FINITE $s$-ARC TRANSITIVE CAYLEY GRAPHS
AND FLAG-TRANSITIVE PROJECTIVE PLANES

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Communicated by John R. Stembridge)

Abstract. In this paper, a characterisation is given of finite $s$-arc transitive
cayley graphs with $s \geq 2$. In particular, it is shown that, for any given integer
$k$ with $k \geq 3$ and $k \neq 7$, there exists a finite set (maybe empty) of $s$-transitive
Cayley graphs with $s \in \{3, 4, 5, 7\}$ such that all $s$-transitive Cayley graphs of
valency $k$ are their normal covers. This indicates that $s$-arc transitive Cayley
graphs with $s \geq 3$ are very rare. However, it is proved that there exist $4$-
arctransitive Cayley graphs for each admissible valency (a prime power plus
one). It is then shown that the existence of a flag-transitive non-Desarguesian
projective plane is equivalent to the existence of a very special arc transitive
normal Cayley graph of a dihedral group.

1. Introduction

A graph $\Gamma$ is a Cayley graph if there exist a group $G$ and a subset $S \subseteq G$ with
$S = S^{-1} := \{s^{-1} | s \in S\}$ such that the vertices of $\Gamma$ may be identified with the
elements of $G$ in such a way that $x$ is connected to $y$ if and only if $yx^{-1} \in S$.
The Cayley graph $\Gamma$ is denoted by $\text{Cay}(G, S)$. Cayley graphs stem from a type of
diagram now called a Cayley color diagram, introduced by Cayley in 1878. In this
paper, we investigate the symmetric Cayley graphs and a relation between them
and finite flag-transitive projective planes.

A graph $\Gamma$ is said to be $(X, s)$-arc transitive if $X \leq \text{Aut}\Gamma$ is transitive on vertices
and $s$-arcs of $\Gamma$ where $s$ is a positive integer. (A sequence $v_0, v_1, \ldots, v_s$ of vertices
of $\Gamma$ is called an $s$-arc if $v_i$ is adjacent to $v_{i+1}$ for $0 \leq i \leq s - 1$ and $v_i \neq v_{i+1}$
for $1 \leq i \leq s - 1$. An $(X, s)$-arc transitive graph is called $(X, s)$-transitive if it
is not $(X, s + 1)$-arc transitive. In particular, if $X = \text{Aut}\Gamma$, then an $(X, s)$-arc
transitive graph and an $(X, s)$-transitive graph are simply called $s$-arc transitive
and $s$-transitive, respectively.

Interest in $s$-arc transitive graphs stems from a seminal result of Tutte in 1947,
who proved that there exist no finite $s$-transitive cubic graphs for $s \geq 6$. Tutte's
Theorem was generalized by Weiss [23] who proved that there exist no finite $s$-
transitive graphs of valency at least 3 for $s = 6$ and $s \geq 8$. Since then, characterizing

Received by the editors August 27, 2003 and, in revised form, September 11, 2003 and Sep-
ember 24, 2003.

2000 Mathematics Subject Classification. Primary 20B15, 20B30, 05C25.

This work was supported by an Australian Research Council Discovery Grant, and a QEII
Fellowship. The author is grateful to the referee for his constructive comments.

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s-arc transitive graphs has received considerable attention in the literature (see, for example, [11, 14, 19, 24]).

Let \( G \) be a group, and let \( \Gamma = \text{Cay}(G, S) \). Then \( \Gamma \) has an automorphism group:

\[
\hat{G} = \{ \hat{g} : x \rightarrow xg \text{ for all } x \in G \mid g \in G \},
\]

consisting of right multiplications of elements \( g \in G \). The subgroup \( \hat{G} \) acts regularly on the vertex set of \( \Gamma \). If \( \hat{G} \) is normal in \( \text{Aut} \Gamma \), then \( \Gamma \) is called a normal Cayley graph; if the core of \( \hat{G} \) in \( \text{Aut} \Gamma \) has index 2 in \( \hat{G} \), then \( \Gamma \) is called a bi-normal Cayley graph. (Recall that the core of a subgroup \( H \) of a group \( X \) is the largest normal subgroup of \( X \) contained in \( H \).) These three classes of Cayley graphs are important for studying s-arc transitive Cayley graphs. Normal Cayley graphs have some very nice properties; refer to [21, 25], and also to Lemma 2.2 and Proposition 2.3. Several nice properties of bi-normal Cayley graphs are given in Lemma 2.4 and Corollary 2.7. Some properties for core-free Cayley graphs are given in Section 3. An interesting result of Section 3 is Proposition 3.2, which says that

“almost all” vertex primitive Cayley graphs are normal Cayley graphs.

Let \( N \triangleleft X \), and let \( \mathcal{B} \) be the set of \( N \)-orbits in \( V \). Then the normal quotient graph \( \hat{G} \) of \( \Gamma \) induced by \( N \) is the graph with vertex set \( \mathcal{B} \) such that \( B;B' \in \mathcal{B} \) are adjacent if and only if some vertex \( u \in B \) is adjacent in \( \Gamma \) to some vertex \( v \in B' \). If \( \Gamma \) and \( \Gamma_N \) have the same valency, then \( \Gamma \) is called a normal cover of \( \Gamma_N \).

