) \leq 1} \sigma(\beta) = \sup_{l(\beta) = 1} \sigma(\beta),$$

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where $\beta$ is a closed geodesic segment transversal to $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of $\beta$. An earthquake measure is called Thurston bounded if it has finite Thurston norm. Let $\mathcal{M}$ be the collection of all Thurston bounded earthquake measures defined on $\mathcal{X}$. $\forall \sigma \in \mathcal{M}$, define

$$V_\sigma(x) = E(\sigma)(x) = \int_{\mathcal{L}} E_{ab}(x)d\sigma(a, b),$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a, b]$ and $E_{ab}(x) = 0$ otherwise. Here we also have the agreement that for each geodesic line $ab$ in $\mathcal{L}$, $[a, b]$ denotes the short arc on $S^1$ in counter-clockwise order. $V_\sigma$ maps each point $x$ on $\mathcal{S}^1$ to a vector tangent to $\mathcal{S}^1$ at $x$, and then it defines a tangent vector field on $\mathcal{S}^1$.

Simply consider the tangent vectors of $S^1$ as complex numbers. A continuous tangent vector field $V$ on $\mathcal{S}^1$ is said to be Zygmund bounded if

$$|V(e^{2\pi i(\theta+t)}) + V(e^{2\pi i(\theta-t)}) - 2V(e^{2\pi i\theta})| \leq M|t|$$

for a constant $M > 0$ and for all $0 \leq \theta < 1$ and $0 < t < \frac{1}{2}$. It is proved in [1] that for any $\sigma \in \mathcal{M}$, $V_\sigma$ is Zygmund bounded; conversely, for any Zygmund bounded tangent vector field $V$ on $\mathcal{S}^1$, there exists a Thurston bounded earthquake measure $\sigma$ such that

$$V(x) = \pi \int_{\mathcal{L}} E_{ab}(x)d\sigma(a, b) \text{ modulo a quadratic polynomial;}$$

furthermore, if two $V$’s differ by a quadratic polynomial, then the corresponding $\sigma$’s are the same. This is the so-called infinitesimal earthquake theorem (see Theorem 5.1 in [1]), which gives a one-to-one correspondence between $\mathcal{M}$ and the space of Zygmund bounded tangent vector fields on $\mathcal{S}^1$.

Motivated by the work of [3], we introduce a norm to measure the Zygmund boundedness of $V$ and show it is equivalent to the Thurston norm of $\sigma$. Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points $a, b, c, d$ on the unit circle $\mathcal{S}^1$ arranged in counter-clockwise order, we define

$$\text{cr}(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)}$$

and

$$V[Q] = \frac{V(b) - V(a)}{b-a} - \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(c)}{d-c} - \frac{V(a) - V(d)}{a-d}.$$}

Then the cross-ratio norm $||V||_{cr}$ of $V$ is defined to be

$$||V||_{cr} = \sup_{\text{cr}(Q)=1} |V[Q]|.$$}

In fact, $||V||_{cr}$ is finite if and only if $V$ is Zygmund bounded.

It is proved in [2] that $V_\sigma$ is the initial derivative to the parameter $t$ of the earthquake curve determined by $t\sigma$, $t \geq 0$. By means of an implicit method, it is deduced in [4] that the cross-ratio norm of $V_\sigma$ is equivalent to the Thurston norm of $\sigma$. On the other hand, it is a general principle that strategies in the study of earthquake theory (see [3] and [2]) transfer to parallel strategies in the infinitesimal theory. In this paper, we construct an intrinsic and direct proof for the following theorem.
**Main Theorem.** There exists a universal constant $C > 0$ such that $\forall \, \sigma \in \mathcal{M}$,
\[
\frac{1}{C} \| \sigma \|_{Th} \leq \| V_\sigma \|_{cr} \leq C \| \sigma \|_{Th}.
\]

2. Proof

We divide the proof into two parts. We first prove that there exists $C > 0$ such that $\| \sigma \|_{Th} \leq \| V_\sigma \|_{cr}$; we then show that there exists $C > 0$ such that $\| V_\sigma \|_{cr} \leq C \| \sigma \|_{Th}$. Before we start the proofs, let us summarize some techniques into lemmas.

