FINITE $s$-ARC TRANSITIVE CAYLEY GRAPHS
AND FLAG-TRANSITIVE PROJECTIVE PLANES

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Abstract. In this paper, a characterisation is given of finite $s$-arc transitive Cayley graphs with $s \geq 2$. In particular, it is shown that, for any given integer $k$ with $k \geq 3$ and $k \neq 7$, there exists a finite set (maybe empty) of $s$-transitive Cayley graphs with $s \in \{3, 4, 5, 7\}$ such that all $s$-transitive Cayley graphs of valency $k$ are their normal covers. This indicates that $s$-arc transitive Cayley graphs with $s \geq 3$ are very rare. However, it is proved that there exist 4-arc transitive Cayley graphs for each admissible valency (a prime power plus one). It is then shown that the existence of a flag-transitive non-Desarguesian projective plane is equivalent to the existence of a very special arc transitive normal Cayley graph of a dihedral group.

1. Introduction

A graph $\Gamma$ is a Cayley graph if there exist a group $G$ and a subset $S \subseteq G$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$ such that the vertices of $\Gamma$ may be identified with the elements of $G$ in such a way that $x$ is connected to $y$ if and only if $yx^{-1} \in S$. The Cayley graph $\Gamma$ is denoted by $\text{Cay}(G, S)$. Cayley graphs stem from a type of diagram now called a Cayley color diagram, introduced by Cayley in 1878. In this paper, we investigate the symmetric Cayley graphs and a relation between them and finite flag-transitive projective planes.

A graph $\Gamma$ is said to be $(X, s)$-arc transitive if $X \subseteq \text{Aut} \Gamma$ is transitive on vertices and $s$-arcs of $\Gamma$ where $s$ is a positive integer. (A sequence $v_0, v_1, \ldots, v_s$ of vertices of $\Gamma$ is called an $s$-arc if $v_i$ is adjacent to $v_{i+1}$ for $0 \leq i \leq s - 1$ and $v_i \neq v_{i+1}$ for $1 \leq i \leq s - 1$.) An $(X, s)$-arc transitive graph is called $(X, s)$-transitive if it is not $(X, s + 1)$-arc transitive. In particular, if $X = \text{Aut} \Gamma$, then an $(X, s)$-arc transitive graph and an $(X, s)$-transitive graph are simply called $s$-arc transitive and $s$-transitive, respectively.

Interest in $s$-arc transitive graphs stems from a seminal result of Tutte in 1947, who proved that there exist no finite $s$-transitive cubic graphs for $s \geq 6$. Tutte’s Theorem was generalized by Weiss [24] who proved that there exist no finite $s$-transitive graphs of valency at least 3 for $s = 6$ and $s \geq 8$. Since then, characterizing
s-arc transitive graphs has received considerable attention in the literature (see, for example, \cite{11,14,19,24}).

Let $G$ be a group, and let $\Gamma = \text{Cay}(G, S)$. Then $\Gamma$ has an automorphism group:

$$\hat{G} = \{ \hat{g} : x \rightarrow xg \text{ for all } x \in G \mid g \in G \},$$

consisting of right multiplications of elements $g \in G$. The subgroup $\hat{G}$ acts regularly on the vertex set of $\Gamma$. If $\hat{G}$ is normal in $\text{Aut}\Gamma$, then $\Gamma$ is called a normal Cayley graph; if the core of $\hat{G}$ in $\text{Aut}\Gamma$ has index 2 in $\hat{G}$, then $\Gamma$ is called a bi-normal Cayley graph, while if $\hat{G}$ is core free in $\text{Aut}\Gamma$, then $\Gamma$ is called a core free Cayley graph. (Recall that the core of a subgroup $H$ of a group $X$ is the largest normal subgroup of $X$ contained in $H$.) These three classes of Cayley graphs are important for studying $s$-arc transitive Cayley graphs. Normal Cayley graphs have some very nice properties; refer to \cite{21,25}, and also to Lemma 2.2 and Proposition 2.3. Several nice properties of bi-normal Cayley graphs are given in Lemma 2.4 and Corollary 2.7. Some properties for core-free Cayley graphs are given in Section 3. An interesting result of Section 3 is Proposition 3.2, which says that

"almost all" vertex primitive Cayley graphs are normal Cayley graphs.

Let $N \triangleleft X$, and let $B$ be the set of $N$-orbits in $V$. Then the normal quotient graph $\Gamma_N$ of $\Gamma$ induced by $N$ is the graph with vertex set $B$ such that $B, B' \in B$ are adjacent if and only if some vertex $u \in B$ is adjacent in $\Gamma$ to some vertex $v \in B'$. If $\Gamma$ and $\Gamma_N$ have the same valency, then $\Gamma$ is called a normal cover of $\Gamma_N$.

Some special classes of 2-arc transitive Cayley graphs have been studied; see \cite{1,2,11,15,18}. One of the main results of this paper is stated in Theorem 1.1, which tells us that $s$-arc transitive Cayley graphs with $s \geq 3$ are rare. Throughout this paper, denote by $\mathcal{G}(s,k)$ the set of core-free $s$-transitive Cayley graphs of valency $k$.

