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# NORMS ON EARTHQUAKE MEASURES AND ZYGMUND FUNCTIONS

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ABSTRACT. The infinitesimal earthquake theorem gives a one-to-one correspondence between Thurston bounded earthquake measures and normalized Zygmund bounded functions. In this paper, we provide an intrinsic proof of a theorem given in an earlier paper by the author; that is, we show that the cross-ratio norm of a Zygmund bounded function is equivalent to the Thurston norm of the earthquake measure in the correspondence.

## 1. INTRODUCTION

Consider the open unit disk $\mathbb{D}$ centered at the origin of the complex plane $\mathbb{C}$ as the hyperbolic plane. A *geodesic lamination* $\mathcal{L}$ in $\mathbb{D}$ is a collection of geodesics that foliate a closed subset $L$ of $\mathbb{D}$. Here $L$ is called the *locus* of $\mathcal{L}$, the geodesics are called the *leaves* of $\mathcal{L}$, the connected components of $\mathbb{D} \setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathcal{L}$ are called the *strata* of $\mathcal{L}$. Let $\mathbb{S}^1$ denote the boundary circle of $\mathbb{D}$, and let $\mathbb{X}$ be the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\textit{the diagonal}\}$ factorized by the equivalence relation $(a,b) \sim (b,a)$. A Borel measure $\sigma$ defined on $\mathbb{X}$ is called an *earthquake measure* if there is a lamination $\mathcal{L}$ such that $\sigma$ is supported on the pairs of the endpoints of the leaves in $\mathcal{L}$.

Let $\mathcal{L}$ be a lamination in $\mathbb{D}$ and $\beta$ a closed geodesic segment of hyperbolic length $\leq 1$. If $\beta$ is transversal to a leaf in $\mathcal{L}$, then $\mathcal{L}$ intersects $\beta$ in a *parallel fashion* in the sense that there are two geodesic lines $l_1$ and $l_2$ among the lines of $\mathcal{L}$ intersecting $\beta$ such that any other line in $\mathcal{L}$ intersecting $\beta$ separates $l_1$ from $l_2$. Suppose the strip $S$ bounded by $l_1$ and $l_2$ in the unit disk is of the form $[a,b] \times [c,d]$, where $a, d$ and $b, c$ are the endpoints of $l_1$ and $l_2$, respectively, and $a, b, c, d$ are arranged on $\mathbb{S}^1$ in counter-clockwise order. We denote by

$$\sigma(\beta) = \sigma([a,b] \times [c,d]).$$

In [5], Thurston defines the *norm* of $\sigma$ to be

$$||\sigma||_{Th} = \sup_{l(\beta)\leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta), \tag{1}$$

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where $\beta$ is a closed geodesic segment transversal to $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of $\beta$. An earthquake measure is called *Thurston bounded* if it has finite Thurston norm. Let $\mathcal{M}$ be the collection of all Thurston bounded earthquake measures defined on $\mathbb{X}$. $\forall\ \sigma \in \mathcal{M}$, define

$$V_\sigma(x) = E(\sigma)(x) = \int\int E_{ab}(x) d\sigma(a, b), \tag{2}$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a, b]$ and $E_{ab}(x) = 0$ otherwise. Here we also have the agreement that for each geodesic line $\overline{ab}$ in $\mathcal{L}$, $[a, b]$ denotes the short arc on $\mathbb{S}^1$ in counter-clockwise order. $V_\sigma$ maps each point $x$ on $\mathbb{S}^1$ to a vector tangent to $\mathbb{S}^1$ at $x$, and then it defines a tangent vector field on $\mathbb{S}^1$.

Simply consider the tangent vectors of $\mathbb{S}^1$ as complex numbers. A continuous tangent vector field $V$ on $\mathbb{S}^1$ is said to be *Zygmund bounded* if

$$|V(e^{2\pi i(\theta+t)}) + V(e^{2\pi i(\theta-t)}) - 2V(e^{2\pi i\theta})| \leq M|t| \tag{3}$$

for a constant $M > 0$ and for all $0 \leq \theta < 1$ and $0 < t < \frac{1}{2}$. It is proved in [1] that for any $\sigma \in \mathcal{M}$, $V_\sigma$ is Zygmund bounded; conversely, for any Zygmund bounded tangent vector field $V$ on $\mathbb{S}^1$, there exists a Thurston bounded earthquake measure $\sigma$ such that

$$V(x) = \pi \int\int E_{ab}(x) d\sigma(a, b) \text{ modulo a quadratic polynomial;} \tag{4}$$

furthermore, if two $V$'s differ by a quadratic polynomial, then the corresponding $\sigma$'s are the same. This is the so-called *infinitesimal earthquake theorem* (see Theorem 5.1 in [1]), which gives a one-to-one correspondence between $\mathcal{M}$ and the space of Zygmund bounded tangent vector fields on $\mathbb{S}^1$.