Let $\sigma$ denote a Thurston bounded earthquake measure, and let $V = V_\sigma$. Let $B$ denote an orientation-preserving Möbius transformation from the upper half-plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and let $\tilde{\sigma} = B_*^{-1}(\sigma)$ be the pullback of $\sigma$ under $B$ (i.e., the pushforward of $\sigma$ under $B^{-1}$). Also, define
\[
(8) \quad \tilde{V}(x) = V_\sigma(x) = E(\tilde{\sigma})(x) = \int \int E_{ab}(x) d\tilde{\sigma}(a,b).
\]

**Lemma 1.**
\[
\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.
\]

**Proof.** Define $x' = B(x)$, $a' = B(a)$ and $b' = B(b)$. By using the identity
\[
[B(a) - B(b)]^2 = (a - b)^2 B'(a) B'(b)
\]
and the assumption that $B$ is orientation-preserving, we have
\[
\frac{[B(x) - B(a)][B(x) - B(b)]}{[B(a) - B(b)]} = \frac{B'(x)(x - a)(x - b)}{(a - b)}.
\]
Then
\[
V(B(x)) = V(x') = \int \int E_{a'b'}(x') d\sigma(a',b') = \int \int \frac{(x' - a')(x' - b')}{a' - b'} d\sigma(a',b')
\]
\[
= \int \int \frac{[B(x) - B(a)][B(x) - B(b)]}{B(a) - B(b)} d\sigma(B(a), B(b))
\]
\[
= \int \int \frac{B'(x)(x - a)(x - b)}{(a - b)} d\tilde{\sigma}(a,b) = B'(x) \tilde{V}(x).
\]
Therefore,
\[
\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.
\]

**Lemma 2.** For any quadruple $Q$ of four points $a, b, c, d$ on the real line or the unit circle in counter-clockwise order,
\[
\tilde{V}[Q] = V[B(Q)] \quad \text{(or \, } V[Q] = \tilde{V}[B^{-1}(Q)]\).
\]

**Proof.** Let $h_t = Id + tV$, where $t$ is a real parameter near zero. Clearly, for each $x$, $h_t(x)$ is differentiable on $t$ with $V = \frac{d}{dt} h_t|_{t=0}$. Define
\[
cr(h_t(Q)) = \frac{[h_t(b) - h_t(a)][h_t(d) - h_t(c)]}{[h_t(c) - h_t(b)][h_t(d) - h_t(a)]}.
\]
Observe first that
\[
V[Q] = \frac{d}{dt} \ln cr(h_t(Q))|_{t=0}.
\]
Let \( \tilde{h}_t = B^{-1} \circ h_t \circ B \). Then \( \tilde{h}_t \) is also differentiable on \( t \), and
\[
\frac{d}{dt} \tilde{h}_t(x)|_{t=0} = \frac{V(B(x))}{B'(x)} = \tilde{V}(x).
\]
Therefore,
\[
\tilde{V}[Q] = \frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0}.
\]
On the other hand,
\[
cr(\tilde{h}_t(Q)) = cr(B^{-1} \circ h_t \circ B(Q)) = cr(h_t \circ B(Q)) = cr(h_t(B(Q)) ,
\]
and then
\[
\frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0} = \frac{d}{dt} \ln cr(\tilde{h}_t(B(Q))|_{t=0} = V[B(Q)].
\]
Hence
\[
\tilde{V}[Q] = V[B(Q)].
\]
\[ \square \]

**Corollary 1.** \( ||\tilde{V}||_{cr} = ||V||_{cr} \).

**Lemma 3.** Assume \( \lambda > 0 \), \(-\infty \leq a < b < c < d \), and \( c \leq s \leq d \leq t \). Let \( V(x) = \lambda E_{s,t}(x) \) and \( Q = \{a,b,c,d\} \). Consider \( V[Q] \) as a function of \( s \) and \( t \). Then \( V[Q] \geq 0 \), and \( V[Q] \) is an increasing function on \( t \) for each fixed \( s \) and a decreasing function on \( s \) for each fixed \( t \).