**Theorem 1.1.** For any integers $s \in \{2,3,4,5,7\}$ and $k \geq 3$, the set $\mathcal{G}(s,k)$ is finite, and for each $s$-transitive Cayley graph $\Gamma$, one of the following statements holds:

(i) $\Gamma$ is a normal or a bi-normal Cayley graph, and either $s = 2$, or $(s,k) = (3,7)$ and $\Gamma$ is bi-normal (so bipartite);

(ii) $\Gamma$ is a normal cover of a member of $\mathcal{G}(s,k)$.

Moreover, $\mathcal{G}(2,k)$ and $\mathcal{G}(3,k)$ are not empty for all $k \geq 3$, and $\mathcal{G}(4,q + 1)$ is not empty for all prime powers $q$; in particular, there exist 4-transitive Cayley graphs of all admissible valencies.

**Remarks.** (a) It is easily shown that complete graphs are members of $\mathcal{G}(2,k)$, and complete bipartite graphs are members of $\mathcal{G}(3,k)$. Some other examples of members in $\mathcal{G}(2,k)$ and $\mathcal{G}(3,k)$ will be constructed in Section 4.

(b) The examples in $\mathcal{G}(4,q + 1)$ with $q$ being a power of a prime are the incidence graphs of Desarguesian projective planes. It is known that the valency of a 4-transitive graph is $r+1$ such that $r$ is a power of a prime; see \cite{24}. Thus, Theorem 1.1 tells us that for all admissible valencies, there exist 4-transitive Cayley graphs. However, we do not know any other examples of 4-transitive Cayley graphs, and we do not have any examples of 5- or 7-transitive Cayley graphs at all.

(c) There have been some results regarding certain special classes of 2-arc transitive normal and bi-normal Cayley graphs; see \cite{2,11,15,20}. However, the general
case is still not quite well-understood. In particular, we do not know any examples of 3-transitive bi-normal Cayley graphs.

We would like to propose the following question and problem.

**Question 1.2.** (a) Do there exist 3-transitive bi-normal Cayley graphs?
(b) Do there exist \(s\)-transitive Cayley graphs for \(s = 5\) and \(s = 7\)?

**Problem 1.3.** (a) Determine members of \(G(s, k)\) for \(k \geq 3\) and \(s \geq 2\).
(b) Give a satisfactory description of bi-normal \(2\)-transitive Cayley graphs.

Let \(\Pi\) be a projective plane of order \(n\) with point set \(\mathcal{P}\) and line set \(\mathcal{L}\). Then \(\Pi\) has \(n^2 + n + 1\) points and \(n^2 + n + 1\) lines such that (a) any two points lie on exactly one line and any two lines intersect at exactly one point, (b) each line contains exactly \(n + 1\) points and each point lies on exactly \(n + 1\) lines. A flag of \(\Pi\) is a pair of point \(p\) and line \(l\) such that \(p\) lies on \(l\). A permutation of flags preserving the incidence of \(\Pi\) is an automorphism of \(\Pi\); that is, the automorphism group of \(\Pi\). A plane \(\Pi\) is called flag-transitive if \(\text{Aut}\Pi\) is transitive on its flags. Let \(\Gamma\) be the incidence graph of \(\Pi\); that is, the vertex set of \(\Gamma\) is \(\mathcal{P} \cup \mathcal{L}\), and the edge set of \(\Gamma\) is the set of flags of \(\Pi\). Then \(\text{Aut}\Gamma = \text{Aut}\Pi\) or \((\text{Aut}\Pi),\mathbb{Z}_2\); so if \(\Pi\) is flag-transitive, then \(\Gamma\) is edge-transitive.

The classical (Desarguesian) projective planes have been well studied, and their incidence graphs give the first family of 4-arc transitive Cayley graphs, presented in Example 4.5. The existence problem of non-Desarguesian flag-transitive projective planes is a long-standing open problem in finite geometry; see, for example, books \[6, 17\], and articles \[7, 12, 22\]. Here we prove that the existence of a non-Desarguesian projective plane is equivalent to the existence of a type of Cayley graph of a dihedral group satisfying very restricted properties.

Let \(n\) be a 2-power such that \(p := n^2 + n + 1\) and \(q := n + 1\) are primes. Let \(G = \langle a \rangle \rtimes \langle z \rangle \cong D_{2p}\), where \(a^2 = a^{-1}\). Let \(\sigma \in \text{Aut}(\langle a \rangle)\) be such that \(\sigma z = z \sigma\) and \(o(\sigma z) = 2q\). Let \(S = \langle az \rangle^{\langle \sigma \rangle} = \{az, a^r z, a^{2r} z, \ldots, a^{r-1} z\}\), and set \(\Gamma(q) = \text{Cay}(G, S)\). It will be shown in Lemma \[5.1\] that \(\Gamma(q)\) has full automorphism group \(\text{Aut}\Gamma(q) = G \rtimes \langle \sigma z \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}\), a Frobenius group of order \(2pq\), and thus \(\Gamma(q)\) is a normal Cayley graph of the dihedral group \(G \cong D_{2p}\). The following theorem shows that the girth of \(\Gamma(q)\) determines the existence of flag-transitive non-Desarguesian projective planes of order \(q - 1\).