Some special classes of 2-arc transitive Cayley graphs have been studied; see [1, 2, 11, 15, 18]. One of the main results of this paper is stated in Theorem 1.1, which tells us that s-arc transitive Cayley graphs with \( s \geq 3 \) are rare. Throughout this paper, denote by \( \mathcal{G}(s,k) \) the set of core-free s-transitive Cayley graphs of valency \( k \).

**Theorem 1.1.** For any integers \( s \in \{2,3,4,5,7\} \) and \( k \geq 3 \), the set \( \mathcal{G}(s,k) \) is finite, and for each s-transitive Cayley graph \( \Gamma \), one of the following statements holds:

(i) \( \Gamma \) is a normal or a bi-normal Cayley graph, and either \( s = 2 \), or \( (s,k) = (3,7) \) and \( \Gamma \) is bi-normal (so bipartite);

(ii) \( \Gamma \) is a normal cover of a member of \( \mathcal{G}(s,k) \).

Moreover, \( \mathcal{G}(2,k) \) and \( \mathcal{G}(3,k) \) are not empty for all \( k \geq 3 \), and \( \mathcal{G}(4,q+1) \) is not empty for all prime powers \( q \); in particular, there exist 4-transitive Cayley graphs of all admissible valencies.

**Remarks.** (a) It is easily shown that complete graphs are members of \( \mathcal{G}(2,k) \), and complete bipartite graphs are members of \( \mathcal{G}(3,k) \). Some other examples of members in \( \mathcal{G}(2,k) \) and \( \mathcal{G}(3,k) \) will be constructed in Section 4.

(b) The examples in \( \mathcal{G}(4,q+1) \) with \( q \) being a power of a prime are the incidence graphs of Desarguesian projective planes. It is known that the valency of a 4-transitive graph is \( r+1 \) such that \( r \) is a power of a prime; see [24]. Thus, Theorem 1.1 tells us that for all admissible valencies, there exist 4-transitive Cayley graphs. However, we do not know any other examples of 4-transitive Cayley graphs, and we do not have any examples of 5- or 7-transitive Cayley graphs at all.

(c) There have been some results regarding certain special classes of 2-arc transitive normal and bi-normal Cayley graphs; see [2, 11, 15, 20]. However, the general...
case is still not quite well-understood. In particular, we do not know any examples of 3-transitive bi-normal Cayley graphs.

We would like to propose the following question and problem.

**Question 1.2.** (a) Do there exist 3-transitive bi-normal Cayley graphs? 
(b) Do there exist s-transitive Cayley graphs for $s = 5$ and $s = 7$?

**Problem 1.3.** (a) Determine members of $G(s, k)$ for $k \geq 3$ and $s \geq 2$. 
(b) Give a satisfactory description of bi-normal 2-transitive Cayley graphs.

Let $\mathcal{P}$ be a projective plane of order $n$ with point set $\mathcal{P}$ and line set $\mathcal{L}$. Then $\Pi$ has $n^2 + n + 1$ points and $n^2 + n + 1$ lines such that (a) any two points lie on exactly one line and any two lines intersect at exactly one point, (b) each line contains exactly $n + 1$ points and each point lies on exactly $n + 1$ lines. A flag of $\Pi$ is a pair of point $p$ and line $l$ such that $p$ lies on $l$. A permutation of flags preserving the incidence of $\Pi$ is an automorphism of $\Pi$, and all automorphisms of $\Pi$ form a group $\text{Aut}(\Pi)$; that is, the automorphism group of $\Pi$. A plane $\Pi$ is called flag-transitive if $\text{Aut}(\Pi)$ is transitive on its flags. Let $\Gamma$ be the incidence graph of $\Pi$; that is, the vertex set of $\Gamma$ is $\mathcal{P} \cup \mathcal{L}$, and the edge set of $\Gamma$ is the set of flags of $\Pi$. Then $\text{Aut}(\Gamma) = \text{Aut}(\Pi)$ or $\text{Aut}(\Pi).\mathbb{Z}_2$; so if $\Pi$ is flag-transitive, then $\Gamma$ is edge-transitive.

The classical (Desarguesian) projective planes have been well studied, and their incidence graphs give the first family of 4-arc transitive Cayley graphs, presented in Example 4.5. The existence problem of non-Desarguesian flag-transitive projective planes is a long-standing open problem in finite geometry; see, for example, books [6, 17], and articles [7, 12, 22]. Here we prove that the existence of a non-Desarguesian projective plane is equivalent to the existence of a type of Cayley graph of a dihedral group satisfying very restricted properties.