**Proof.** It is easy to check that \( \frac{\partial}{\partial t} E_{s,t}(x) > 0 \) and \( \frac{\partial}{\partial s} E_{s,t}(x) < 0 \) for each \( s < x < t \). Clearly,
\[
V[Q] = \frac{V(b) - V(a)}{b-a} - \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(c)}{d-c} - \frac{V(a) - V(d)}{a-d} = \frac{V(d)}{d-c} - \frac{V(d)}{d-a} = \left(1 - \frac{1}{d-c} - \frac{1}{d-a}\right)\lambda E_{s,t}(d).
\]
Since \( \frac{1}{d-c} - \frac{1}{d-a} > 0 \),
\[
\frac{\partial}{\partial t} V[Q] > 0 \quad \text{and} \quad \frac{\partial}{\partial s} V[Q] < 0.
\]
Therefore \( V[Q] \) is an increasing function on \( t \) for each fixed \( s \) and is a decreasing function on \( s \) for each fixed \( t \). \[ \square \]

**Lemma 4.** Assume \( \lambda > 0 \), \(-\infty \leq a < b < c < d \leq \infty \), and \( b \leq s \leq c \) and \( t \geq d \). Let \( V(x) = \lambda E_{s,t}(x) \) and \( Q = \{a,b,c,d\} \). Consider \( V[Q] \) as a function of \( s \) and \( t \). Then \( V[Q] \geq 0 \), and \( V[Q] \) is increasing on \( s \) for each fixed \( t \) and also increasing on \( t \) for each fixed \( s \).

**Proof.** It is straightforward to check
\[
V[Q] = \lambda \left(-\frac{d-b}{(c-b)(d-c)} E_{s,t}(c) + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d)\right)
\]
\[
= \lambda \left[-\frac{d-b}{(c-b)(d-c)} \frac{(d-s)(c-t)}{s-t} + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d) \frac{(d-s)(d-t)}{s-t}\right].
\]
Then
\[
\frac{\partial}{\partial s} V[Q] = \lambda \left[ \frac{(d-b)(c-s)^2}{(c-b)(d-c)(s-t)^2} + \frac{c-a}{(d-c)(d-a)(s-t)^2} \right]
= \frac{\lambda (t-d)^2}{(s-t)^2 (d-c)} \left[ \frac{d-b}{s-b} + \frac{c-a}{d-a} \right] - \frac{c-a}{d-a}.
\]

Clearly, \(\frac{d-b}{s-b} > 1\) and \(\frac{c-a}{d-a} < 1\). Hence \(\frac{\partial}{\partial s} V[Q] > 0\) when \(t > d\) and \(\frac{\partial}{\partial s} V[Q] \geq 0\) when \(t = d\). Therefore \(V[Q]\) is an increasing function on \(s\) for each fixed \(t\).

Similarly,
\[
\frac{\partial}{\partial t} V[Q] = \lambda \left[ \frac{(d-b)(c-s)^2}{(c-b)(d-c)(s-t)^2} + \frac{c-a}{(d-c)(d-a)(s-t)^2} \right]
= \frac{\lambda (c-s)^2}{(s-t)^2 (d-c)} \left[ \frac{d-b}{c-b} + \frac{c-a}{d-a} \right] - \frac{c-a}{d-a}.
\]

Since \(a < b \leq s \leq c < d\),
\[
\frac{d-s}{c-s} \geq \frac{d-b}{c-b} > 1 \quad \text{and} \quad \frac{d-s}{c-s} > \frac{d-a}{c-a} > 1.
\]

