PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 1, Pages 229–238
S 0002-9939(04)07555-0
Article electronically published on July 26, 2004

BOUNDED AND COMPLETELY BOUNDED LOCAL DERIVATIONS FROM CERTAIN COMMUTATIVE SEMISIMPLE BANACH ALGEBRAS

EBRAHIM SAMEI

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We show that for a locally compact group G, every completely bounded local derivation from the Fourier algebra $A(G)$ into a symmetric operator $A(G)$-module or the operator dual of an essential $A(G)$-bimodule is a derivation. Moreover, for amenable G we show that the result is true for all operator $A(G)$-bimodules. In particular, we derive a new proof to a result of N. Spronk that $A(G)$ is always operator weakly amenable.

The concept of local derivations was introduced by R. V. Kadison in [10]. An operator D from a Banach algebra A into a Banach A-bimodule X is a *local derivation* if for each $a \in A$ there is a derivation D_a from A into X such that $D(a) = D_a(a)$. Kadison showed that if A is a von Neumann algebra and X is a dual Banach bimodule, then all bounded local derivations from A into X are derivations. B. E. Johnson in [9] extended Kadison's result and showed that every local derivation from a C^*-algebra A into any Banach A-bimodule is a derivation. He showed that it is enough to establish the result for the commutative regular Banach algebra $C_0(\mathbb{R})$. For $C_0(\mathbb{R})$, he first studied "local operators" and "local multipliers" from this algebra and then deduced results about local derivations. However, $C_0(\mathbb{R})$ is very well-behaved; it is a commutative C^*-algebra so that it is amenable and all the derivations from it into any Banach $C_0(\mathbb{R})$-bimodule (including those D_a considered above) are automatically continuous. In this paper, we exploit Johnson's approach and investigate local derivations from another family of commutative regular Banach algebras which do not necessarily have the above properties. To compensate for this, we look more into the "local structure" of these algebras, and we show that most of the bounded local operators and all of the bounded local multipliers from these algebras are multipliers (sections 2 and 3). This provides us with the necessary tools to establish our main results on the bounded local derivations (section 4).

In section 5, we apply the results we have obtained in the previous sections to the Fourier algebra $A(G)$ of a locally compact group G. Since $A(G)$ is the predual of the von Neumann algebra $VN(G)$, it has a natural operator space structure [4]. We show that every completely bounded local derivation from $A(G)$ into the

Received by the editors June 13, 2003 and, in revised form, September 30, 2003.
2000 *Mathematics Subject Classification.* Primary 46L07, 47B47.
Key words and phrases. Local derivations, Fourier algebra, operator space, operator weak amenability.

operator dual of an essential $A(G)$-module is a derivation. Moreover, for amenable G we show that the above result is true for all operator $A(G)$-bimodules.

Recently, N. Spronk [16] has shown that for any locally compact group G, $A(G)$ is operator weakly amenable. We give an alternative proof to Spronk's result and then prove a stronger statement that every completely bounded local derivation from $A(G)$ into any symmetric operator $A(G)$-module is zero (Proposition 4.1 and Corollary 4.4; see also Theorem 5.2(i)).

For more results on local derivations on the other families of operator algebras see [2] and [11].

1. Preliminaries

We have followed [3] in using some of the basic terminology not defined here. Let A be a commutative regular semisimple Banach algebra with the carrier space Φ. Throughout this paper we always regard A as a function algebra on its carrier space. Let F be a closed subset of Φ. Then F is *a set of synthesis* for A if there is a unique closed ideal in A whose hull is F. Put

$$J_F(A) = \{t \in A \mid t = 0 \text{ on } F\}$$

and

$$J_F^0(A) = \{t \in A \mid t \text{ has a compact support disjoint from } F\}.$$

Then $J_F(A)$ is the largest and $J_F^0(A)$ is the smallest ideal in A whose hull is F (see, for example, [13, 25D. Theorem]). So, F is a set of synthesis for A if and only if $J_F^0(A)$ is dense in $J_F(A)$.

Let $V(A) = A \otimes^{\gamma} A$ be the Banach space projective tensor product of A. Then $V(A)$ is a commutative regular Banach algebra with the carrier space $\Phi \times \Phi$ [1, Proposition 42.19 and Corollary 23.9]. If, in addition, $V(A)$ is semisimple (this happens, for example, when A satisfies the Grothendieck approximation property [18]), then we can consider the synthesis problem for the closed subsets of $\Phi \times \Phi$. In particular, the diagonal, that is $\Delta = \{(\varphi, \varphi) \mid \varphi \in \Phi\}$, is a set of synthesis for $V(A)$ if and only if $J_{\Delta}^0(V(A))$ is dense in $J_{\Delta}(V(A))$. Let π be the multiplication operator from $V(A)$ into A specified by $\pi(a \otimes b) = ab$. Then, with the assumption of the semisimplicity of $V(A)$, it is easy to verify that $J_{\Delta}(V(A))= \ker \pi$.