Motivated by the work of [3], we introduce a norm to measure the Zygmund boundedness of $V$ and show it is equivalent to the Thurston norm of $\sigma$. Given a quadruple $Q = \{a, b, c, d\}$ consisting of four points $a, b, c, d$ on the unit circle $\mathbb{S}^1$ arranged in counter-clockwise order, we define

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} \tag{5}$$

and

$$V[Q] = \frac{V(b) - V(a)}{b-a} - \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(c)}{d-c} - \frac{V(a) - V(d)}{a-d}. \tag{6}$$

Then the *cross-ratio norm* $||V||_{cr}$ of $V$ is defined to be

$$||V||_{cr} = \sup_{cr(Q)=1} |V[Q]|. \tag{7}$$

In fact, $||V||_{cr}$ is finite if and only if $V$ is Zygmund bounded.

It is proved in [2] that $V_\sigma$ is the initial derivative to the parameter $t$ of the earthquake curve determined by $t\sigma$, $t \geq 0$. By means of an implicit method, it is deduced in [4] that the cross-ratio norm of $V_\sigma$ is equivalent to the Thurston norm of $\sigma$. On the other hand, it is a general principle that strategies in the study of earthquake theory (see [3] and [2]) transfer to parallel strategies in the infinitesimal theory. In this paper, we construct an intrinsic and direct proof for the following theorem.

**Main Theorem.** *There exists a universal constant* $C > 0$ *such that* $\forall\ \sigma \in \mathcal{M}$,

$$\frac{1}{C}||\sigma||_{Th} \leq ||V_\sigma||_{cr} \leq C||\sigma||_{Th}.$$

## 2. Proof

We divide the proof into two parts. We first prove that there exists $C > 0$ such that $||\sigma||_{Th} \leq C||V_\sigma||_{cr}$; we then show that there exists $C > 0$ such that $||V_\sigma||_{cr} \leq C||\sigma||_{Th}$. Before we start the proofs, let us summarize some techniques into lemmas.

Let $\sigma$ denote a Thurston bounded earthquake measure, and let $V = V_\sigma$. Let $B$ denote an orientation-preserving Möbius transformation from the upper half-plane $\mathbb{H}$ or the unit open disk $\mathbb{D}$ onto $\mathbb{D}$, and let $\tilde{\sigma} = B_*^{-1}(\sigma)$ be the pullback of $\sigma$ under $B$ (i.e., the pushforward of $\sigma$ under $B^{-1}$). Also, define

$$\tilde{V}(x) = V_{\tilde{\sigma}}(x) = E(\tilde{\sigma})(x) = \int\int E_{ab}(x)d\tilde{\sigma}(a,b). \tag{8}$$

**Lemma 1.**

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.$$

*Proof.* Define $x' = B(x)$, $a' = B(a)$ and $b' = B(b)$. By using the identity

$$[B(a) - B(b)]^2 = (a-b)^2 B'(a)B'(b)$$

and the assumption that $B$ is orientation-preserving, we have

$$\frac{[B(x) - B(a)][B(x) - B(b)]}{[B(a) - B(b)]} = \frac{B('x)(x-a)(x-b)}{(a-b)}.$$

Then

$$\begin{aligned} V(B(x)) &= V(x') = \int\int E_{a'b'}(x')d\sigma(a',b') = \int\int \frac{(x'-a')(x'-b')}{a'-b'}d\sigma(a',b') \\ &= \int\int \frac{[B(x) - B(a)][B(x) - B(b)]}{B(a) - B(b)}d\sigma(B(a), B(b)) \\ &= \int\int \frac{B'(x)(x-a)(x-b)}{(a-b)}d\tilde{\sigma}(a,b) = B'(x)\tilde{V}(x). \end{aligned}$$

Therefore,

$$\tilde{V}(x) = \frac{V(B(x))}{B'(x)}.$$

□

**Lemma 2.** *For any quadruple* $Q$ *of four points* $a, b, c, d$ *on the real line or the unit circle in counter-clockwise order,*

$$\tilde{V}[Q] = V[B(Q)] \ \ (\text{or } \ V[Q] = \tilde{V}[B^{-1}(Q)]).$$

*Proof.* Let $h_t = Id + tV$, where $t$ is a real parameter near zero. Clearly, for each $x$, $h_t(x)$ is differentiable on $t$ with $V = \frac{d}{dt}h_t|_{t=0}$. Define

$$cr(h_t(Q)) = \frac{[h_t(b) - h_t(a)][h_t(d) - h_t(c)]}{[h_t(c) - h_t(b)][h_t(d) - h_t(a)]}.$$