Then
\[
-1 + \left( \frac{d-s}{c-s} \frac{d-b}{c-b} \right) \left( \frac{d-s}{c-s} \frac{d-a}{c-a} \right) > 0.
\]

Hence \(\frac{\partial}{\partial t} V[Q] > 0\) when \(b \leq s < c\) and \(\frac{\partial}{\partial t} V[Q] = 0\) when \(s = c\). Therefore \(V[Q]\) is also an increasing function on \(t\) for each fixed \(s\). \(\square\)

**Theorem 1.** There exists a universal constant \(C > 0\) such that
\[
||\sigma||_{TH} \leq C||V_\sigma||_{cr}.
\]

In fact, we can take \(C = \frac{(1+e)^2}{-3e+6+e} = C||V_\sigma||_{cr}.
\]

**Proof.** Let \(D\) denote a closed disk in \(\mathbb{D}\) of hyperbolic diameter 1, \(l_1\) and \(l_2\) denote the lines in the lamination \(L\) of \(\sigma\) that bound all the lines in \(L\) intersecting \(D\). Let \(\beta\) denote the geodesic perpendicular to both \(l_1\) and \(l_2\) (in the case that \(l_1\) and \(l_2\) share at least one endpoint, we let \(\beta\) be a geodesic perpendicular to \(l_1\) such that the hyperbolic length of the segment on \(\beta\) between \(l_1\) and \(l_2\) is less than or equal to \(1/2\)). Label the endpoints of \(\beta\) by \(x\) and \(y\) so that the arc \([x,y]\) from \(x\) to \(y\) going in the counter-clockwise direction is not longer than the arc \([y,x]\) from \(y\) to \(x\). Let \(B : \mathbb{D} \rightarrow \mathbb{H}\) be the Möbius transformation that maps \(x\) to \(0\), \(y\) to \(\infty\), the arc \([x,y]\) to the positive half real line, and the geodesics \(l_1\) and \(l_2\) to the geodesics connecting \(-1\) to \(1\). Clearly, \(cr(Q) = cr(Q') = 1\). Let \(\tilde{V}\) denote the pushforward of \(\sigma\) under \(B\) and \(\tilde{V} = E(\tilde{\sigma})\). It is clear that \(||\tilde{\sigma}||_{TH} = ||\sigma||_{TH}\), and Lemma \(\ref{thm:thm_1}\) implies \(||\tilde{V}||_{cr} = ||V||_{cr}\).

Assume \(a = 1, b = \infty, c = -s\) and \(d = \frac{1}{s}\). Denote by \(Q = \{a,b,c,d\}\) and \(Q' = B^{-1}(Q) = \{a',b',c',d'\}\). Clearly, \(cr(Q) = cr(Q') = 1\), and Lemma \(\ref{thm:thm_1}\) implies \(\tilde{V}[Q] = V[Q']\). Denote by \(u'\) and \(v'\) the other endpoints of \(l_1\) and \(l_2\) such that \(a',u',b',c',v',d'\) are arranged on \(S^1\) in counter-clockwise order. We divide the lines
in the lamination $\mathcal{L}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\mathcal{L}$ intersecting $D$, $\mathcal{L}_b$ denote the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $[u', v')$ to points in $(b', c']$, and $\mathcal{L}_d$ the collection of the lines in $\mathcal{L} \setminus \mathcal{L}_m$ connecting points in $(v', d')$ to points in $(d', a')$. Denote by $\sigma_i = \sigma_{|\mathcal{L}_i}$ and $V_i = E(\sigma_i)$ for $i = m, b, d$. By the linearity of the operator $E$, we have

$$V[Q'] = V_m[Q'] + V_b[Q'] + V_d[Q'].$$ 

By Lemma 4, if we move the weights of the geodesic lines in the lamination $\mathcal{L}$

$$\text{then the value of } V_m[Q'].$$ 

Denote by $\tilde{\sigma}_m$ the pushforward of $\sigma_m$ under $B$ and $\tilde{V}_m = E(\tilde{\sigma}_m)$. By Lemma 3, $V_m[Q'] = \tilde{V}_m[B(Q')] = \tilde{V}_m[Q]$.