Let $n$ be a 2-power such that $p := n^2 + n + 1$ and $q := n + 1$ are primes. Let $G = \langle a \rangle \times \langle z \rangle \cong D_{2p}$, where $a^2 = a^{-1}$. Let $\sigma \in \text{Aut}(\langle a \rangle)$ be such that $\sigma z = z \sigma$ and $o(\sigma) = 2q$. Let

$$S = \langle az \rangle^{\langle \sigma \rangle} = \{az, a^2z, a^3z, \ldots, a^{q-1}z\},$$

and set $\Gamma(q) = \text{Cay}(G, S)$. It will be shown in Lemma 5.1 that $\Gamma(q)$ has full automorphism group $\text{Aut}(\Gamma(q)) = G \rtimes \langle \sigma z \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2q$, a Frobenius group of order $2pq$, and thus $\Gamma(q)$ is a normal Cayley graph of the dihedral group $G \cong D_{2p}$. The following theorem shows that the girth of $\Gamma(q)$ determines the existence of flag-transitive non-Desarguesian projective planes of order $q - 1$.

**Theorem 1.4.** Using the notation defined above, there exists a flag-transitive non-Desarguesian projective plane of order $n$ if and only if the girth of the Cayley graph $\Gamma(q)$ is equal to 6.

As widely believed, all flag-transitive projective planes are Desarguesian, and hence we make the following conjecture; see Lemma 5.1.

**Conjecture 1.5.** For each prime power $q$, the graph $\Gamma(q)$ defined above has diameter 4 and girth 4.

Theorem 1.4 tells us that if this conjecture is true, then there exist no finite non-Desarguesian flag-transitive projective planes.
2. AUTOMORPHISM GROUPS OF CAYLEY GRAPHS: NORMALITY

It is known that a graph $\Gamma$ is a Cayley graph of a group $G$ if and only if its automorphism group contains a subgroup that is isomorphic to $G$ and acts regularly on vertices; see, for example, [3, Proposition 16.3]. It is hence natural to use the regular subgroup to describe properties of $\Gamma$ and $\text{Aut}\Gamma$.

2.1. Normal Cayley graphs. Let $G$ be a finite group, and let $\Gamma = \text{Cay}(G, S)$. Let

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$  

Let $1$ be the vertex of $\Gamma$ corresponding to the identity of $G$. It follows from the definition that each element of $\text{Aut}(G, S)$ induces an automorphism of $\Gamma$ fixing $1$, and so $\text{Aut}(G, S) \leq (\text{Aut}\Gamma)_1$. Moreover, we have the following statement.

**Lemma 2.1** ([8]). For a Cayley graph $\Gamma = \text{Cay}(G, S)$, the following holds:

$$\text{N}_{\text{Aut}}(\hat{G}) = \hat{G} \rtimes \text{Aut}(G, S).$$

The subgroup $\text{Aut}(G, S)$ plays an important role in the study of Cayley graphs; refer to [8, 21, 25]. Some structural information may be read out from $\text{Aut}(G, S)$; for instance, if $\text{Aut}(G, S)$ is transitive on $S$, then the Cayley graph $\Gamma$ is $\text{N}_{\text{Aut}}(\hat{G})$-arc transitive; if $\text{Aut}(G, S)$ is 2-transitive on $S$, then $\Gamma$ is $(\text{N}_{\text{Aut}}(\hat{G}), 2)$-arc transitive.

For a group $X$ with $\hat{G} \leq X \leq \text{Aut}\Gamma$, if $\hat{G}$ is normal in $X$, then $X_1 \leq \text{Aut}(G, S)$, and $\Gamma$ is called an $X$-normal Cayley graph of $G$. In particular, if $X = \text{Aut}\Gamma$, then $\Gamma$ is a normal Cayley graph, and $\text{Aut}\Gamma = \hat{G} \rtimes \text{Aut}(G, S)$. The action of $X$ on the vertices of $\Gamma$ behaves in a very nice way:

**Lemma 2.2.** Let $\Gamma = \text{Cay}(G, S)$, and let $X \leq \text{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of $X$. Then $X = \hat{G} \rtimes X_1$, and for an element $x = \hat{g}y \in X$, where $\hat{g} \in \hat{G}$ and $y \in X_1$, and for any vertex $h \in G$, we have

$$h^x = h^{\hat{g}y} = (h\hat{g})^y = y^{-1}h\hat{g}y.$$  

This property implies the following result about $s$-arc transitive Cayley graphs, which was obtained in [13].

**Proposition 2.3.** Let $\Gamma = \text{Cay}(G, S)$, and let $X \leq \text{Aut}\Gamma$ be such that $\hat{G}$ is a normal subgroup of $X$. Then $\Gamma$ is not $(X, 3)$-arc transitive.

2.2. Bi-normal Cayley graphs. Let $\Gamma = \text{Cay}(G, S)$, and let $\hat{G} \leq X \leq \text{Aut}\Gamma$. In this section we study the important case where $\hat{G}$ is not normal in $X$ but $\text{core}_X(\hat{G})$ is “big”. A Cayley graph $\text{Cay}(G, S)$ is called an $X$-bi-normal Cayley graph of $G$ if the core $\text{core}_X(\hat{G})$ has index 2 in $\hat{G}$. Hence if $X = \text{Aut}\Gamma$, then an $X$-bi-normal Cayley graph is a bi-normal Cayley graph.

For an $X$-normal Cayley graph $\Gamma = \text{Cay}(G, S)$, $X_1$ acts on the vertex set $G$ by conjugation, and so if $\Gamma$ is connected, then $X_1$ is faithful on $S$. For an $X$-bi-normal Cayley graph $\text{Cay}(G, S)$, the $X_1$-action only on half of its vertices is by conjugation; however, $X_1$ is still faithful on $S$ under certain conditions, as shown below.