**Theorem 1.4.** Using the notation defined above, there exists a flag-transitive non-Desarguesian projective plane of order \(n\) if and only if the girth of the Cayley graph \(\Gamma(q)\) is equal to 6.

As widely believed, all flag-transitive projective planes are Desarguesian, and hence we make the following conjecture; see Lemma \[5.1\].

**Conjecture 1.5.** For each prime power \(q\), the graph \(\Gamma(q)\) defined above has diameter 4 and girth 4.

Theorem \[1.4\] tells us that if this conjecture is true, then there exist no finite non-Desarguesian flag-transitive projective planes.
2. Automorphism groups of Cayley graphs: Normality

It is known that a graph $\Gamma$ is a Cayley graph of a group $G$ if and only if its automorphism group contains a subgroup that is isomorphic to $G$ and acts regularly on vertices; see, for example, [3] Proposition 16.3. It is hence natural to use the regular subgroup to describe properties of $\Gamma$ and $\text{Aut}\Gamma$.

2.1. Normal Cayley graphs. Let $G$ be a finite group, and let $\Gamma = \text{Cay}(G,S)$. Let

$$\text{Aut}(G,S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$ 

Let $\mathbb{1}$ be the vertex of $\Gamma$ corresponding to the identity of $G$. It follows from the definition that each element of $\text{Aut}(G,S)$ induces an automorphism of $\Gamma$ fixing $\mathbb{1}$, and so $\text{Aut}(G,S) \leq (\text{Aut}\Gamma)_1$. Moreover, we have the following statement.

Lemma 2.2 (S). For a Cayley graph $\Gamma = \text{Cay}(G,S)$, the following holds:

$$\text{N}_{\text{Aut}}(\hat{G}) = \hat{G} \rtimes \text{Aut}(G,S).$$

The subgroup $\text{Aut}(G,S)$ plays an important role in the study of Cayley graphs; refer to [S, 21, 25]. Some structural information may be read out from $\text{Aut}(G,S)$; for instance, if $\text{Aut}(G,S)$ is transitive on $S$, then the Cayley graph $\Gamma$ is $N_{\text{Aut}}(\hat{G})$-arc transitive; if $\text{Aut}(G,S)$ is 2-transitive on $S$, then $\Gamma$ is $(N_{\text{Aut}}(\hat{G}),2)$-arc transitive.

For a group $X$ with $\hat{G} \leq X \leq \text{Aut}\Gamma$, if $\hat{G}$ is normal in $X$, then $X_1 \leq \text{Aut}(G,S)$, and $\Gamma$ is called an $X$-normal Cayley graph of $G$. In particular, if $X = \text{Aut}\Gamma$, then $\Gamma$ is a normal Cayley graph, and $\text{Aut}\Gamma = \hat{G} \rtimes \text{Aut}(G,S)$. The action of $X$ on the vertices of $\Gamma$ behaves in a very nice way:

Lemma 2.2. Let $\Gamma = \text{Cay}(G,S)$, and let $X \leq \text{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of $X$. Then $X = \hat{G} \rtimes X_1$, and for an element $x = \hat{g}y \in X$, where $\hat{g} \in \hat{G}$ and $y \in X_1$, and for any vertex $h \in G$, we have

$$h^x = h^{\hat{g}y} = (h\hat{g})^y = y^{-1}h\hat{g}y.$$ 

This property implies the following result about $s$-arc transitive Cayley graphs, which was obtained in [13].

Proposition 2.3. Let $\Gamma = \text{Cay}(G,S)$, and let $X \leq \text{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of $X$. Then $\Gamma$ is not $(X,3)$-arc transitive.

2.2. Bi-normal Cayley graphs. Let $\Gamma = \text{Cay}(G,S)$, and let $\hat{G} \leq X \leq \text{Aut}\Gamma$. In this section we study the important case where $\hat{G}$ is not normal in $X$ but $\text{core}_X(\hat{G})$ is “big”. A Cayley graph $\text{Cay}(G,S)$ is called an $X$-bi-normal Cayley graph of $G$ if the core $\text{core}_X(\hat{G})$ has index 2 in $\hat{G}$. Hence if $X = \text{Aut}\Gamma$, then an $X$-bi-normal Cayley graph is a bi-normal Cayley graph.

For an $X$-normal Cayley graph $\Gamma = \text{Cay}(G,S)$, $X_1$ acts on the vertex set $G$ by conjugation, and so if $\Gamma$ is connected, then $X_1$ is faithful on $S$. For an $X$-bi-normal Cayley graph $\text{Cay}(G,S)$, the $X_1$-action only on half of its vertices is by conjugation; however, $X_1$ is still faithful on $S$ under certain conditions, as shown below.