By Lemma 4, if we move the weights of the geodesic lines in the lamination $\tilde{\mathcal{L}}_m$ of $\tilde{\sigma}_m$ to the geodesic line connecting $-1$ to $s$, then the value of $V_m[[b, c, d, a]]$ is possibly increased, and hence the value of $\tilde{V}_m[Q] = -\tilde{V}_m[[b, c, d, a]]$ is possibly decreased. Therefore

$$\tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q],$$

where $\lambda = \sigma(\mathcal{L}_m)$. It is easy to check that

$$(\lambda E_{-1,s})[Q] = \frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d}$$

and

$$\frac{2E_{-1,s}(d) - E_{-1,s}(a)}{a - d} = \frac{-3s^2 + 6s + 1}{(1 + s)^2} \geq \frac{-3e^2 + 6e + 1}{(1 + e)^2} > 0$$

for $1 \leq s \leq e$. Letting $C = \frac{(1 + e)^2}{-3e^2 + 6e + 1}$, we have

$$\|V\|_{cr} \geq V[Q'] \geq V_m[Q'] = \tilde{V}_m[Q] \geq (\lambda E_{-1,s})[Q] \geq \frac{\lambda}{C}.$$ 

Hence

$$\lambda \leq C\|V\|_{cr},$$

which implies

$$\|\sigma\|_{Th} \leq C\|V\|_{cr}.\]$$


\textbf{Theorem 2.} There exists a universal constant $C > 0$, independent of $\sigma$, such that

$$\|V_\sigma\|_{cr} \leq C\|\sigma\|_{Th}.$$ 

In fact, we can take $C = C_0 + 2C_1C_2$, where $C_0$ is the smallest positive integer greater than or equal to $\ln(3 + 2\sqrt{2})$, $C_1 = \frac{e}{e - 1}$, and $C_2$ is the smallest positive integer greater than or equal to $\ln(e + \sqrt{e^2 - 1})$.  

\[\Box\]
Remark. The inequality (10) was obtained in [1] for a different cross-ratio norm on \( V \) through a complex method which considered the holomorphic differential arising from the third derivative of \( V + iH(V) \), where \( H(V) \) denotes the Hilbert transformation of \( V \). The proof of the inequality (10) in this paper is purely real. In addition, our Theorem [1] and that inequality in [1] imply that the two cross-ratio norms on \( V \) are actually equivalent (see [2]).

Lemma 5. Let \( Q \) be a quadruple consisting of four points \( a, b, c, d \) on \( \mathbb{R}^1 \) or \( S^1 \) arranged in counter-clockwise order. Take \( \frac{|dz|}{y} \) as the hyperbolic metric on the upper half-plane \( \mathbb{H} \). Then \( \text{cr}(Q) = 1 \) if and only if the geodesic \( \overline{ac} \) from \( a \) to \( c \) is perpendicular to the geodesic \( \overline{bd} \) from \( b \) to \( d \), and if and only if the hyperbolic distance from \( \overline{ab} \) to \( \overline{cd} \) (or \( \overline{bc} \) to \( \overline{da} \)) is equal to \( \ln(3 + 2 \sqrt{2}) \).

Proof. It is straightforward (see [2] for details). \( \square \)

Lemma 6. Consider the upper half-plane \( \mathbb{H} \). Let \( l_n \) denote the geodesic connecting \(-e^{-n}\) to \( e^{-n}\) for each \( n \in \{0\} \cup \mathbb{N} \) and \( L \) the lamination consisting of \( l_n \)'s. Suppose that \( \sigma \) is an earthquake measure supported on \( L \), and let \( \lambda_n = \sigma(l_n) \). Let \( Q = \{1, \infty, -1, 0\} \), and

\[
V(x) = \int \int E_{a,b}(x) d\sigma(a,b).
\]