Let G be a locally compact group with a fixed left Haar measure. Given a function f on G the left translation of f by $x \in G$ is denoted by $(l_x f)(y) = f(xy)$. Let $P(G)$ be the set of all continuous positive definite functions on G, and let $B(G)$ be its linear span. The space $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest C^*-norm. With pointwise multiplication and the dual norm, $B(G)$ is a commutative regular semisimple Banach algebra. The Fourier algebra $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$. It is shown in [6] that $A(G)$ is a commutative regular semisimple Banach algebra whose carrier space is G. Also, if λ is the left regular representation of G on $L^2(G)$ then, up to isomorphism, $A(G)$ is the unique predual of $VN(G)$, the von Neumann algebra generated by the representation λ.

2. Local operators

Let A be a commutative regular semisimple Banach algebra with the carrier space Φ, and let X be a Banach left (right) A-module. For $x \in X$, the annihilator

$\mathrm{Ann}(x)$ of x is

$$\mathrm{Ann}(x) = \{a \in A \mid \ ax = 0 \ (xa = 0)\}.$$

It is clearly a closed ideal in A, and its hull is called the *support* of x, denoted by $\mathrm{supp}\, x$. In the case $X = A$ where we regard A as a Banach (left or right) A-module on itself, the support of an element $a \in A$ will be the closure of $\{t \in \Phi \mid a(t) \neq 0\}$. An operator $T \colon A \longrightarrow X$ is called a *local operator* if $\mathrm{supp}\, T(a) \subseteq \mathrm{supp}\, a$ for all $a \in A$.

The following is a well-known result (see, for example, [13, 25.C Lemma]).

Lemma 2.1. *Let A be a commutative regular semisimple Banach algebra with the carrier space Φ. For every compact subset K and closed subset C of Φ with $K \cap C = \emptyset$, there is an element a in A such that $a = 1$ on K and $a = 0$ on C.*

The following lemma is the modification of [6, Proposition 4.4].

Lemma 2.2. *Let A be a commutative regular semisimple Banach algebra with the carrier space Φ, let X be a Banach left (right) A-module, and let $x \in X$. Then $t \in \mathrm{supp}\, x$ if and only if for every compact neighborhood V of t, there is an element $a \in A$, with the support in V, such that $ax \neq 0$ $(xa \neq 0)$.*

Proof. We prove the lemma in the case of a left module. The other case can be proved similarly. Let $t \in \mathrm{supp}\, x$, and assume that there is a compact neighborhood V of t such that for every $a \in A$, with $\mathrm{supp}\, a \subseteq V$, we have $ax = 0$. By Lemma 2.1, there is $b \in A$ such that $\mathrm{supp}\, b \subseteq V$ and $b(t) \neq 0$. Thus, for every $a \in A$, $\mathrm{supp}\,(ab) \subseteq V$ and so $abx = 0$. In particular, if we take $a \in A$ such that $a = 1$ on V, then $bx = abx = 0$. So $b \in \mathrm{Ann}(x)$, and since $t \in \mathrm{supp}\, x$, $b(t) = 0$, which is a contradiction. For the converse, let $t \in \Phi$ with the given property and $a \in A$ such that $a(t) \neq 0$. We will show that $ax \neq 0$. There is a compact neighborhood V of t and $\delta > 0$ such that $|a(v)| \geq \delta > 0$ for all $v \in V$. Because of the regularity of A and [15, Theorem 3.6.15], there is $b \in A$ such that $ab = 1$ on V. Let $c \in A$ be a function whose support is in V such that $cx \neq 0$. Then $abcx = cx$; therefore, $ax \neq 0$. □

Definition 2.3. A commutative regular semisimple Banach algebra A is called a *Tauberian algebra* if the elements with compact support are dense in A (see [15]). We denote the set of all such elements by A_c.

We recall that a Banach left [right] A-module X is *essential* if it is the closure of $AX = \mathrm{span}\{ax \mid a \in A, x \in X\}$ $[XA = \mathrm{span}\{xa \mid a \in A, x \in X\}]$, and a Banach A-bimodule X is *essential* if it is essential both as a Banach left and right A-module. If A is a Tauberian algebra then, by Lemma 2.1, it is easy to see that $A_c = A_c^2$, and so $A = \overline{A^2}$. Hence A is essential as a Banach A-bimodule on itself.