**Lemma 2.4.** Let $\Gamma$ be a connected bipartite graph with biparts $U$ and $U'$. Assume that $X \leq \text{Aut}\Gamma$ is transitive on the vertex set $V\Gamma$, and assume furthermore that $X$ has a normal subgroup $N \leq X^+$ such that $N$ is intransitive on $V\Gamma$, $N_v$ is transitive on $\Gamma(v)$ for some vertex $v \in U$, and $N$ has a normal subgroup that is regular on both $U$ and $U'$. Then $N_v$ acts faithfully on $\Gamma(v)$.  

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Proof. Choose \( v \in U \), and let \( R = MN_v^{[1]} \). Since \( U \) is an orbit of \( N \) and \( M \) acts transitively on \( U \), we have \( N = MN_v \). Since \( M \) is normal in \( N \) and \( N_v^{[1]} \) is normal in \( N_v \), it follows that \( R \) is normal in \( N \) and that \( R = MN_w^{[1]} \) (and hence \( R_w \leq N_w^{[1]} \)) for all \( w \in U \). Now let \( \{u, v\} \) be an arbitrary edge. We may assume that \( u \in U \) and \( v \in \overline{U} \). Let \( x \in R_u \). Since \( R = MN_v^{[1]} \), we have \( x = yz \) where \( y \in M \) and \( z \in N_v^{[1]} \). Since \( N_v^{[1]} \leq R_u \), we have \( z \in R_u \). It then follows that \( y \in M \cap R_u \). By the assumption that \( M \) is regular on \( U \), we have \( M \cap R_u = 1 \). Thus \( y = 1 \). We conclude that \( x = z \in N_v^{[1]} \), and so \( R_u \leq N_v^{[1]} \leq R_v \). Since \( \{u, v\} \) is arbitrary, it follows that \( R_u \leq N_v^{[1]} \) for every \( u \in U' \). Thus \( R_u \leq N_v^{[1]} \) for every \( u \in VT \). Since \( \Gamma \) is connected, it follows that \( R_{uv} = 1 \) for every edge \( \{u, v\} \). Therefore, \( N_v^{[1]} = 1 \) for every vertex \( u \in VT \). 

By inspecting the classification of finite 2-transitive permutation groups (see [4]), the statement of the next lemma is easily obtained.

**Lemma 2.5.** Let \( T \) be a 2-transitive permutation group on a set \( \Omega \) of degree \( n \). Then for two points \( \omega, \omega' \in \Omega \), the point stabilizer \( T_{\omega,\omega'} \) has a transitive permutation representation of degree \( n - 1 \) if and only if \( \soc(T) = A_7 \) or \( S_7 \) and \( n = 7 \).

The next result tells us that for a \((G, s)\)-arc transitive graph, if the vertex stabilizer is faithful on the neighborhood, then \( s \) is small.

**Proposition 2.6.** Let \( \Gamma \) be an \((X, s)\)-transitive graph of valency \( k \), where \( G \leq \Aut\Gamma \) and \( s \geq 1 \). Assume that \( X_v \) is faithful on \( \Gamma(v) \). Then either \( s \leq 2 \), or \((k, s) = (7, 3) \) and \( X_v \cong A_7 \) or \( S_7 \).

Proof. By Lemma 2.4, the vertex stabilizer \( X_v \) acts faithfully on \( \Gamma(v) \); that is, \( X_v \cong X_v^{(\Gamma(v))} \), and thus \( X_v \) is a 2-transitive permutation group on \( \Gamma(v) \). Suppose furthermore that \( \Gamma \) is \((X, 3)\)-arc transitive. Then for distinct vertices \( u, w \in \Gamma(v) \), the stabilizer \( X_{uvw} \) of the 2-arc \( (u, v, w) \) is transitive on \( \Gamma(w) \setminus \{v\} \). In particular, \( X_{uvw} \) has a transitive permutation representation of degree \( k - 1 \). Note that \( X_{uvw} \) is the stabilizer of \( u, w \) of the 2-transitive permutation group \( X_v \cong X_v^{(\Gamma(v))} \). Thus by Lemma 2.3, we conclude that \( X_v \cong A_7 \) or \( S_7 \), \( X_{uw} \cong A_6 \) of \( S_6 \), and \( X_{uwv} \cong A_5 \) or \( S_5 \), respectively. In particular, \( k = 7 \), completing the proof of the proposition. 

Combining Lemma 2.4 and Proposition 2.6, we have the following statement for bi-normal Cayley graphs.

**Corollary 2.7.** Let \( \Gamma = \text{Cay}(G, S) \) be a connected graph of valency \( k \). Assume that \( \hat{G} < X \leq \Aut\Gamma \) is such that \( \Gamma \) is \( X \)-bi-normal and \((X, 2)\)-arc transitive. Then \( X_1 \) is faithful on \( \Gamma(1) \), and furthermore, either

(i) \( \Gamma \) is not \((X, 3)\)-arc transitive, or

(ii) \( k = 7 \), \( X_1 \cong A_7 \) or \( S_7 \), and \( \Gamma \) is \((X, 3)\)-transitive.