There exists a constant \( C_1 > 0 \) such that

\[
0 \leq V[Q] \leq C_1 \max_{n \geq 0} \lambda_n.
\]

Proof. Define \( \lambda = \max_{n \geq 0} \lambda_n \), \( a_n = -e^{-n} \) and \( b_n = e^{-n} \). Clearly, \( V(1) = V(\infty) = V(-1) = 0 \), and then \( V[Q] = 2V(0) \). We need to work out \( V(0) \), that is,

\[
0 \leq V(0) = \int \int E_{a_n,b_n}(0) d\sigma(a_n,b_n) = \sum_{n \geq 0} \lambda_n a_n b_n = \sum_{n \geq 0} \lambda_n \frac{e^{-n}}{2} \leq \frac{1}{e-1} \lambda.
\]

Let \( C_1 = \frac{e}{e-1} \). Then \( 0 \leq V[Q] = 2V(0) \leq C_1 \lambda \). \( \square \)

We reduce the proof of Theorem 2 to Propositions 1 and 2. Let \( L \) be the lamination that supports \( \sigma \). Given a quadruple \( Q \) of four points \( a, b, c, d \) on \( S^1 \) in counter-clockwise order with \( \text{cr}(Q) = 1 \), we first assume that three points \( a, b \) and \( c \) belong to the same stratum \( \mathcal{A} \) and estimate \( V[Q] \) in this case. By a Möbius change of coordinates and Lemma 2 we may assume that \( a = 1, b = \infty, c = -1, \) and \( d = 0 \). We will show that there is a constant \( C \) such that

\[
0 \leq V[Q] \leq C ||\sigma||_{TH}.
\]

Let \( x_n \) denote the point \(-e^{-n}\) and \( y_n \) the point \( e^{-n}\) on the real axis for each \( n \in \{0\} \cup \mathbb{N} \). Let \( L' \) denote the collection of the lines in \( L \) that connect points of the interval \([-1, 0)\) to points of \((0, 1)\), \( L^-_n \) the collection of the lines in \( L' \) that connect points of \([x_0, x_1)\) to points of \((0, y_0)\), and \( L^+_n \) the collection of the lines in \( L' \) that connect points of \([x_0, 0)\) to points of \((y_1, y_0)\). Finally, let \( L'_0 = L^-_0 \cup L^+_0 \). Then any line in \( L' \setminus L'_0 \) must connect a point in \([x_1, 0)\) to a point in \((0, y_1)\). Inductively, for each \( n \in \mathbb{N} \), let \( L^-_n \) denote the collection of the lines in \( L' \setminus (L'_0 \cup L'_1 \cup \cdots \cup L'_{n-1}) \) that connect points of \([x_n, x_{n+1})\) to points of \((0, y_n)\), and \( L^+_n \) the collection of the lines in \( L' \setminus (L'_0 \cup L'_1 \cup \cdots \cup L'_{n-1}) \) that connect points of \([x_n, 0)\) to points of \((y_{n+1}, y_n)\), and \( L_n = L^-_n \cup L^+_n \). We have the following three lemmas.
Lemma 7. For each \( n \in \{0\} \cup \mathbb{N} \), any line in \( L_n \) must connect a point in \([x_n,0]\) to a point in \((0,y_n]\).

Proof. It can be easily proved by an induction on \( n \).

Lemma 8. There exists a constant \( C_2 > 0 \), independent of \( h \) and \( \sigma \), such that \( \sigma(L_n) \leq C_2 ||\sigma||_{Th} \) for any \( n \in \{0\} \cup \mathbb{N} \).