Theorem 2.4. *Let A be a Tauberian algebra such that $V(A)$ is semisimple. Suppose that the diagonal is a set of synthesis for $V(A)$, and let X be an essential Banach right (left) A-module. Then every bounded local operator T from A into X^* is a multiplier.*

Proof. We prove the theorem in the case of a right module. The other case can be proved similarly. First consider the case $X = A$, where we are considering A as a right module on itself, and let $T \colon A \longrightarrow A^*$ be a bounded local operator. Define

$$\tilde{T} \colon V(A) \longrightarrow \mathbb{C} \quad \text{by} \quad \tilde{T}(a \otimes b) = \langle T(a), b\rangle \quad (a, b \in A).$$

Pick a and b in A_c such that supp $a \cap$ supp $b = \emptyset$. By Lemma 2.1, there is $e \in A$ such that $e = 1$ on supp b, so that $b = be$. Since T is a local operator, supp $bT(a) \subseteq$ supp $b \cap$ supp $T(a) = \emptyset$. So by the Tauberian theorem [6, Theorem 3.34], $\operatorname{Ann} bT(a) = 0$. That means $bT(a) = ebT(a) = 0$ and hence $\tilde{T}(a \otimes b) = \langle bT(a)\ ,\ e\rangle = 0$. So $\tilde{T} = 0$ on

$$J = \overline{\text{span}}\{a \otimes b \in V(A) \mid a \text{ and } b \text{ have disjoint compact supports}\}.$$

But J is a closed two-sided ideal whose hull is the diagonal. So by the hypothesis, $J = J_\Delta(V(A)) = \ker \pi$. Therefore, for all $a, b, c \in A$, $\tilde{T}(ac \otimes b) = \tilde{T}(a \otimes cb)$. Thus

$$\langle T(ca),\ b\rangle = \langle T(ac),\ b\rangle = \langle T(a),\ cb\rangle = \langle T(a),\ bc\rangle = \langle cT(a),\ b\rangle.$$

So T is a multiplier.

Now we consider the general case. Let X be a Banach right A-module and $x \in X$. Define:

$$K_x \colon X^* \longrightarrow A^* \ \ \langle K_x(x^*),\ a\rangle = \langle x^*,\ xa\rangle \quad (a \in A, x^* \in X^*).$$

It is easy to see that K_x is a bounded left A-module homomorphism. So $K_x \circ T$ is a bounded local operator from A into A^* and so it is a (right) multiplier. Therefore, $K_x(T(ab)) - aT(b)) = 0$ for all a and b in A. So for each $c \in A$, we have

$$\langle T(ab) - aT(b),\ xc\rangle = 0.$$

The final result follows by the essentiality of X. □

3. Local multipliers

Let A be a commutative regular semisimple Banach algebra, and let X be a Banach left A-module. An operator $T\colon A \longrightarrow X$ is a *right multiplier* if for each $a, b \in A$, $T(ab) = aT(b)$. A *local right multiplier* is an operator $T\colon A \longrightarrow X$ such that for each $a \in A$ there is a right multiplier $T_a\colon A \longrightarrow X$ with $T(a) = T_a(a)$. Similarly, we can define (*local*) *left multipliers* for Banach right A-modules. In any case, we have $\operatorname{Ann}(a) \subseteq \operatorname{Ann}(T_a(a)) = \operatorname{Ann}(T(a))$; so supp $T(a) \subseteq$ supp a. Thus, a local multiplier is a local operator.

Theorem 3.1. *Let A be a Tauberian algebra such that $V(A)$ is semisimple. Suppose that the diagonal is a set of synthesis for $V(A)$, and let X be a Banach* (*right or left*) *A-module. Then every bounded local multiplier T from A into X is a multiplier.*

Proof. We prove this for the local right multipliers. The other case can be proved similarly. Since every bounded local multiplier is a bounded local operator, a similar argument to what we have made in the proof of Theorem 2.4 (by replacing X with X^{**}) shows that

$$cT(ab) - caT(b) = 0 \quad (a, b, c \in A).$$

Take $a \in A_c$ and $c \in A$ such that $c = 1$ on supp a. So $ca = a$. Since T is a local multiplier, there is a right multiplier M from A into X such that $T(ab) = M(ab)$. Hence,

$$0 = cT(ab) - caT(b) = cM(ab) - aT(b) = M(cab) - aT(b) = T(ab) - aT(b).$$

The final result follows by the density of A_c in A. □

4. Local derivations

Throughout this section, A is a commutative regular semisimple Banach algebra with the carrier space Φ. Let X be a Banach A-bimodule. An operator D from A into X is called a *local derivation* if for each $a \in A$, there is a derivation D_a from A into X such that $D(a) = D_a(a)$.