3. **Core-free Cayley graphs and their group duals**

Let \( \Gamma = \text{Cay}(G, S) \), and let \( X \leq \Aut\Gamma \) contain \( \hat{G} \). Assume furthermore that \( \hat{G} \) is not normal in \( X \). If the core of \( \hat{G} \) in \( X \) is trivial, then \( \Gamma \) is a core-free Cayley graph for if \( \hat{G} \) is core-free in \( X \), then \( \hat{G} \) is core-free in \( \Aut\Gamma \).
Suppose now that \( \hat{G} \) is core-free in \( X \). Then \( X \) has an exact factorization with core-free factors:

\[
X = \hat{G}H, \text{ with } H, \hat{G} \text{ core-free, and } \hat{G} \cap H = 1,
\]

where \( H = X_1 \). The action of \( X \) on \([X : \hat{G}]\) gives rise to Cayley graphs of the group \( H \), called dual Cayley graphs of \( \Gamma \). In this case, we observe that the order of \( X \) and hence the order of \( G \) are up-bounded in terms of the order of \( X_1 \); that is,

\[
\hat{G} < X \leq \operatorname{Sym}(|X_1|).
\]

It is known that for many important classes of graphs, the order of \( X_1 \) is further up-bounded in terms of the valency: for example, cubic edge-transitive graphs, 2-arc transitive graphs, and vertex-primitive graphs. This therefore gives an up-bound of \(|G|\) and \(|X|\) in terms of the valency. This implies that \( X \) has two faithful transitive permutation representations, that is, on \([X : \hat{G}]\) and on \([X : H]\), and the statements in the following lemma are obviously true.

**Lemma 3.1.** Let \( \Gamma \) be a Cayley graph of \( G \), and let \( X \leq \operatorname{Aut} \Gamma \) be such that \( \hat{G} < X \) and \( \operatorname{core}_X(\hat{G}) = 1 \). Then \( X \) has a faithful transitive permutation representation on the set \([X : \hat{G}]\), and \( X_1 \) is a regular subgroup of \( X \) acting on \([X : \hat{G}]\). In particular, the order \(|X|\) is up-bounded in terms of \(|X_1|\).

Each generalized orbital graph of \( X \) on the set \([X : \hat{G}]\) is therefore a Cayley graph of the group \( H = X_1 \), which is a dual Cayley graph of \( \text{Cay}(G, S) \). Considering the dual permutation representation of a vertex-primitive automorphism group of a Cayley graph, we have the following result.

**Proposition 3.2.** For every given integer \( k \), there are only finitely many vertex-primitive Cayley graphs of valency \( k \) that are not normal Cayley graphs.

**Proof.** Let \( \Gamma = \text{Cay}(G, S) \) be such that \( X = \operatorname{Aut} \Gamma \) is primitive on \( V \Gamma \). It follows that \( \hat{G} \) in \( X \) is either normal or core-free. By \([3]\), the order of \( X_1 \) is up-bounded in terms of the valency of an orbital graph of \( X \) on \( V \Gamma \), where \( 1 \) is the identity of \( G \). Thus \(|X_1|\) is up-bounded in terms of the valency of \( \Gamma \). If \( \Gamma \) is core-free, then \( X \) has a faithful representation on \([X : \hat{G}]\), which is of degree \(|X_1|\). Hence \(|X| \leq |X_1|!\), and so the order of \( X \) is up-bounded in terms of the valency of \( \Gamma \), and so the number of vertex-primitive Cayley graphs of valency \( k \) that are not normal Cayley graphs is up-bounded in terms of \( k \).

This shows that, although they are very special, normal Cayley graphs are not very rare. Similarly, we have the next proposition regarding 2-arc transitive Cayley graphs.

**Proposition 3.3.** For any given positive integer \( k \), there are at most finitely many core-free 2-arc transitive Cayley graphs of valency \( k \).

**Proof.** Let \( \Gamma = \text{Cay}(G, S) \) be an \((X, 2)\)-arc transitive graph of valency \( k = |S| \) such that \( \hat{G} < X \leq \operatorname{Aut} \Gamma \) and \( \hat{G} \) is core-free in \( X \). By \([23]\), the order of \( X_1 \) is up-bounded in terms of the valency \( k \), where \( 1 \) is the identity of \( G \). Since \( \hat{G} \) is core-free in \( X \), the group \( X \) has a faithful transitive representation on \([X : \hat{G}]\), which is of degree \(|X_1| = |X : \hat{G}|\). Hence \(|X| \leq |X_1|!\), and so the order of \( X \) is up-bounded in terms of \( k \). In particular, the number of core-free 2-arc transitive Cayley graphs of valency \( k \) is up-bounded in terms of \( k \).
4. Examples

In this section, we construct examples of $s$-arc transitive core-free Cayley graphs, where $s \geq 2$.