Observe first that

$$V[Q] = \frac{d}{dt}\ln cr(h_t(Q))|_{t=0}.$$

Let $\tilde{h}_t = B^{-1} \circ h_t \circ B$. Then $\tilde{h}_t$ is also differentiable on $t$, and

$$\frac{d}{dt}\tilde{h}_t(x)|_{t=0} = \frac{V(B(x))}{B'(x)} = \tilde{V}(x).$$

Therefore,

$$\tilde{V}[Q] = \frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0}.$$

On the other hand,

$$cr(\tilde{h}_t(Q)) = cr(B^{-1} \circ h_t \circ B(Q)) = cr(h_t \circ B(Q)) = cr(h_t(B(Q))),$$

and then

$$\frac{d}{dt} \ln cr(\tilde{h}_t(Q))|_{t=0} = \frac{d}{dt} \ln cr(h_t(B(Q)))|_{t=0} = V[B(Q)].$$

Hence

$$\tilde{V}[Q] = V[B(Q)].$$

□

**Corollary 1.** $||\tilde{V}||_{cr} = ||V||_{cr}$.

**Lemma 3.** *Assume $\lambda > 0$, $-\infty \leq a < b < c < d$, and $c \leq s \leq d \leq t$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$, and $V[Q]$ is an increasing function on $t$ for each fixed $s$ and a decreasing function on $s$ for each fixed $t$.*

*Proof.* It is easy to check that $\frac{\partial}{\partial t} E_{s,t}(x) > 0$ and $\frac{\partial}{\partial s} E_{s,t}(x) < 0$ for each $s < x < t$. Clearly,

$$\begin{aligned} V[Q] &= \frac{V(b) - V(a)}{b-a} - \frac{V(c) - V(b)}{c-b} + \frac{V(d) - V(c)}{d-c} - \frac{V(a) - V(d)}{a-d} \\ &= \frac{V(d)}{d-c} - \frac{V(d)}{d-a} = (\frac{1}{d-c} - \frac{1}{d-a})\lambda E_{s,t}(d). \end{aligned}$$

Since $\frac{1}{d-c} - \frac{1}{d-a} > 0$,

$$\frac{\partial}{\partial t} V[Q] > 0 \text{ and } \frac{\partial}{\partial s} V[Q] < 0.$$

Therefore $V[Q]$ is an increasing function on $t$ for each fixed $s$ and is a decreasing function on $s$ for each fixed $t$. □

**Lemma 4.** *Assume $\lambda > 0$, $-\infty \leq a < b < c < d \leq \infty$, and $b \leq s \leq c$ and $t \geq d$. Let $V(x) = \lambda E_{s,t}(x)$ and $Q = \{a, b, c, d\}$. Consider $V[Q]$ as a function of $s$ and $t$. Then $V[Q] \geq 0$, and $V[Q]$ is increasing on $s$ for each fixed $t$ and also increasing on $t$ for each fixed $s$.*

*Proof.* It is straightforward to check

$$\begin{aligned} V[Q] &= \lambda[-\frac{d-b}{(c-b)(d-c)} E_{s,t}(c) + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d)] \\ &= \lambda[-\frac{d-b}{(c-b)(d-c)} \frac{(c-s)(c-t)}{s-t} + \frac{c-a}{(d-c)(d-a)} E_{s,t}(d) \frac{(d-s)(d-t)}{s-t}]. \end{aligned}$$

Then

$$\begin{aligned}\frac{\partial}{\partial s}V[Q] &= \lambda[\frac{(d-b)}{(c-b)(d-c)}\frac{(t-c)^2}{(s-t)^2} - \frac{(c-a)}{(d-c)(d-a)}\frac{(t-d)^2}{(s-t)^2}] \\ &= \frac{\lambda(t-d)^2}{(s-t)^2(d-c)}[\frac{d-b}{c-b}(\frac{t-c}{t-d})^2 - \frac{c-a}{d-a}].\end{aligned}$$

Clearly, $\frac{d-b}{c-b}(\frac{t-c}{t-d})^2 > 1$ and $\frac{c-a}{d-a} < 1$. Hence $\frac{\partial}{\partial s}V[Q] > 0$ when $t > d$ and $\frac{\partial}{\partial s}V[Q] \geq 0$ when $t = d$. Therefore $V[Q]$ is an increasing function on $s$ for each fixed $t$.