We recall that a Banach A-bimodule X is called a symmetric Banach A-module if for all $a \in A$ and $x \in X$, $ax = xa$. We say that A is weakly amenable if every bounded derivation from A into A^* is zero, or equivalently, every bounded derivation from A into any symmetric Banach A-module is zero [3, Theorem 2.8.63].

Proposition 4.1. *Let A be a Tauberian algebra such that $V(A)$ is semisimple and the diagonal is a set of synthesis for $V(A)$. Then A is weakly amenable.*

Proof. Let $D\colon A \longrightarrow A^*$ be a bounded derivation. Define the bounded operator $\tilde{D}\colon V(A) \longrightarrow \mathbb{C}$ specified by

$$\tilde{D}(a \otimes b) = \langle D(a), b\rangle \quad (a, b \in A).$$

Take a and b in A_c such that a and b have disjoint (compact) supports. By Lemma 2.1, there is an e in A such that $e = 1$ on supp a and $e = 0$ on supp b. So $ea = a$ and $eb = 0$. Hence

$$\begin{aligned}\langle D(a), b\rangle &= \langle D(ea), b\rangle = \langle D(e)a + eD(a), b\rangle \\ &= \langle D(e), ab\rangle + \langle D(a), be\rangle = 0.\end{aligned}$$

Therefore, by a similar argument to the one made in the proof of Theorem 2.4, D is a multiplier. So for all $a, c \in A$ we have $aD(c) = D(ac) = aD(c) + D(a)c$. Thus $D(a)c = 0$. Hence $D(a) = 0$, since $\overline{A^2} = A$ by the remark made after the Definition 2.3. So D is zero. □

For any subset E of Φ we let

$$K(E) = \{a \in A \mid a = 0 \text{ on } E\},$$

and for any two subsets I_1 and I_2 we let $V_0(I_1, I_2)$ be the closed linear span in $V(A)$ of the elements $a_1 \otimes a_2$ where $a_i \in K(\Phi \setminus I_i)$, $i = 1, 2$. It is easy to check that $V_0(I_1, I_2)$ is a Banach A-submodule of $V(A)$. The following lemma is a modification of [9, Lemma 5.2]. We include the proof for the sake of completeness.

Lemma 4.2. *Let I_1 and I_2 be subsets of Φ and $\theta \in V(A)^*$.*
(i) If $a \in K(I_1)$, then $\theta a \in V_0(I_1, I_2)^{\perp}$.
(ii) If $a \in K(I_2)$, then $a\theta \in V_0(I_1, I_2)^{\perp}$.
(iii) If $a \in A$ and $a = 1$ on I_1, then $\theta - \theta a \in V_0(I_1, I_2)^{\perp}$.
(iv) If $a \in A$ and $a = 1$ on I_2, then $\theta - a\theta \in V_0(I_1, I_2)^{\perp}$.

Proof. Let $c_i \in K(\Phi \setminus I_i)$, $i = 1, 2$. For (i) we have $\langle \theta a, c_1 \otimes c_2\rangle = \langle \theta, ac_1 \otimes c_2\rangle = 0$. For (iii) we have $\langle \theta - \theta a, c_1 \otimes c_2\rangle = \langle \theta, (c_1 - ac_1) \otimes c_2\rangle = 0$ because $c_1 = ac_1$. The other two statements can be proved similarly. □

Theorem 4.3. *Let A be a Tauberian algebra such that $V(A)$ is semisimple. Suppose that the diagonal is a set of synthesis for $V(A)$, and let X be an essential Banach A-bimodule. Then every bounded local derivation D from A into X^* is a derivation. Moreover, if A has a bounded approximate identity, then the statement of the theorem is true for all Banach A-bimodules.*

Proof. Consider first the case $X = V(A)$. Let $D\colon A \longrightarrow V(A)^*$ be a bounded local derivation, I_1 and I_2 be disjoint compact subsets of Φ and

$$q\colon V(A)^* \longrightarrow V(A)^*/V_0(I_1, I_2)^\perp$$

be the natural quotient map. Put $\tilde{D} = q \circ D$. Since q is a bounded A-module homomorphism, $\tilde{D}$ is a bounded local derivation. Now, let $b_1 \in K(I_1)$ and define

$$T_1\colon A \longrightarrow V(A)^*/V_0(I_1, I_2)^\perp \;\; T_1(a) = \tilde{D}(ab_1) \quad (a \in A).$$