Proof. For each \( n \in \{0\} \cup \mathbb{N} \), let \( l_n \) denote the geodesic line connecting the point \( x_n \) to the point \( y_n \). Also, for any \( n \in \mathbb{N} \), let \( l_n^+ \) denote the geodesic connecting the point \( x_n \) to 0, and \( l_n^- \) the geodesic connecting 0 to the point \( y_n \). The hyperbolic distance from \( l_n \) to \( l_{n+1}^- \) (or \( l_{n+1}^+ \)), \( n \in \{0\} \cup \mathbb{N} \), is equal to a constant, that is equal to \( \ln(e + \sqrt{e^2 - 1}) \). Let \( C_2 \) denote the smallest positive integer that is greater than or equal to \( \ln(e + \sqrt{e^2 - 1}) \). Then

\[
\sigma(L_n) \leq C_2 ||\sigma||_{Th}
\]

for each \( n \in \{0\} \cup \mathbb{N} \).

Let \( \hat{V} \) be the same map as defined in Lemma 4 with \( \lambda_n = \sigma(L_n) \).

Lemma 9. We have the following inequality:

\[
V[Q] \leq \hat{V}[Q].
\]

Proof. Define \( \sigma_n = \sigma|_{L_n} \) and \( V_n = E(\sigma_n) \). Let \( \hat{\sigma}_n \) denote the atomic earthquake measure with weight \( \lambda_n \) supported on the geodesic \( l_n \) and let \( \hat{V}_n = E(\hat{\sigma}_n) \). By the linearity of the operator \( E \),

\[
V[Q] = \sum_{n=0}^{\infty} V_n[Q] \text{ and } \hat{V}[Q] = \sum_{n=0}^{\infty} \hat{V}_n[Q].
\]

By Lemma 3 and Lemma 7, if we move the weights of the geodesic lines in \( L_n \) to the geodesic line \( l_n \), we only increase \( V_n[Q] \), that is, \( V_n[Q] \leq \hat{V}_n[Q] \). Therefore

\[
V[Q] \leq \hat{V}[Q].
\]

Proposition 1. If \( cr(Q) = 1 \) and \( a, b, c \) belong to the same stratum of an earthquake measure \((E,L)\), then

\[
0 \leq V_\sigma[Q] \leq C_1 C_2 ||\sigma||_{Th}.
\]

Proposition 2. Suppose \( cr(Q) = 1 \), and assume that there exists at least one geodesic line in the lamination \( L \) of \( \sigma \) that separates the vertices \( a \) and \( b \) from the vertices \( c \) and \( d \). Then

\[
|V_\sigma[Q]| \leq (C_0 + 2C_1 C_2)||\sigma||_{Th}.
\]

Proof. Given two points \( x \) and \( y \) on the unit circle, we use \([x,y]\) to denote the arc on \( S^1 \) from \( x \) to \( y \) in the counter-clockwise direction. We divide the geodesic lines in \( L \) that affect \( V[Q] \) into five groups. Let \( L_m \) denote the collection of the geodesic lines in \( L \) that connect points of the arc \([d,a]\) to points of the arc \([b,c]\). Let \( L_a \) denote the collection of the lines in \( L \) that connect points of the arc \((d,a)\) to points of the arc \((a,b)\), \( L_b \) the collection of the lines in \( L \) that connect points of the arc \((a,b)\) to points of the arc \((b,c)\), \( L_c \) the collection of the lines in \( L \) that connect
Figure 1. Five subcollections of $\mathcal{L}$ in the proof of Proposition 2

points of the arc $(b, c)$ to points of the arc $(c, d)$, and finally $\mathcal{L}_d$ the collection of the lines in $\mathcal{L}$ that connect points of the arc $(c, d)$ to points of the arc $(d, a)$ (see Figure 1). Let $\sigma_i$ denote the restriction of $\sigma$ on $\mathcal{L}_i$ and $V_i = E(\sigma_i)$, where $i = m, a, b, c, d$. Clearly $||\sigma_i||_{Th} \leq ||\sigma||_{Th}$ and