Similarly,

$$\begin{aligned}\frac{\partial}{\partial t}V[Q] &= \lambda[-\frac{(d-b)}{(c-b)(d-c)}\frac{(c-s))^2}{(s-t)^2} + \frac{(c-a)}{(d-c)(d-a)}\frac{(d-s)^2}{(s-t)^2}] \\ &= \frac{\lambda(c-s)^2}{(s-t)^2(d-c)}[-\frac{d-b}{c-b} + \frac{c-a}{d-a}(\frac{d-s}{c-s})^2] \\ &= \frac{\lambda(c-s)^2}{(s-t)^2(d-c)}\frac{d-b}{c-b}[-1 + (\frac{d-s}{c-s}/\frac{d-b}{c-b})(\frac{d-s}{c-s}/\frac{d-a}{c-a})].\end{aligned}$$

Since $a < b \leq s \leq c < d$,

$$\frac{d-s}{c-s} \geq \frac{d-b}{c-b} > 1 \ \text{ and } \ \frac{d-s}{c-s} > \frac{d-a}{c-a} > 1.$$

Then

$$-1 + (\frac{d-s}{c-s}/\frac{d-b}{c-b})(\frac{d-s}{c-s}/\frac{d-a}{c-a}) > 0.$$

Hence $\frac{\partial}{\partial t}V[Q] > 0$ when $b \leq s < c$ and $\frac{\partial}{\partial t}V[Q] = 0$ when $s = c$. Therefore $V[Q]$ is also an increasing function on $t$ for each fixed $s$. □

**Theorem 1.** *There exists a universal constant $C > 0$ such that*

$$||\sigma||_{Th} \leq C||V_\sigma||_{cr}. \tag{9}$$

*In fact, we can take $C = \frac{(1+e)^2}{-3e^2+6e+1}$.*

*Proof.* Let $D$ denote a closed disk in $\mathbb{D}$ of hyperbolic diameter 1, $l_1$ and $l_2$ denote the lines in the lamination $\mathcal{L}$ of $\sigma$ that bound all the lines in $\mathcal{L}$ intersecting $D$. Let $\beta$ denote the geodesic perpendicular to both $l_1$ and $l_2$ (in the case that $l_1$ and $l_2$ share at least one endpoint, we let $\beta$ be a geodesic perpendicular to $l_1$ such that the hyperbolic length of the segment on $\beta$ between $l_1$ and $l_2$ is less than or equal to $\frac{1}{2}$). Label the endpoints of $\beta$ by $x$ and $y$ so that the arc $[x, y]$ from $x$ to $y$ going in the counter-clockwise direction is not longer than the arc $[y, x]$ from $y$ to $x$. Let $B : \mathbb{D} \to \mathbb{H}$ be the Möbius transformation that maps $x$ to 0, $y$ to $\infty$, the arc $[x, y]$ to the positive half real line, and the geodesics $l_1$ and $l_2$ to the geodesics connecting $-1$ to 1 and $-s$ to $s$ with $s > 1$ (in the case that $l_1$ and $l_2$ share at least one endpoint, then $B(l_1)$ or $B(l_2)$ connects $-s$ to 1 or $-1$ to $s$ with $s \geq 1$). Since the hyperbolic distance between $l_1$ and $l_2$ is less than or equal to 1, $s \leq e$ (in the case that $l_1$ and $l_2$ share one endpoint, the requirment on $\beta$ also implies that $s \leq e$). Let $\tilde{\sigma}$ denote the pushforward of $\sigma$ under $B$ and $\tilde{V} = E(\tilde{\sigma})$. It is clear that $||\tilde{\sigma}||_{Th} = ||\sigma||_{Th}$, and Lemma 2 implies $||\tilde{V}||_{cr} = ||V||_{Th}$.

Assume $a = 1$, $b = \infty$, $c = -s$ and $d = \frac{1-s}{2}$. Denote by $Q = \{a, b, c, d\}$ and $Q' = B^{-1}(Q) = \{a', b', c', d'\}$. Clearly, $cr(Q) = cr(Q') = 1$, and Lemma 2 implies $\tilde{V}[Q] = V[Q']$. Denote by $u'$ and $v'$ the other endpoints of $l_1$ and $l_2$ such that $a', u', b', c', v', d'$ are arranged on $\mathbb{S}^1$ in counter-clockwise order. We divide the lines in the lamination $\mathcal{L}$ that affect the value of $V[Q']$ into three groups. Let $\mathcal{L}_m$ denote the collection of the lines in $\mathcal{L}$ intersecting $D$, $\mathcal{L}_b$ denote the collection of the lines in $\mathcal{L}\setminus\mathcal{L}_m$ connecting points in $[u',b')$ to points in $(b',c']$, and $\mathcal{L}_d$ the collection of the lines in $\mathcal{L}\setminus\mathcal{L}_m$ connecting points in $[v',d')$ to points in $(d',a']$. Denote by $\sigma_i=\sigma|_{\mathcal{L}_i}$ and $V_i=E(\sigma_i)$ for $i=m,b,d$. By the linearity of the operator $E$, we have