Since $\tilde{D}$ is a local derivation, for each $a \in A$, there is a derivation $S\colon A \longrightarrow V(A)^*/V_0(I_1, I_2)^\perp$ such that $\tilde{D}(ab_1) = S(ab_1)$. So, by Lemma 4.2(i), we have

$$T_1(a) = S(ab_1) = aS(b_1) + S(a)b_1 = aS(b_1).$$

Thus, T_1 is a bounded local right multiplier, and so it is a right multiplier, by Theorem 3.1. Hence for all $a, c \in A$,

$$\tilde{D}(acb_1) = a\tilde{D}(cb_1). \tag{4.1}$$

Similarly, we can show that for all $a, c \in A$ and $b_2 \in K(I_2)$,

$$\tilde{D}(acb_2) = \tilde{D}(cb_2)a. \tag{4.2}$$

Let $a, c \in A_c$, and let U be a compact neighborhood in Φ such that $I_2 \subseteq U$ and $U \cap I_1 = \emptyset$. By Lemma 2.1, there are b, e and b_1 in A such that $b = 1$ on $\operatorname{supp} a \cup \operatorname{supp} c \cup U \cup I_1$, $e = 1$ on I_2 and $e = 0$ outside of U, and finally $b_1 = 0$ on I_1 and $b_1 = 1$ on I_2. Put $b_2 = b - b_1$. Then

$$b_i \in K(I_i),\; ab = a,\; bc = c,\; \text{and } eb = e. \tag{4.3}$$

Since $\tilde{D}$ is a local derivation, there is a derivation Δ from A into $V(A)^*/V_0(I_1, I_2)^\perp$ such that $\tilde{D}(b^2) = \Delta(b^2)$. So by Lemma 4.2 (iii) and (iv), $\tilde{D}(b^2) = \Delta(b^2) = e\Delta(b^2) = \Delta(eb^2) - \Delta(e)b^2 = \Delta(e) - \Delta(e) = 0$. Thus $\tilde{D}(b^2) = 0$. On the other hand, from (4.1), (4.2) and (4.3) we have

$$\tilde{D}(a) = \tilde{D}(ab^2) = \tilde{D}(ab(b_1 + b_2)) = \tilde{D}(abb_1) + \tilde{D}(abb_2) = a\tilde{D}(bb_1) + \tilde{D}(bb_2)a.$$

But $\tilde{D}(bb_1) + \tilde{D}(bb_2) = \tilde{D}(b^2) = 0$. So, if we put $\theta = \tilde{D}(bb_1)$, then

$$\tilde{D}(a) = a\theta - \theta a. \tag{4.4}$$

Similarly, we can show that (4.4) holds with the same θ if we replace a by either c or ac. Therefore, $\tilde{D}(ac) = a\tilde{D}(c) + \tilde{D}(a)c$. Since a and c were arbitrary elements in A_c, by the density, we can conclude that $\tilde{D}$ is a derivation into $V(A)^*/V_0(I_1, I_2)^\perp$.

Consider δD given by $\delta D(a, b) = aD(b) - D(ab) + D(a)b$. It is a 2-cocycle from A with values in $V(A)^*$. However, because $\tilde{D}$ is a derivation, δD maps into $V_0(I_1, I_2)^\perp$, and since this holds for all the choices of I_1 and I_2, δD maps into

$$(\overline{\operatorname{span}}\{V_0(I_1, I_2) \mid I_1 \text{ and } I_2 \text{ are disjoint compact sets}\})^\perp,$$

which is $J_\Delta(V(A))^\perp$, by the assumption that Δ, the diagonal, is a set of synthesis for $V(A)$. On the other hand, $J_\Delta(V(A))^\perp = (\ker \pi)^\perp \cong (V(A)/\ker \pi)^*$. So δD maps into $(V(A)/\ker \pi)^*$, which is the dual of a symmetric essential Banach A-module. Fix $b \in A$ and define a bounded operator $\mathcal{D}\colon A \longrightarrow (V(A)/\ker \pi)^*$ by

$$\mathcal{D}(a) = \delta D(a, b) \quad (a \in A).$$

We claim that $\mathcal{D}$ is a local operator. Let $a \in A$ and $t \notin \operatorname{supp} a$. So there is a compact neighborhood V of t such that $\operatorname{supp} a \cap V = \emptyset$. Take $c \in A$ with supp $c \subseteq V$. By Lemma 2.1, there is $e \in A$ such that $e = 1$ on V and $e = 0$ on supp a. Then, since $V(A)/\ker \pi$ is symmetric and $ca = 0$,

$$c\mathcal{D}(a) = ce\mathcal{D}(a) = c\delta D(a,b)e = -cD(ab)e + cD(b)eb. \tag{4.5}$$