It is obvious that a complete graph $K_n$ of $n$ vertices with $n \geq 4$ has automorphism group isomorphic to $S_n$ and is a 2-transitive Cayley graph of an arbitrary group of order $n$. Also, it is easy to see that a complete bipartite graph $K_{n,n}$ with $n \geq 3$ has automorphism group $S_n \wr S_2$, and is a 3-transitive Cayley graph of valency $n$. This shows that $G(2,k)$ and $G(3,k)$ are nonempty for all $k \geq 3$. Except for these “trivial” examples, various other $s$-arc transitive core-free Cayley graphs will be constructed in this section.

Let $X$ be a group, and let $H$ be a core-free subgroup of $X$. Let $[X : H] = \{Hx \mid x \in X\}$. For an element $g \in X$ such that $g^2 \in H$, a coset graph

$$\Gamma := \text{Cos}(X, H, HgH)$$

is defined as the graph with vertex set $V = [X : H]$ such that $Hx$ is adjacent to $Hy$ if and only if $yx^{-1} \in HgH$. We have the following well-known properties.

Lemma 4.1. Let $\Gamma = \text{Cos}(X, H, HgH)$. Then

(i) $\Gamma$ is connected if and only if $(H, g) = X$;

(ii) $\Gamma$ is $(X, 2)$-arc transitive if and only if $H$ is 2-transitive on $[H : H \cap H^g]$.

The first example gives a family of 2-arc transitive core-free Cayley graphs of prime-power valency.

Example 4.2. Let $X = S_{p^r}$ where $p$ is a prime, acting on $\Omega = \{1, 2, \ldots, p^r\}$. Then $X$ has two subgroups $G, H$ such that $G = X_{1,2} \cong S_{p^r-2}$ and $H \cong AGL_1(p^r) \cong \mathbb{Z}_p^r \rtimes \mathbb{Z}_{p^r-1}$ is 2-transitive on $\Omega$. It follows that $G \cap H = 1$ and $X = GH$. Write $H = N \rtimes \langle z \rangle$, where $N \cong \mathbb{Z}_p^r$ is regular on $\Omega$, and $z = (1, 2, 3, \ldots, p^r-1)$. It is easily shown that there exists an involution $g \in X$ such that $z^g = z^{-1}$ and $(H, g) = X$. Let $\Gamma = \text{Cos}(X, H, HgH)$. Then $\Gamma$ is a connected 2-arc transitive graph of valency $p^r$ and is a Cayley graph of $G$.

The next two examples were constructed in [16] as examples of quasiprimitive 2-arc transitive graphs of a certain type. Here we prove they are Cayley graphs.

Example 4.3. Let $\Gamma$ be a graph constructed in Example 4.2 with $r = 1$ and $p \geq 5$. Then $\Gamma$ is a 2-arc transitive Cayley graph of $S_{p-2}$. Let $l$ be a divisor of $(p-1)/2$, and let

$$Y = (T_1 \times T_2 \times \cdots \times T_1) \rtimes (O \times L),$$

where $T_i \cong A_p$, $O \cong \mathbb{Z}_2$ normalizes but does not centralize $T_i$ for each $i$, and $L = ((1, 2, 3, \ldots, l)) \cong \mathbb{Z}_l$ permutes cyclically the direct factors $T_1, T_2, \ldots, T_l$.

It is shown in [16] that $Y$ has a subgroup $K \cong (\mathbb{Z}_p \times \mathbb{Z}_{p-1}) \rtimes \mathbb{Z}_l$ and an element $g \in Y$ such that the coset graph $\Gamma := \text{Cos}(Y, K, KgK)$ is connected and $(Y, 2)$-arc transitive of valency $p$. It is now easily shown that the subgroup $R \times T_2 \times \cdots \times T_1$, where $S_{p-2} \cong R < T_1$, acts regularly on $VT$. Hence $\Gamma$ is a Cayley graph.

The following example presents a family of 3-arc transitive Cayley graphs.

Example 4.4. Let $T = \text{PSL}(2,q)$, where $q = 2^e$ with $e \geq 2$, and let $H = \mathbb{Z}_{2^e}^2 \rtimes D$, where $D \cong \mathbb{Z}_{2^e-1}$. Let $f \in T$ be such that $(D, f) \cong D_{2(q^{e-1})}$. Then $\Sigma = \text{Cos}(T, H, HfH) \cong K_{q+1}$ is $(T, 2)$-arc transitive. Let

$$Y = (T_1 \times T_2 \times \cdots \times T_{q-1}) \rtimes L,$$
where \( T_i \cong A_p \), and \( L = \langle (1, 2, 3, \ldots, q - 1) \rangle \cong \mathbb{Z}_{q - 1} \) permutes cyclically the direct factors \( T_1, T_2, \ldots, T_{q - 1} \).

It is shown in [10] that \( Y \) has a subgroup \( K \cong (\mathbb{Z}_q \times \mathbb{Z}_{q - 1}) \times \mathbb{Z}_{q - 1} \) and an element \( g \in Y \) such that the coset graph \( \Gamma := \text{Cos}(Y, K, KgK) \) is connected and \( (Y, 3) \)-arc transitive of valency \( p \). It is now easily shown that the subgroup \( R \times T_2 \times \cdots \times T_l \), where \( \mathbb{Z}_{q + 1} \cong R < T_1 \), acts regularly on \( V_T \). Hence \( \Gamma \) is a Cayley graph.