Lemma 2.4. Let $\Gamma$ be a connected bipartite graph with biparts $U$ and $U'$. Assume that $X \leq \text{Aut}\Gamma$ is transitive on the vertex set $V\Gamma$, and assume furthermore that $X$ has a normal subgroup $N \leq X^+$ such that $N$ is intransitive on $V\Gamma$, $N_v$ is transitive on $\Gamma(v)$ for some vertex $v \in U$, and $N$ has a normal subgroup that is regular on both $U$ and $U'$. Then $N_v$ acts faithfully on $\Gamma(v)$. 

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Proof. Choose \( v \in U \), and let \( R = MN_v^{[1]} \). Since \( U \) is an orbit of \( N \) and \( M \) acts transitively on \( U \), we have \( N = MN_v \). Since \( M \) is normal in \( N \) and \( N_v^{[1]} \) is normal in \( N_v \), it follows that \( R \) is normal in \( N \) and that \( R = MN_w^{[1]} \) (and hence \( R_w \leq N_w^{[1]} \)) for all \( w \in U \). Now let \( \{u, v\} \) be an arbitrary edge. We may assume that \( u \in U \) and \( v \in U' \). Let \( x \in R_u \). Since \( R = MN_v^{[1]} \), we have \( x = yz \) where \( y \in M \) and \( z \in N_v^{[1]} \). Since \( N_v^{[1]} \leq R_u \), we have \( z \in R_u \). It then follows that \( y \in M \cap R_u \). By the assumption that \( M \) is regular on \( U \), we have \( M \cap R_u = 1 \). Thus \( y = 1 \). We conclude that \( x = z \in N_v^{[1]} \), and so \( R_u \leq N_v^{[1]} \leq R_v \). Since \( \{u, v\} \) is arbitrary, it follows that \( R_u \leq N_v^{[1]} \) for every \( u \in U' \). Thus \( R_u \leq N_v^{[1]} \) for every \( u \in VT \). Since \( \Gamma \) is connected, it follows that \( R_u v = 1 \) for every edge \( \{u, v\} \). Therefore, \( N_v^{[1]} = 1 \) for every vertex \( u \in VT \).

By inspecting the classification of finite 2-transitive permutation groups (see [4]), the statement of the next lemma is easily obtained.

Lemma 2.5. Let \( T \) be a 2-transitive permutation group on a set \( \Omega \) of degree \( n \). Then for two points \( \omega, \omega' \in \Omega \), the point stabilizer \( T_{\omega,\omega'} \) has a transitive permutation representation of degree \( n - 1 \) if and only if \( \text{soc}(T) = A_7 \) or \( S_7 \) and \( n = 7 \).

The next result tells us that for a \((G, s)\)-arc transitive graph, if the vertex stabilizer is faithful on the neighborhood, then \( s \) is small.

Proposition 2.6. Let \( \Gamma \) be an \((X, s)\)-transitive graph of valency \( k \), where \( G \leq \text{Aut}\Gamma \) and \( s \geq 1 \). Assume that \( X_v \) is faithful on \( \Gamma(v) \). Then either \( s \leq 2 \), or \((k, s) = (7, 3)\) and \( X_v \cong A_7 \) or \( S_7 \).

Proof. By Lemma 2.5, the vertex stabilizer \( X_v \) acts faithfully on \( \Gamma(v) \); that is, \( X_v \cong X_v^{\Gamma(v)} \), and thus \( X_v \) is a 2-transitive permutation group on \( \Gamma(v) \). Suppose furthermore that \( \Gamma \) is \((X, 3)\)-arc transitive. Then for distinct vertices \( u, w \in \Gamma(v) \), the stabilizer \( X_{uvw} \) of the 2-arc \((u, v, w)\) is transitive on \( \Gamma(w) \setminus \{v\} \). In particular, \( X_{uvw} \) has a transitive permutation representation of degree \( k - 1 \). Note that \( X_{uvw} \) is the stabilizer of \( u, w \) of the 2-transitive permutation group \( X_v \cong X_v^{\Gamma(v)} \). Thus by Lemma 2.6, we conclude that \( X_v \cong A_7 \) or \( S_7 \), \( X_{uw} \cong A_6 \) or \( S_6 \), and \( X_{uvw} \cong A_5 \) or \( S_5 \), respectively. In particular, \( k = 7 \), completing the proof of the proposition.

Combining Lemma 2.4 and Proposition 2.6, we have the following statement for bi-normal Cayley graphs.

Corollary 2.7. Let \( \Gamma = \text{Cay}(G, S) \) be a connected graph of valency \( k \). Assume that \( \hat{G} < X \leq \text{Aut}\Gamma \) is such that \( \Gamma \) is \( X \)-bi-normal and \((X, 2)\)-arc transitive. Then \( X_1 \) is faithful on \( \Gamma(1) \), and furthermore, either

(i) \( \Gamma \) is not \((X, 3)\)-arc transitive, or
(ii) \( k = 7 \), \( X_1 \cong A_7 \) or \( S_7 \), and \( \Gamma \) is \((X, 3)\)-transitive.