On the other hand, let $h \in A$ be any element such that $ch = eh = 0$, and let $\Delta\colon A \longrightarrow V(A)^*$ be a derivation such that $D(h) = \Delta(h)$. Then $cD(h)e = c\Delta(h)e = \Delta(ch)e - \Delta(c)he = 0$. Thus, from (4.5), $c\mathcal{D}(a) = 0$, and so by Lemma 2.2, $t \notin$ supp $\mathcal{D}(a)$, i.e., $\mathcal{D}$ is a bounded local operator; so by Lemma 2.4, it is a multiplier. Hence, for all $a, b, c \in A$, $\delta D(ac, b) = a\delta D(c, b)$. So

$$D(acb) - D(ac)b = aD(cb) - aD(c)b. \tag{4.6}$$

Now, take $a, b \in A_c$ and $c \in A$ such that $c = 1$ on supp $a \cup$supp b. Then from (4.6),

$$D(ab) - D(a)b - aD(b) = -aD(c)b. \tag{4.7}$$

However, D is a local derivation; so, there is a derivation N from A into $V(A)^*$ such that $D(c) = N(c)$. So

$$aD(c)b = aN(c)b = N(ac)b - N(a)cb = N(a)b - N(a)b = 0.$$

Hence, from (4.7), $\delta D(a, b) = 0$ for all $a, b \in A_c$. Therefore, by the density, $\delta D = 0$ and so D is a derivation.

We now consider the general case. Let $x \in X$, and define $L_x\colon X^* \longrightarrow V(A)^*$ by

$$\langle L_x(x^*),\ a \otimes b\rangle = \langle x^*,\ axb\rangle \quad (a, b \in A, x^* \in X^*).$$

It is easy to check that L_x is a bounded A-bimodule homomorphism, and hence $L_x \circ D$ is a bounded local derivation into $V(A)^*$. Thus $L_x(\delta D(c, e)) = 0$ for all $c, e \in A$ and $x \in X$. So $\langle \delta D(c, e),\ axb\rangle = 0$ for all $a, b \in A$ and $x \in X$. Thus, by the essentiality of X, $\delta D = 0$, showing that D is a derivation.

Finally, suppose that A has a bounded approximate identity, X is a Banach A-bimodule and $D\colon A \longrightarrow X$ is a bounded local derivation. By a similar argument to the one made above (by replacing X with X^{**}), we can show that for all $a, b, c, d \in A$,

$$c\,\delta D(a, b)\,d = 0. \tag{4.8}$$

Put $Y = XA$. By Cohen's factorization theorem [1, Theorem 11.10], Y is a closed submodule of X. Let q be the natural quotient map from X onto X/Y. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be a bounded approximate identity for A. For each $\alpha \in \Lambda$ define $T_\alpha\colon A \longrightarrow X/Y$ by

$$T_\alpha(a) = q(D(ae_\alpha)) \quad (a \in A).$$

It is easy to see that T_α is a bounded local right multiplier; so it is a right multiplier by Theorem 3.1. Hence, for all $a, b \in A$ and $\alpha \in \Lambda$, we have $q(D(abe_\alpha)) = aq(D(be_\alpha))$. By letting $\alpha \longrightarrow \infty$, we see that $q(D(ab) - aD(b)) = 0$. So $D(ab) - aD(b) \in XA$. Hence $\delta D(a, b) \in XA$. So, by Cohen's factorization theorem, there is $e \in A$ and $x \in X$ such that $\delta D(a, b) = xe$. So if we put $d = e_\alpha$ in (4.8) and let $\alpha \longrightarrow \infty$ we will get $c\delta D(a, b) = 0$. Similarly, by letting $Y = AX$, we can show that $\delta D(a, b) = 0$ for all $a, b \in A$. So D is a derivation. □

Corollary 4.4. *Let A be a Tauberian algebra such that $V(A)$ is semisimple. Suppose that the diagonal is a set of synthesis for $V(A)$, and let X be a symmetric Banach A-module. Then every bounded local derivation D from A into X is zero.*