For the lamination $\mathcal{L}_d$, three points $a, b, c$ are contained in the same stratum. By Proposition 1, we have

$$0 \leq V_d[Q] \leq C_1 C_2 ||\sigma_d||_{Th} \leq C_1 C_2 ||\sigma||_{Th}.$$ 

For the lamination $\mathcal{L}_h$, three points $c, d, a$ are contained in the same stratum. Again by Proposition 1 we have

$$0 \leq V_b[[c, d, a, b]] \leq C_1 C_2 ||\sigma_b||_{Th} \leq C_1 C_2 ||\sigma||_{Th}.$$ 

Therefore,

$$0 \leq V_b[Q] = V_b[[c, d, a, b]] \leq C_1 C_2 ||\sigma||_{Th}.$$ 

Similarly, we obtain

$$-C_1 C_2 ||\sigma||_{Th} \leq V_a[Q] \leq 0 \quad \text{and} \quad -C_1 C_2 ||\sigma||_{Th} \leq V_c[Q] \leq 0.$$

It remains to consider $V_m[Q]$. Because of Lemma 2, by a Möbius change of coordinates, we may assume $a = -\infty$, $b = -1$, $c = 0$, $d = 1$. Then the geodesic lines in $\mathcal{L}_m$ connect the points of $[-1, 0]$ to the points of $[1, +\infty]$. By Lemma 4 if we move all the lines in $\mathcal{L}_m$ to the geodesic line from $0$ to $\infty$ without changing the weights of the lines in $\mathcal{L}_m$ to obtain a new measure $\sigma'_m$, then $V'_m[Q] = E(\sigma'_m)[Q]$ is possibly bigger, that is, $V_m[Q] \leq V'_m[Q]$. Clearly, $V'_m(x) = 0$ for $x \in [-\infty, 0]$ and $V'_m(x) = \sigma(\mathcal{L}_m)x$ for $x \in (0, +\infty)$. Then $V'_m[Q] = V'_m(1) = \sigma(\mathcal{L}_m)$. If we let $C_0$
denote the smallest integer $\geq \ln(3 + 2\sqrt{2})$, then by Lemma 3, $\sigma(L_m) \leq C_0||\sigma||_{Th}$.

Therefore,

$$V_m[Q] \leq V_m''[Q] = \sigma(L_m) \leq C_0||\sigma||_{Th}.$$ 

Again by Lemma 3, if we move all the lines in $L_m$ to the geodesic line from $-1$ to 1 without changing the weights of the lines in $L_m$ to obtain a new measure $\sigma_m''$, then $V_m''[Q] = E(\sigma_m'')[Q]$ is possibly smaller, that is, $V_m[Q] \geq V_m''[Q]$. Clearly, $V_m''(x) = 0$ for $x \notin (-1, 1)$ and $V_m''(x) = \sigma(L_m)(x+1)(1-x)$ for $x \in (-1, 1)$. Then $V_m''[Q] = -2V_m''(0) = -\sigma(L_m)$. Therefore,

$$V_m[Q] \geq V_m''[Q] = -\sigma(L_m) \geq -C_0||\sigma||_{Th}.$$ 

Collecting together these estimates, we obtain

$$-(C_0 + 2C_1C_2)||\sigma||_{Th} \leq V_a[Q] + V_c[Q] + V_m[Q] + V_b[Q] + V_d[Q] \leq (C_0 + 2C_1C_2)||\sigma||_{Th};$$

that is,

$$|V[Q]| \leq (C_0 + 2C_1C_2)||\sigma||_{Th}.$$ 

\[ \square \]

Propositions 1 and 2 imply Theorem 2.

Proof. Consider the pattern of the geodesic lines in $L$ with respect to $Q$, there either exists a line in $L$ separating two adjacent vertices in $Q$ from the other two or no such line exists. The former case is treated by Proposition 2 and the fact that $V([b, c, d, a]) = -V[Q]$; the latter one is treated by Proposition 1 and the summability of $V[Q] = E(\sigma)[Q]$ over four subcollections of $L$ that contain three vertices of $Q$ in the same stratum.

Finally, Theorems 1 and 2 imply our Main Theorem. \[ \square \]

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REFERENCES