Finally, we present a family of 4-arc transitive Cayley graphs.

**Example 4.5.** Let \( \Gamma \) be the incidence graph of a Desarguesian projective plane \( II \) of order \( n \). Then \( n \) is a prime power, and \( \text{Aut} II = \text{PGL}(3, n) \); see [6]. It is known that the incidence graph \( \Gamma \) is 4-arc transitive of valency \( n + 1 \), and \( \text{Aut} \Gamma = \text{Aut} II \). Now the Singer subgroup \( C \cong \mathbb{Z}_{n^2 + n + 1} \) is regular on both the point set \( P \) and the line set \( L \). There exists an involution \( \tau \in \text{Aut} \Gamma \) such that \( \tau \) normalizes \( C \) (refer to [10], Lemma 3.2). Hence \( \text{Aut} \Gamma \) has a subgroup \( G := \langle C, \tau \rangle \cong \mathbb{Z}_{n^2 + n + 1} \times \mathbb{Z}_2 \) that is regular on the vertices of the graph \( \Gamma \). So \( \Gamma \) is a Cayley graph of a group isomorphic to \( G \).

5. Proofs of Theorems 1.1 and 1.4

In this section, we prove Theorems 1.1 and 1.4.

**Proof of Theorem 1.1**. Let \( \Gamma = \text{Cay}(G, S) \) be an \( (X, s) \)-arc transitive graph of valency \( k \) with vertex set \( V \), where \( s \geq 2 \). Assume furthermore that \( \hat{G} < X \). Let \( C = \text{core}_X(\hat{G}) \).

Assume first that \( C = \hat{G} \). Then \( \hat{G} \) is normal in \( X \), and hence by Proposition 2.3, we have that \( s = 2 \), as in part (i) of Theorem 1.1.

Assume that \( C \) has exactly two orbits in \( V \). Then \( \hat{G} \) is bi-normal in \( X \), and \( \Gamma \) is bipartite. Hence by Corollary 2.7, we have that either \( s = 2 \), or \( (s, k) = (3, 7) \), as in part (i) of Theorem 1.1.

Finally, assume that \( C \) has at least three orbits in \( V \). Then \( \hat{G}/C \) is core-free in \( X/C \) and regular on the vertex set of \( \Gamma \). \( X/C \leq \text{Aut} \Gamma \), and then \( \Gamma \) is an \( (X/C, s) \)-arc transitive graph of valency \( k \). Thus \( \Gamma \) is a member of \( \mathcal{G}(s, k) \), and \( \Gamma \) is a normal cover of \( C \), as in part (ii) of Theorem 1.1.

The existence of graphs in \( \mathcal{G}(s, k) \) with \( s \leq 4 \) is justified by the examples given in Section 4. This completes the proof of Theorem 1.1.

To prove Theorem 1.4, we first investigate the graphs \( \Gamma(q) \). Let \( n \) be a prime power such that \( p := n^2 + n + 1 \) and \( q := n + 1 \) are primes. Let

\[
G = D_{2p} = \langle a \rangle \rtimes \langle z \rangle,
\]

where \( a^2 = a^{-1} \). Let \( \sigma \in \text{Aut}(\langle a \rangle) \) be such that \( \sigma z = z \sigma \) and \( o(\sigma z) = 2q \). Let

\[
S = (az)^{\langle \sigma \rangle} = \{az, a^2z, a^3z, \ldots, a^{q-1} z\},
\]

and set \( \Gamma(q) = \text{Cay}(G, S) \).

**Lemma 5.1.** Let \( \Gamma = \Gamma(q) \), and let \( d(\Gamma) \) and \( g(\Gamma) \) be the diameter and the girth of \( \Gamma \), respectively. Then

(i) \( \Gamma \) is of valency \( q \), and \( \text{Aut} \Gamma = \langle a \rangle \rtimes \langle z \sigma \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{2q} ; \)

(ii) either \( (d(\Gamma), g(\Gamma)) = (3, 6) \), or \( g(\Gamma) = 4 \).
Lemma 2.1, we have that $\text{Aut} = G$. In particular, $A^+$ is a primitive permutation group of degree $p$. Thus by Burnside’s theorem, either $A^+$ is affine or an almost simple group. It then follows from the result of [9] that either $\langle a \rangle \triangleleft A^+$, or $\text{soc}(A^+) = \text{PSL}(d, r)$ with $p = \frac{r^2 - 1}{r - 1}$, or $A_p$. Suppose that $\text{soc}(A^+) = \text{PSL}(d, r)$. Then the cyclic group $\langle a \rangle$ is a Singer cycle of $\text{PSL}(d, r)$, and so $\text{N}_{\text{PSL}(d, r)}(\langle a \rangle) \leq \langle a \rangle \rtimes \mathbb{Z}_d$. Thus the order $o(\sigma)$ divides $d$, which is not possible. Suppose that $\text{soc}(A^+) = A_p$. Then $A_1 = A_{p-1}$. Since the smallest transitive permutation representation of $A_{p-1}$ is $p - 1$, it follows that $G$ has valence at least $p - 1$, which is a contradiction. Thus $\langle a \rangle$ is normal in $A^+$. It then follows that $\text{Aut} = G \rtimes \langle \sigma \rangle$, as in part (i).