Proof. First consider the case $X = A^*$. Since X is the dual of an essential Banach A-bimodule, by Theorem 4.3, D is a bounded derivation and therefore, by Proposition 4.1, $D = 0$. For the general case, by a similar argument to the one made in the proof of Theorem 2.4 (by replacing X with X^{**}) we have

$$bD(a) = 0 \quad (a, b \in A). \tag{4.1}$$

Now, let $a, b \in A_c$ and take $c \in A$ such that $c = 1$ on $\operatorname{supp} a \cup \operatorname{supp} b$. So $ac = a$ and $bc = b$. Since D is a local derivation, there is a derivation S from A into X such that $D(ab) = S(ab)$. Hence,

$$D(ab) = S(abc) = abS(c) + S(ab)c = aS(c)b + D(ab)c.$$

But $D(ab)c = 0$ by (4.1), and $aS(c)b = S(ac)b - S(a)cb = S(a)b - S(a)b = 0$. So $D = 0$ on A_c^2. Therefore, by the density, $D = 0$. □

Corollary 4.5. *Let G be a locally compact group such that it has an abelian subgroup of finite index. Then every bounded local derivation from $A(G)$ into any Banach $A(G)$-bimodule is a derivation.*

Proof. By a result of Losert in [14], $A(G)\widehat{\otimes}A(G) \cong A(G \times G)$. Thus, since the diagonal $\Delta(G)$ is a closed subgroup of $G \times G$, by [17, Theorem 3] $\Delta(G)$ is a set of synthesis for $V(A(G))$. So we have the result from Theorem 4.3. □

5. Completely bounded local derivations on the Fourier algebra

Let H be a Hilbert space, and let $B(H)$ denote the space of all bounded linear operators on H. For each $n \in \mathbb{N}$, there is a natural operator norm $\|\cdot\|_n$ on the $n \times n$ matrix space $M_n(B(H)) \cong B(H^n)$. This family of norms $\{\|\cdot\|_n\}$ is the *operator matrix norm* on $B(H)$. An *operator space* is a (normed closed in this paper) linear subspace of $B(H)$ together with the operator matrix norms inherited from $B(H)$. Let V and W be two operator spaces and $\phi\colon V \longrightarrow W$ a linear map. For each $n \in \mathbb{N}$, ϕ induces a linear map $\phi_n\colon M_n(V) \longrightarrow M_n(W)$ defined by

$$\phi_n([v_{ij}]) = [\phi(v_{ij})]$$

for $[v_{ij}] \in M_n(V)$. The *completely bounded norm* of ϕ is

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| \mid n \in \mathbb{N}\}.$$

Then ϕ is *completely bounded* (resp. *completely contractive, completely isometric*) if $\|\phi\|_{cb} < \infty$ (resp. $\|\phi\|_{cb} \le 1$, *each* ϕ_n is an isometry). We let $CB(V, W)$ denote the space of all completely bounded maps from V into W. It is shown in [4] that there is a natural operator space structure on $CB(V, W)$ obtained by identifying $M_n(CB(V, W)) \cong CB(V, M_n(W))$. Thus for every operator space V, its Banach dual space $V^* = B(V, \mathbb{C}) = CB(V, \mathbb{C})$ is again an operator space and is called the *operator dual* of V. If we let $V\hat{\otimes}W$ be the *operator projective tensor product* of V and W (see [4, Chapter 7]), then there is a complete isometry $CB(V, W^*) \cong (V\hat{\otimes}W)^*$ given by

$$\langle \widetilde{T}(v), w\rangle = \langle T, v \otimes w\rangle \quad (v \in V,\ w \in W,\ T \in (V\hat{\otimes}W)^*).$$

Also, if Z is an operator space, then there are natural complete isometric isomorphisms $V\hat{\otimes}W \cong W\hat{\otimes}V$ and $(V\hat{\otimes}W)\hat{\otimes}Z \cong V\hat{\otimes}(W\hat{\otimes}Z)$.

Let A be a Banach algebra that is additionally an operator space. A is called a *completely contractive Banach algebra* if the multiplication $m\colon A\hat{\otimes}A \longrightarrow A$ is completely contractive. Moreover, if X is a Banach A-bimodule, then X is called an *operator A-bimodule* if X is an operator space and the A-bimodule operations

$$A\hat{\otimes}X \longrightarrow X; a \otimes x \longmapsto ax$$

and

$$X\hat{\otimes}A \longrightarrow X;\ x \otimes a \longmapsto xa$$

are completely bounded. It is easy to check that there is a natural operator A-bimodule structure on X^*. A is called *operator weakly amenable* if every completely bounded derivation from A into A^* is inner [7]. By [7, Proposition 3.2], for A commutative, this is equivalent to saying that every completely bounded derivation from A into any symmetric operator A-module is zero. The following lemma is well known and easy to check; so we omit the proof.