$$V[Q']=V_m[Q']+V_b[Q']+V_d[Q'].$$

By Lemma 3,

$$V_d[Q']=V_d[\{a',b',c',d'\}]\geq 0 \text{ and } V_b[\{c',d',a',b'\}]\geq 0.$$

Then

$$V_b[Q']=V_b[\{a',b',c',d'\}]=-V_b[\{b',c',d',a'\}]=V_b[\{c',d',a',b'\}]\geq 0.$$

Therefore,

$$V[Q']\geq V_m[Q'].$$

To complete the proof, we need to work out an explicit lower bound for $V_m[Q']$. Denote by $\tilde{\sigma}_m$ the pushforward of $\sigma_m$ under $B$ and $\tilde{V}_m=E(\tilde{\sigma}_m)$. By Lemma 2,

$$V_m[Q']=\tilde{V}_m[B(Q')]=\tilde{V}_m[Q].$$

By Lemma 4, if we move the weights of the geodesic lines in the lamination $\tilde{\mathcal{L}}_m$ of $\tilde{\sigma}_m$ to the geodesic line connecting $-1$ to $s$, then the value of $\tilde{V}_m[\{b,c,d,a\}]$ is possibly increased, and hence the value of $\tilde{V}_m[Q]=-\tilde{V}_m[\{b,c,d,a\}]$ is possibly decreased. Therefore

$$\tilde{V}_m[Q]\geq(\lambda E_{-1,s})[Q],$$

where $\lambda=\sigma(\mathcal{L}_m)$. It is easy to check that

$$(\lambda E_{-1,s})[Q]=\lambda\frac{2E_{-1,s}(d)-E_{-1,s}(a)}{a-d}$$

and

$$\frac{2E_{-1,s}(d)-E_{-1,s}(a)}{a-d}=\frac{-3s^2+6s+1}{(1+s)^2}\geq\frac{-3e^2+6e+1}{(1+e)^2}>0$$

for $1\leq s\leq e$. Letting $C=\frac{(1+e)^2}{-3e^2+6e+1}$, we have

$$||V||_{cr}\geq V[Q']\geq V_m[Q']=\tilde{V}_m[Q]\geq(\lambda E_{-1,s})[Q]\geq\frac{\lambda}{C}.$$

Hence

$$\lambda\leq C||V||_{cr},$$

which implies

$$||\sigma||_{Th}\leq C||V||_{cr}.$$

□

**Theorem 2.** *There exists a universal constant $C>0$, independent of $\sigma$, such that*

$$||V_\sigma||_{cr}\leq C||\sigma||_{Th}. \tag{10}$$

*In fact, we can take $C=C_0+2C_1C_2$, where $C_0$ is the smallest positive integer greater than or equal to $\ln(3+2\sqrt{2})$, $C_1=\frac{e}{e-1}$, and $C_2$ is the smallest positive integer greater than or equal to $\ln(e+\sqrt{e^2-1})$.*

*Remark.* The inequality (10) was obtained in [1] for a different cross-ratio norm on $V$ through a complex method which considered the holomorphic differential arising from the third derivative of $V + iH(V)$, where $H(V)$ denotes the Hilbert transformation of $V$. The proof of the inequality (10) in this paper is purely real. In addition, our Theorem 1 and that inequality in [1] imply that the two cross-ratio norms on $V$ are actually equivalent (see [4]).