3. Core-free Cayley graphs and their group duals

Let \( \Gamma = \text{Cay}(G, S) \), and let \( X \leq \text{Aut}\Gamma \) contain \( \hat{G} \). Assume furthermore that \( \hat{G} \) is not normal in \( X \). If the core of \( \hat{G} \) in \( X \) is trivial, then \( \Gamma \) is a core-free Cayley graph for if \( \hat{G} \) is core-free in \( X \), then \( \hat{G} \) is core-free in \( \text{Aut}\Gamma \).
Suppose now that $\hat{G}$ is core-free in $X$. Then $X$ has an exact factorization with core-free factors:

$$X = \hat{G}H,$$

where $H = X_1$. The action of $X$ on $[X : \hat{G}]$ gives rise to Cayley graphs of the group $H$, called dual Cayley graphs of $\Gamma$. In this case, we observe that the order of $X$ and hence the order of $G$ are up-bounded in terms of the order of $X_1$; that is,

$$\hat{G} < X \leq \text{Sym}(|X_1|).$$

It is known that for many important classes of graphs, the order of $X_1$ is further up-bounded in terms of the valency: for example, cubic edge-transitive graphs, 2-arc transitive graphs, and vertex-primitive graphs. This therefore gives an up-bound of $|G|$ and $|X|$ in terms of the valency. This implies that $X$ has two faithful transitive permutation representations, that is, on $[X : \hat{G}]$ and on $[X : H]$, and the statements in the following lemma are obviously true.

**Lemma 3.1.** Let $\Gamma$ be a Cayley graph of $G$, and let $X \leq \text{Aut}\Gamma$ be such that $\hat{G} < X$ and $\text{core}_X(\hat{G}) = 1$. Then $X$ has a faithful transitive permutation representation on the set $[X : \hat{G}]$, and $X_1$ is a regular subgroup of $X$ acting on $[X : \hat{G}]$. In particular, the order $|X|$ is up-bounded in terms of $|X_1|$.

Each generalized orbital graph of $X$ on the set $[X : \hat{G}]$ is therefore a Cayley graph of the group $H = X_1$, which is a dual Cayley graph of $\text{Cay}(G, S)$. Considering the dual permutation representation of a vertex-primitive automorphism group of a Cayley graph, we have the following result.

**Proposition 3.2.** For every given integer $k$, there are only finitely many vertex-primitive Cayley graphs of valency $k$ that are not normal Cayley graphs.

**Proof.** Let $\Gamma = \text{Cay}(G, S)$ be such that $X = \text{Aut}\Gamma$ is primitive on $V \Gamma$. It follows that $\hat{G}$ in $X$ is either normal or core-free. By [3], the order of $X_1$ is up-bounded in terms of the valency of an orbital graph of $X$ on $V \Gamma$, where $1$ is the identity of $G$. Thus $|X_1|$ is up-bounded in terms of the valency of $\Gamma$. If $\Gamma$ is core-free, then $X$ has a faithful representation on $[X : \hat{G}]$, which is of degree $|X_1|$. Hence $|X| \leq |X_1|!$, and so the order of $X$ is up-bounded in terms of the valency of $\Gamma$, and so the number of vertex-primitive Cayley graphs of valency $k$ that are not normal Cayley graphs is up-bounded in terms of $k$.

This shows that, although they are very special, normal Cayley graphs are not very rare. Similarly, we have the next proposition regarding 2-arc transitive Cayley graphs.

**Proposition 3.3.** For any given positive integer $k$, there are at most finitely many core-free 2-arc transitive Cayley graphs of valency $k$.

**Proof.** Let $\Gamma = \text{Cay}(G, S)$ be an $(X, 2)$-arc transitive graph of valency $k = |S|$ such that $\hat{G} < X \leq \text{Aut}\Gamma$ and $\hat{G}$ is core-free in $X$. By [23], the order of $X_1$ is up-bounded in terms of the valency $k$, where $1$ is the identity of $G$. Since $\hat{G}$ is core-free in $X$, the group $X$ has a faithful transitive representation on $[X : \hat{G}]$, which is of degree $|X_1| = |X : \hat{G}|$. Hence $|X| \leq |X_1|!$, and so the order of $X$ is up-bounded in terms of $k$. In particular, the number of core-free 2-arc transitive Cayley graphs of valency $k$ is up-bounded in terms of $k$.\[\square\]
4. Examples

In this section, we construct examples of $s$-arc transitive core-free Cayley graphs, where $s \geq 2$.

It is obvious that a complete graph $K_n$ of $n$ vertices with $n \geq 4$ has an automorphism group isomorphic to $S_n$ and is a 2-transitive Cayley graph of an arbitrary group of order $n$. Also, it is easy to see that a complete bipartite graph $K_{n,n}$, with $n \geq 3$ has an automorphism group $S_n \wr S_2$, and is a 3-transitive Cayley graph of valency $n$. This shows that $G(2,k)$ and $G(3,k)$ are nonempty for all $k \geq 3$. Except for these “trivial” examples, various other $s$-arc transitive core-free Cayley graphs will be constructed in this section.