Suppose that the girth $g(\Gamma)$ of $G$ is at least 6. Since $|\Gamma(1)| = n + 1$, we have that $|\Gamma(1)| = n^2 + n$. Furthermore, since $|\Gamma| = 2(n^2 + n + 1)$, we conclude that $\Gamma_3(1)| \leq |\Gamma| - (1 + (n + 1) + (n^2 + n)) = n^2$. It then follows that $g(\Gamma) = 6$, $\Gamma_3(1)| = n^2$, and the diameter $d(\Gamma) = 3$. Suppose finally that the girth $g(\Gamma)$ of $G$ is less than 6. Then since $\Gamma$ is bipartite, $g(\Gamma) = 4$. \hfill $\square$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\Pi$ be a projective plane of order $n$, where $n$ is a natural integer. Let $\Gamma$ be the incidence graph of $\Pi$; that is, the vertex set of $\Gamma$ is $\mathcal{P} \cup \mathcal{L}$, and the edges of $\Gamma$ is the set of flags of $\Pi$. Then $\Gamma$ is a bipartite graph with bipartite $\mathcal{P}$ and $\mathcal{L}$, and either $\text{Aut} = \text{Aut}(\Gamma)$, or $\text{Aut} = (\text{Aut}(\Pi)) \rtimes \mathbb{Z}_2$.

Assume that $\Pi$ is not Desarguesian, and assume that $\Pi$ is flag-transitive. By [12], $n$ is a 2-power, both $q := n + 1$ and $p := n^2 + n + 1$ are primes, and $\text{Aut}(\Pi) = \langle a \rangle \rtimes \langle \sigma \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ is a Frobenius group, where $o(a) = p$ and $o(\sigma) = q$.

The subgroup $\text{Aut}(\Pi)$ acts transitively on the edge set of $\Gamma$, and furthermore, the cyclic subgroup $\langle a \rangle$ of $\text{Aut}(\Pi)$ is regular on each of the bipartite $\mathcal{P}$ and $\mathcal{L}$. Thus by [10] Lemma 3.2, there exists an element $z \in \text{Aut}(\Pi)$ of order 2 and $\text{Aut}(\Pi) \rtimes \mathbb{Z}_2 = \langle a, \sigma, z \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, and the group $G := \langle a, z \rangle$ is regular on the vertex set $\mathcal{P} \cap \mathcal{L}$. Thus $G$ may be visualized as a Cayley graph of $G$, say $\Gamma = \text{Cay}(G, S)$ for some subset $S \subset G$. Since $\Gamma$ is the incidence graph of the projective plane $\Pi$, $\Gamma$ has girth 6 and diameter 3. It follows that $G$ is not abelian and that $\text{Aut}(\Gamma)$ is a Frobenius group. In particular, $G$ is a dihedral group of order $2p$.

Now $G$ is normal in $\text{Aut}(\Pi)$, and hence $G$ is a normal Cayley graph of $G$. By Lemma 2.1 we have that $\text{Aut}(\Gamma) = G \rtimes \text{Aut}(G, S)$. Hence $\text{Aut}(\Gamma)_1 = \text{Aut}(G, S) \cong \mathbb{Z}_q$. Since all subgroups of $\text{Aut}(\Gamma)$ of order $q$ are conjugate, we may assume that $\text{Aut}(G, S) = \langle \sigma \rangle$. Since $\Gamma$ is bipartite, we conclude that $S \cap \langle a \rangle = \emptyset$, and so all elements of $S$ are involutions. Thus $a^i z \in S$ for some $i$ with $1 \leq i \leq p - 1$. Let $a^r = a^s$ for some integer $r$. Then $\overline{S} = \langle a^i z \rangle = \{a^i z, a^{2i} z, \ldots, a^{(r-1)i} z\}$, and so $\Gamma \cong \Gamma(q)$, as defined before.

Conversely, assume that $\Gamma(q)$ has girth 6 for some $q$. By Lemma 5.1, the diameter of $\Gamma(q)$ equals 3. Let $\mathcal{P}$ and $\mathcal{L}$ be the bipartition of $\Gamma(q)$, and call the vertices in $\mathcal{P}$ points and the vertices in $\mathcal{L}$ lines. Then it is easily shown that the incidence structure $\Pi = (\mathcal{P}, \mathcal{L})$ is a projective plane of order $q - 1$. By Lemma 5.1 we have $\text{Aut}(\Pi(q)) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$. It is known that a Desarguesian projective plane of order $q - 1$ has automorphism group $\text{PGL}(3, q - 1)$; see, for example, [6]. So $\Pi$ is not Desarguesian. \hfill $\square$
A remark on Question 1.2(b). A 5-arc transitive Cayley graph is constructed by Xu, Fang, Wang and Xu in On cubic s-arc transitive Cayley graphs of finite simple groups, to appear in European Journal of Combinatorics. Also M. Conder announced that he has constructed some infinite families of s-arc transitive Cayley graphs for $s = 5$ or $7$.

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