**Lemma 5.** *Let $Q$ be a quadruple consisting of four points $a, b, c, d$ on $\mathbb{R}^1$ or $\mathbb{S}^1$ arranged in counter-clockwise order. Take $\frac{|dz|}{y}$ as the hyperbolic metric on the upper half-plane $\mathbb{H}$. Then $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ from $a$ to $c$ is perpendicular to the geodesic $\overline{bd}$ from $b$ to $d$, and if and only if the hyperbolic distance from $\overline{ab}$ to $\overline{cd}$ (or $\overline{bc}$ to $\overline{da}$) is equal to $\ln(3 + 2\sqrt{2})$.*

*Proof.* It is straightforward (see [3] for details). □

**Lemma 6.** *Consider the upper half-plane $\mathbb{H}$. Let $l_n$ denote the geodesic connecting $-e^{-n}$ to $e^{-n}$ for each $n \in \{0\} \cup \mathbb{N}$ and $\mathcal{L}$ the lamination consisting of $l_n$'s. Suppose that $\sigma$ is an earthquake measure supported on $\mathcal{L}$, and let $\lambda_n = \sigma(l_n)$. Let $Q = \{1, \infty, -1, 0\}$, and*

$$V(x) = \int\int E_{a,b}(x) d\sigma(a, b).$$

*There exists a constant $C_1 > 0$ such that*

$$0 \leq V[Q] \leq C_1 \max_{n \geq 0} \lambda_n.$$

*Proof.* Define $\lambda = \max_{n\geq 0} \lambda_n$, $a_n = -e^{-n}$ and $b_n = e^{-n}$. Clearly, $V(1) = V(\infty) = V(-1) = 0$, and then $V[Q] = 2V(0)$. We need to work out $V(0)$, that is,

$$0 \leq V(0) = \int\int E_{a_n,b_n}(0) d\sigma(a_n, b_n) = \sum_{n\geq 0} \lambda_n \frac{a_n b_n}{b_n - a_n} = \sum_{n\geq 0} \lambda_n \frac{e^{-n}}{2} \leq \frac{1}{2} \frac{e}{e-1} \lambda.$$

Let $C_1 = \frac{e}{e-1}$. Then $0 \leq V[Q] = 2V(0) \leq C_1 \lambda$. □

We reduce the proof of Theorem 2 to Propositions 1 and 2. Let $\mathcal{L}$ be the lamination that supports $\sigma$. Given a quadruple $Q$ of four points $a, b, c, d$ on $\mathbb{S}^1$ in counter-clockwise order with $cr(Q) = 1$, we first assume that three points $a$, $b$ and $c$ belong to the same stratum $A$ and estimate $V[Q]$ in this case. By a Möbius change of coordinates and Lemma 2, we may assume that $a = 1, b = \infty, c = -1$, and $d = 0$. We will show that there is a constant $C$ such that

$$0 \leq V[Q] \leq C||\sigma||_{Th}.$$

Let $x_n$ denote the point $-e^{-n}$ and $y_n$ the point $e^{-n}$ on the real axis for each $n \in \{0\} \cup \mathbb{N}$. Let $\mathcal{L}'$ denote the collection of the lines in $\mathcal{L}$ that connect points of the interval $[-1, 0)$ to points of $(0, 1]$, $\mathcal{L}_0^-$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, x_1)$ to points of $(0, y_0]$, and $\mathcal{L}_0^+$ the collection of the lines in $\mathcal{L}'$ that connect points of $[x_0, 0)$ to points of $(y_1, y_0]$. Finally, let $\mathcal{L}_0 = \mathcal{L}_0^- \cup \mathcal{L}_0^+$. Then any line in $\mathcal{L}' \setminus \mathcal{L}_0$ must connect a point in $[x_1, 0)$ to a point in $(0, y_1]$. Inductively, for each $n \in \mathbb{N}$, let $\mathcal{L}_n^-$ denote the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{n-1})$ that connect points of $[x_n, x_{n+1})$ to points of $(0, y_n]$, and $\mathcal{L}_n^+$ the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \mathcal{L}_{n-1})$ that connect points of $[x_n, 0)$ to points of $(y_{n+1}, y_n]$, and $\mathcal{L}_n = \mathcal{L}_n^- \cup \mathcal{L}_n^+$. We have the following three lemmas.

**Lemma 7.** *For each $n \in \{0\} \cup \mathbb{N}$, any line in $\mathcal{L}_n$ must connect a point in $[x_n, 0)$ to a point in $(0, y_n]$.*

*Proof.* It can be easily proved by an induction on $n$. □

**Lemma 8.** *There exists a constant $C_2 > 0$, independent of $h$ and $\sigma$, such that $\sigma(\mathcal{L}_n) \leq C_2 ||\sigma||_{Th}$ for any $n \in \{0\} \cup \mathbb{N}$.*