Let $T$ be a group, and let $S$ be a core-free subgroup of $T$. Then $S$ has automorphism group $\text{Aut}(S)$. Let $Z$ be a group, where $Z_s = \{Z \mid x \in X\}$. For an element $g \in Z$ such that $g^2 \in S$, a coset graph

$$\Gamma := \text{Cos}(Z, H, HgH)$$

is defined as the graph with vertex set $V = \{Z : H\}$ such that $Hx$ is adjacent to $Hy$ if and only if $yx^{-1} \in HgH$. We have the following well-known properties.

Lemma 4.1. Let $\Gamma = \text{Cos}(X, H, HgH)$. Then

(i) $\Gamma$ is connected if and only if $(H, g) = X$;

(ii) $\Gamma$ is $(X, 2)$-arc transitive if and only if $H$ is $2$-transitive on $[H : H \cap Hg]$.

The first example gives a family of 2-arc transitive core-free Cayley graphs of prime-power valency.

Example 4.2. Let $X = S_p^r$ where $p$ is a prime, acting on $\Omega = \{1, 2, \ldots, p^r\}$. Then $X$ has two subgroups $G, H$ such that $G = X_{1,2} \cong S_{p^r-2}$ and $H \cong \text{AGL}_1(p^r) \cong \mathbb{Z}_p^r \times \mathbb{Z}_{p^r-1}$ is 2-transitive on $\Omega$. It follows that $G \cap H = 1$ and $X = GH$. Write $H = N \rtimes \langle z \rangle$, where $N \cong \mathbb{Z}_p^r$ is regular on $\Omega$, and $z = (1, 2, 3, \ldots, p^r-1)$. It is easily shown that there exists an involution $g \in X$ such that $z^g = z^{-1}$ and $(H, g) = X$. Let $\Gamma = \text{Cos}(X, H, HgH)$. Then $\Gamma$ is a connected 2-arc transitive graph of valency $p^r$ and is a Cayley graph of $G$. \hfill \Box

The next two examples were constructed in [16] as examples of quasiprimitive 2-arc transitive graphs of a certain type. Here we prove they are Cayley graphs.

Example 4.3. Let $\Gamma$ be a graph constructed in Example 4.2 with $r = 1$ and $p \geq 5$. Then $\Gamma$ is a 2-arc transitive Cayley graph of $S_{p-2}$. Let $l$ be a divisor of $(p - 1)/2$, and let

$$Y = (T_1 \times T_2 \times \cdots \times T_l) \ltimes (O \times L),$$

where $T_i \cong A_p$, $O \cong \mathbb{Z}_2$ normalizes but does not centralize $T_i$ for each $i$, and $L = ((1, 2, 3, \ldots, l)) \cong \mathbb{Z}_l$ permutes cyclically the direct factors $T_1, T_2, \ldots, T_l$.

It is shown in [16] that $Y$ has a subgroup $K = (\mathbb{Z}_p \times \mathbb{Z}_{p-1}) \times \mathbb{Z}_l$ and an element $g \in Y$ such that the coset graph $\Gamma := \text{Cos}(Y, K, KgK)$ is connected and $(Y, 2)$-arc transitive of valency $p$. It is now easily shown that the subgroup $R \times T_2 \times \cdots \times T_l$, where $S_{p-2} \cong R < T_1$, acts regularly on $VT$. Hence $\Gamma$ is a Cayley graph. \hfill \Box

The following example presents a family of 3-arc transitive Cayley graphs.

Example 4.4. Let $T = \text{PSL}(2, q)$, where $q = 2^e$ with $e \geq 2$, and let $H = \mathbb{Z}_2^e \ltimes D$, where $D \cong \mathbb{Z}_{2^e-1}$. Let $f \in T$ be such that $(D, f) \cong D_{2^e}$. Then $\Sigma = \text{Cos}(T, H, HfH) \cong \text{K}_r$ is $(T, 2)$-arc transitive. Let

$$Y = (T_1 \times T_2 \times \cdots \times T_{q-1}) \ltimes L,$$
where $T_i \cong A_p$, and $L = \langle (1, 2, 3, \ldots, q - 1) \rangle \cong \mathbb{Z}_{q-1}$ permutes cyclically the direct factors $T_1, T_2, \ldots, T_q$.

It is shown in [10] that $Y$ has a subgroup $K \cong (\mathbb{Z}_q \times \mathbb{Z}_{q-1}) \times \mathbb{Z}_{q-1}$ and an element $g \in Y$ such that the coset graph $\Gamma := \text{Cos}(Y, K, KgK)$ is connected and $(Y, 3)$-arc transitive of valency $p$. It is now easily shown that the subgroup $R \times T_2 \times \cdots \times T_l$, where $\mathbb{Z}_{q+1} \cong R < T_1$, acts regularly on $VT$. Hence $\Gamma$ is a Cayley graph. \hfill \Box

Finally, we present a family of 4-arc transitive Cayley graphs.

**Example 4.5.** Let $\Gamma$ be the incidence graph of a Desarguesian projective plane $\Pi$ of order $n$. Then $n$ is a prime power, and $\text{Aut}\Pi = \text{PGL}(3, n)$; see [10]. It is known that the incidence graph $\Gamma$ is 4-arc transitive of valency $n + 1$, and $\text{Aut}(\Gamma) = \text{Aut}(\Pi, Z_2) = \text{Aut}(\text{PSL}(3, n))$. Now the Singer subgroup $C \cong Z_{n^2+n+1}$ is regular on both the point set $P$ and the line set $L$. There exists an involution $\tau \in \text{Aut}(\Pi)\backslash\text{Aut}(\Gamma)$ such that $\tau$ normalizes $C$ (refer to [10, Lemma 3.2]). Hence $\text{Aut}(\Gamma)$ has a subgroup $G := \langle C, \tau \rangle \cong Z_{n^2+n+1} \rtimes Z_2$ that is regular on the vertices of the graph $\Gamma$. So $\Gamma$ is a Cayley graph of a group isomorphic to $G$. \hfill \Box

5. **Proofs of Theorems 1.1 and 1.4**

In this section, we prove Theorems 1.1 and 1.4.

**Proof of Theorem 1.1.** Let $\Gamma = \text{Cay}(G, S)$ be an $(X, s)$-arc transitive graph of valency $k$ with vertex set $V$, where $s \geq 2$. Assume furthermore that $\hat{G} < X$. Let $C = \text{core}_X(\hat{G})$.

Assume first that $C = \hat{G}$. Then $\hat{G}$ is normal in $X$, and hence by Proposition 2.6 we have that $s = 2$, as in part (i) of Theorem 1.1.

Assume that $C$ has exactly two orbits in $V$. Then $\hat{G}$ is bi-normal in $X$, and $\Gamma$ is bipartite. Hence by Corollary 2.7, we have that either $s = 2$, or $(s, k) = (3, 7)$, as in part (i) of Theorem 1.1.

Finally, assume that $C$ has at least three orbits in $V$. Then $\hat{G}/C$ is core-free in $X/C$ and regular on the vertex set of $\Gamma_C$, $X/C \leq \text{Aut}(\Gamma_C)$, and $\Gamma_C$ is an $(X/C, s)$-arc transitive graph of valency $k$. Thus $\Gamma_C$ is a member of $\mathcal{G}(s, k)$, and $\Gamma$ is a normal cover of $\Gamma_C$, as in part (ii) of Theorem 1.1.

The existence of graphs in $\mathcal{G}(s, k)$ with $s \leq 4$ is justified by the examples given in Section 4. This completes the proof of Theorem 1.1. \hfill \Box

To prove Theorem 1.4, we first investigate the graphs $\Gamma(q)$. Let $n$ be a prime power such that $p := n^2 + n + 1$ and $q := n + 1$ are primes. Let $G = D_{2p} = \langle a \rangle \rtimes \langle z \rangle$, where $a^2 = a^{-1}$. Let $\sigma \in \text{Aut}(\langle a \rangle)$ be such that $\sigma z = z \sigma$ and $o(\sigma z) = 2q$. Let $S = \langle az \rangle^\sigma = \{az, a^2z, a^3z, \ldots, a^{q-1}z\}$, and set $\Gamma(q) = \text{Cay}(G, S)$.

**Lemma 5.1.** Let $\Gamma = \Gamma(q)$, and let $d(\Gamma)$ and $g(\Gamma)$ be the diameter and the girth of $\Gamma$, respectively. Then

(i) $\Gamma$ is of valency $q$, and $\text{Aut}(\Gamma) = \langle a \rangle \rtimes (z \sigma) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q}$;
(ii) either $(d(\Gamma), g(\Gamma)) = (3, 6)$, or $g(\Gamma) = 4$. 

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Proof. The graph $\Gamma$ is a bipartite graph of valency $q$ with biparts $\Delta$ and $\Delta'$ say. Write $A = \text{Aut}(\Gamma)$ and $A^+ = A + \Delta = A G$. Then $A \geq G \times \langle \sigma \rangle$, and $A^+ \geq \langle a \rangle \times \langle \sigma \rangle \cong \mathbb{Z}_q \rtimes \mathbb{Z}_q$. In particular, $A^+$ is a primitive permutation group of degree $p$. Thus by Burnside’s theorem, either $A^+$ is affine or an almost simple group. It then follows from the result of [9] that either $\langle a \rangle \triangleleft A^+$, or $\text{soc}(A^+) = \text{PSL}(d, r)$ with $p = \frac{r-1}{r-1}$, or $A_p$. Suppose that $\text{soc}(A^+) = \text{PSL}(d, r)$. Then the cyclic group $\langle a \rangle$ is a Singer cycle of $\text{PSL}(d, r)$, and so $\text{N}_{\text{PSL}(d, r)}(\langle a \rangle) \leq \langle a \rangle \rtimes \mathbb{Z}_d$. Thus the order $o(\sigma)$ divides $d$, which is not possible. Suppose that $\text{soc}(A^+) = A_p$. Then $A_1 = A_{p-1}$. Since the smallest transitive permutation representation of $A_{p-1}$ is $p - 1$, it follows that $\Gamma$ has valency at least $p - 1$, which is a contradiction. Thus $\langle a \rangle$ is normal in $A^+$. It then follows that $\text{Aut}(\Gamma) = G \times \langle \sigma \rangle$, as in part (i).