*Proof.* For each $n \in \{0\} \cup \mathbb{N}$, let $l_n$ denote the geodesic line connecting the point $x_n$ to the point $y_n$. Also, for any $n \in \mathbb{N}$, let $l_n^-$ denote the geodesic connecting the point $x_n$ to 0, and $l_n^+$ the geodesic connecting 0 to the point $y_n$. The hyperbolic distance from $l_n$ to $l_{n+1}^-$ (or $l_{n+1}^+$), $n \in \{0\} \cup \mathbb{N}$, is equal to a constant, that is equal to $\ln(e + \sqrt{e^2 - 1})$. Let $C_2$ denote the smallest positive integer that is greater than or equal to $\ln(e + \sqrt{e^2 - 1})$. Then

$$\sigma(\mathcal{L}_n) \leq C_2 ||\sigma||_{Th}$$

for each $n \in \{0\} \cup \mathbb{N}$. □

Let $\tilde{V}$ be the same map as defined in Lemma 6 with $\lambda_n = \sigma(\mathcal{L}_n)$.

**Lemma 9.** *We have the following inequality:*

$$V[Q] \leq \tilde{V}[Q].$$

*Proof.* Define $\sigma_n = \sigma|_{\mathcal{L}_n}$ and $V_n = E(\sigma_n)$. Let $\tilde{\sigma}_n$ denote the atomic earthquake measure with weight $\lambda_n$ supported on the geodesic $l_n$ and let $\tilde{V}_n = E(\tilde{\sigma}_n)$. By the linearity of the operator $E$,

$$V[Q] = \sum_{n=0}^{\infty} V_n[Q] \text{ and } \tilde{V}[Q] = \sum_{n=0}^{\infty} \tilde{V}_n[Q].$$

By Lemma 3 and Lemma 7, if we move the weights of the geodesic lines in $\mathcal{L}_n$ to the goedesic line $l_n$, we only increase $V_n[Q]$, that is, $V_n[Q] \leq \tilde{V}_n[Q]$. Therefore

$$V[Q] \leq \tilde{V}[Q].$$

□

Lemmas 9, 6 and 8 imply the following proposition.

**Proposition 1.** *If $cr(Q) = 1$ and $a, b, c$ belong to the same stratum of an earthquake measure $(E, \mathcal{L})$, then*

$$0 \leq V_\sigma[Q] \leq C_1 C_2 ||\sigma||_{Th}.$$

**Proposition 2.** *Suppose $cr(Q) = 1$, and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ of $\sigma$ that separates the vertices $a$ and $b$ from the vertices $c$ and $d$. Then*

$$|V_\sigma[Q]| \leq (C_0 + 2C_1C_2)||\sigma||_{Th}.$$

*Proof.* Given two points $x$ and $y$ on the unit circle, we use $[x, y]$ to denote the arc on $\mathbb{S}^1$ from $x$ to $y$ in the counter-clockwise direction. We divide the geodesic lines in $\mathcal{L}$ that affect $V[Q]$ into five groups. Let $\mathcal{L}_m$ denote the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$. Let $\mathcal{L}_a$ denote the collection of the lines in $\mathcal{L}$ that connect points of the arc $(d, a)$ to points of the arc $(a, b)$, $\mathcal{L}_b$ the collection of the lines in $\mathcal{L}$ that connect points of the arc $(a, b)$ to points of the arc $(b, c)$, $\mathcal{L}_c$ the collection of the lines in $\mathcal{L}$ that connect

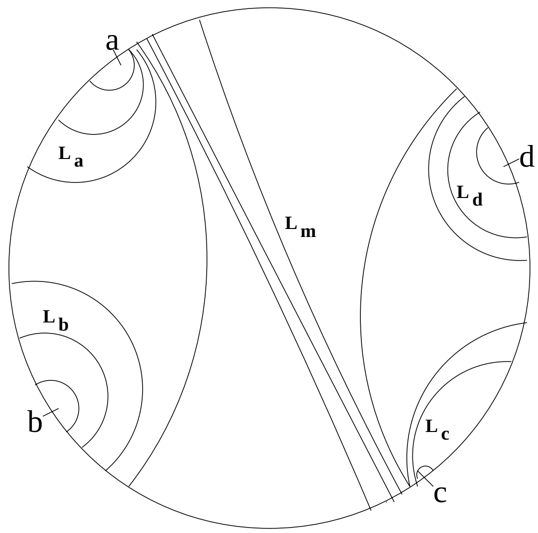


FIGURE 1. Five subcollections of $\mathcal{L}$ in the proof of Proposition 2

points of the arc $(b, c)$ to points of the arc $(c, d)$, and finally $\mathcal{L}_d$ the collection of the lines in $\mathcal{L}$ that connect points of the arc $(c, d)$ to points of the arc $(d, a)$ (see Figure 1). Let $\sigma_i$ denote the restriction of $\sigma$ on $\mathcal{L}_i$ and $V_i = E(\sigma_i)$, where $i = m, a, b, c, d$. Clearly $||\sigma_i||_{Th} \leq ||\sigma||_{Th}$ and