Suppose that the girth $g(\Gamma)$ of $\Gamma$ is at least 6. Since $|\Gamma(\overline{1})| = n + 1$, we have that $|\Gamma(\overline{1})| = n^2 + n$. Furthermore, since $|\Gamma| = 2(n^2 + n + 1)$, we conclude that $|\Gamma(\overline{1})| \leq |\Gamma| - (1 + (n + 1) + (n^2 + n)) = n^2$. It then follows that $g(\Gamma) = 6$, $|\Gamma(\overline{1})| = n^2$, and the diameter $d(\Gamma) = 3$. Suppose finally that the girth $g(\Gamma)$ of $\Gamma$ is less than 6. Then since $\Gamma$ is bipartite, $g(\Gamma) = 4$. \hfill $\Box$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\Pi$ be a projective plane of order $n$, where $n$ is a natural integer. Let $\Gamma$ be the incidence graph of $\Pi$; that is, the vertex set of $\Gamma$ is $\mathcal{P} \cup \mathcal{L}$, and the edge set of $\Gamma$ is the set of flags of $\Pi$. Then $\Gamma$ is a bipartite graph with biparts $\mathcal{P}$ and $\mathcal{L}$, and either $\text{Aut}(\Gamma) = \text{Aut}(\Pi)$, or $\text{Aut}(\Gamma) = (\text{Aut}(\Pi)).\mathbb{Z}_2$.

Assume that $\Pi$ is not Desarguesian, and assume that $\Pi$ is flag-transitive. By [12] (or refer to [7]), $n$ is a 2-power, both $q := n + 1$ and $p := n^2 + n + 1$ are primes, and $\text{Aut}(\Pi) = \langle a \rangle \times \langle \sigma \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is a Frobenius group, where $o(a) = p$ and $o(\sigma) = q$.

The subgroup $\text{Aut}(\Pi)$ acts transitively on the edge set of $\Gamma$, and furthermore, the cyclic subgroup $\langle a \rangle$ of $\text{Aut}(\Pi)$ is regular on each of the biparts $\mathcal{P}$ and $\mathcal{L}$. Thus by [10] Lemma 3.2, there exists an element $z \in \text{Aut}(\Gamma)$ interchanging points and lines of $\Pi$. It follows that $\text{Aut}(\Gamma) = (\text{Aut}(\Pi)).\mathbb{Z}_2 = \langle a, \sigma, z \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, and the subgroup $G := \langle a, z \rangle$ is regular on the vertex set $\mathcal{P} \cap \mathcal{L}$. Thus $\Gamma$ may be viewed as a Cayley graph of $G$, say $\Gamma = \text{Cay}(G, S)$ for some subset $S \subset G$.

Since $\Gamma$ is the incidence graph of the projective plane $\Pi$, $\Gamma$ has girth 6 and diameter 3. It follows that $G$ is not abelian and that $\text{Aut}(\Gamma)$ is a Frobenius group. In particular, $G$ is a dihedral group of order $2p$.

Now $G$ is normal in $\text{Aut}(\Gamma)$, and hence $\Gamma$ is a normal Cayley graph of $G$. By Lemma 2.1 we have that $\text{Aut}(\Gamma) = G \rtimes \text{Aut}(G, S)$. Hence $\langle \text{Aut}(\Gamma) \rangle = \text{Aut}(G, S) \cong \mathbb{Z}_q$. Since all subgroups of $\text{Aut}(\Gamma)$ of order $q$ are conjugate, we may assume that $\text{Aut}(G, S) = \langle \sigma \rangle$. Since $\Gamma$ is bipartite, we conclude that $S \cap \langle a \rangle = \emptyset$, and so all elements of $S$ are involutions. Thus $a^iz \in S$ for some $i$ with $1 \leq i \leq p - 1$. Let $a^\sigma = a^r$ for some integer $r$. Then $S = (a^iz)^{\langle \sigma \rangle} = \{a^iz, a^{ri}z, \ldots, a^{ri}z^i \}z\}, and so $\Gamma \cong \Gamma(q)$, as defined before.

Conversely, assume that $\Gamma(q)$ has girth 6 for some $q$. By Lemma 5.1 the diameter of $\Gamma(q)$ equals 3. Let $\mathcal{P}$ and $\mathcal{L}$ be the biparts of $\Gamma(q)$, and call the vertices in $\mathcal{P}$ points and the vertices in $\mathcal{L}$ lines. Then it is easily shown that the incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ is a projective plane of order $q - 1$. By Lemma 5.1 we have $\text{Aut}(\Gamma(q)) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$. It is known that a Desarguesian projective plane of order $q - 1$ has automorphism group $\text{PGL}(3, q - 1)$; see, for example, [6]. So $\Pi$ is not Desarguesian. \hfill $\Box$
A remark on Question 1.2(b). A 5-arc transitive Cayley graph is constructed by Xu, Fang, Wang and Xu in On cubic s-arc transitive Cayley graphs of finite simple groups, to appear in European Journal of Combinatorics. Also M. Conder announced that he has constructed some infinite families of s-arc transitive Cayley graphs for s = 5 or 7.

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