$$V[Q] = V_m[Q] + V_a[Q] + V_b[Q] + V_c[Q] + V_d[Q].$$

For the lamination $\mathcal{L}_d$, three points $a, b, c$ are contained in the same stratum. By Proposition 1, we have

$$0 \leq V_d[Q] \leq C_1C_2||\sigma_d||_{Th} \leq C_1C_2||\sigma||_{Th}.$$

For the lamination $\mathcal{L}_b$, three points $c, d, a$ are contained in the same stratum. Again by Proposition 1, we have

$$0 \leq V_b[\{c, d, a, b\}] \leq C_1C_2||\sigma_b||_{Th} \leq C_1C_2||\sigma||_{Th}.$$

Therefore,

$$0 \leq V_b[Q] = V_b[\{c, d, a, b\}] \leq C_1C_2||\sigma||_{Th}.$$

Similarly, we obtain

$$-C_1C_2||\sigma||_{Th} \leq V_a[Q] \leq 0 \text{ and} - C_1C_2||\sigma||_{Th} \leq V_c[Q] \leq 0.$$

It remains to consider $V_m[Q]$. Because of Lemma 2, by a Möbius change of coordinates, we may assume $a = -\infty$, $b = -1$, $c = 0$, $d = 1$. Then the geodesic lines in $\mathcal{L}_m$ connect the points of $[-1, 0]$ to the points of $[1, +\infty]$. By Lemma 4, if we move all the lines in $\mathcal{L}_m$ to the geodesic line from 0 to $\infty$ without changing the weights of the lines in $\mathcal{L}_m$ to obtain a new measure $\sigma'_m$, then $V'_m[Q] = E(\sigma'_m)[Q]$ is possibly bigger, that is, $V_m[Q] \leq V'_m[Q]$. Clearly, $V'_m(x) = 0$ for $x \in [-\infty, 0]$ and $V'_m(x) = \sigma(\mathcal{L}_m)x$ for $x \in (0, +\infty)$. Then $V'_m[Q] = V'_m(1) = \sigma(\mathcal{L}_m)$. If we let $C_0$

denote the smallest integer $\geq \ln(3+2\sqrt{2})$, then by Lemma 5, $\sigma(\mathcal{L}_m) \leq C_0||\sigma||_{Th}$. Therefore,

$$V_m[Q] \leq V'_m[Q] = \sigma(\mathcal{L}_m) \leq C_0||\sigma||_{Th}.$$

Again by Lemma 4, if we move all the lines in $\mathcal{L}_m$ to the geodesic line from $-1$ to 1 without changing the weights of the lines in $\mathcal{L}_m$ to obtain a new measure $\sigma''_m$, then $V''_m[Q] = E(\sigma''_m)[Q]$ is possibly smaller, that is, $V_m[Q] \geq V''_m[Q]$. Clearly, $V''_m(x) = 0$ for $x \notin (-1,1)$ and $V''_m(x) = \sigma(\mathcal{L}_m)\frac{(x+1)(1-x)}{2}$ for $x \in (-1,1)$. Then $V''_m[Q] = -2V''_m(0) = -\sigma(\mathcal{L}_m)$. Therefore,

$$V_m[Q] \geq V''_m[Q] = -\sigma(\mathcal{L}_m) \geq -C_0||\sigma||_{Th}.$$

Collecting together these estimates, we obtain

$$-(C_0+2C_1C_2)||\sigma||_{Th} \leq V_a[Q]+V_c[Q]+V_m[Q]+V_b[Q]+V_d[Q] \leq (C_0+2C_1C_2)||\sigma||_{Th};$$

that is,

$$|V[Q]| \leq (C_0 + 2C_1C_2)||\sigma||_{Th}.$$

□

Propositions 1 and 2 imply Theorem 2.

*Proof.* Consider the pattern of the geodesic lines in $\mathcal{L}$ with respect to $Q$, there either exists a line in $\mathcal{L}$ separating two adjacent vertices in $Q$ from the other two or no such line exists. The former case is treated by Proposition 2 and the fact that $V[\{b,c,d,a\}] = -V[Q]$; the latter one is treated by Proposition 1 and the summability of $V[Q] = E(\sigma)[Q]$ over four subcollections of $\mathcal{L}$ that contain three vertices of $Q$ in the same stratum. □

Finally, Theorems 1 and 2 imply our Main Theorem.

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