Lemma 5.1. *Let A be a completely contractive Banach algebra, let X be an operator A-bimodule, and let $x \in X$. Then, the operators K_x and L_x defined by*

$$K_x\colon X^* \longrightarrow A^*, \langle K_x(x^*), a\rangle = \langle x^*, xa\rangle \quad (a \in A, x^* \in X^*),$$
$$L_x\colon X^* \longrightarrow (A\hat{\otimes}A)^*, \langle L_x(x^*),\ a \otimes b\rangle = \langle x^*,\ axb\rangle \quad (a, b \in A, x^* \in X^*)$$

are completely bounded.

Let G be a locally compact group. Then, the von Neumann algebra $VN(G) \subset B(L^2(G))$ is an operator space. Thus $A(G)$, regarded as the operator predual of $VN(G)$, has a natural operator space structure that makes it a completely contractive Banach algebra [4, Chapter 16].

Theorem 5.2. *Suppose that G is a locally compact group. Then*

(i) Every completely bounded local derivation from $A(G)$ into any symmetric operator $A(G)$-module is zero. In particular, $A(G)$ is operator weakly amenable.

(ii) Every completely bounded local derivation from $A(G)$ into the operator dual of an essential $A(G)$-bimodule is a derivation.

(iii) If G is amenable, then the statement in (ii) is true for all operator $A(G)$-bimodules.

Proof. By replacing "Banach algebra" with "completely contractive Banach algebra", "bounded" with "completely bounded", "projective tensor product" with "operator projective tensor product", "Banach module" with "operator module" and "weakly amenable" with "operator weakly amenable", we see that the quantized versions of Theorems 2.4 and 3.1, Proposition 4.1, Theorem 4.3 and Corollary 4.4 are valid (see [4, Chapters 3, 4 and 7] and Lemma 5.1 for the details). On the other hand, it is shown in [5] that there is a complete isometry $A(G)\hat{\otimes}A(G) \cong A(G\times G)$. This map is also an algebraic isomorphism. So $A(G)\hat{\otimes}A(G)$ is semisimple and, since the diagonal $\Delta(G) = \{(g, g) \mid g \in G\}$ is a closed subgroup of $G \times G$, by [17, Theorem 3] $\Delta(G)$ is a set of synthesis for $A(G)\hat{\otimes}A(G)$. Hence, by the quantized versions of Theorem 4.3 and Corollary 4.4, the statements (i)-(iii) above follow. We note that in (iii) we are using the fact that if G is amenable, then $A(G)$ has a bounded approximate identity [12]. □

Forrest and Wood have shown in [7, Theorem 4.5] that if G is a locally compact group such that it has an abelian subgroup of finite index, then every bounded linear mapping from $A(G)$ into any operator space is completely bounded. So (by some modification) Corollary 4.5 follows from Theorem 5.2.

Acknowledgment

The author would like to express his deep gratitude to Professor F. Ghahramani for his invaluable and constructive comments during this work. The author also would like to thank the referee for helpful comments which have improved the paper.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, New York, Springer-Verlag 1973. MR54:11013

[2] Randel L. Crist, *Local derivations on operator algebras*, Journal of Functional Analysis **135** (1996), 76-92. MR96m:46128

[3] H. G. Dales, *Banach algebras and automatic continuity*, New York, Oxford University Press, 2000. MR2002e:46001

[4] E. G. Effros and Z.-J. Ruan, *Operator spaces*, London Math. Soc. Monograghs, New series, vol. 23, Oxford University Press, New York, 2000. MR2002a:46082

[5] E. G. Effros and Z.-J. Ruan, *On approximation properties for operator spaces*, Internat. J. Math. **1** (1990), 163-187. MR92g:46089

[6] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181-236. MR37:4208

[7] B. E. Forrest and P. J. Wood, *Cohomology and the operator space structure of the Fourier algebra and its second dual*, Indiana Math. J. **50** (2001), 1217-1240. MR2003d:43007

[8] C. S. Herz, *Harmonic synthesis for subgroups.* Ann. Inst. Fourier, Grenoble **23** (1973), no. 3, 91-123. MR50:7956

[9] B. E. Johnson, *Local derivations on C^*-algebras are derivations*, Trans. Amer. Math. Soc. **353** (2000), 313-325. MR2002c:46132

[10] R. V. Kadison, *Local derivations*, J. Algebra **130** (1990), 494-509. MR91f:46092

[11] D. Larson and A. Sourour, *Local derivations and local automorphisims of $B(X)$*, Proc. Sympos. Pure Math. **51** (1990), 187-194. MR91k:47106

[12] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A-B **266** (1968), 1180-1182. MR39:362

[13] L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, Van Nostrand, 1953. MR14:883c

[14] V. Losert, *On tensor products of Fourier algebras*, Arch. Math. **43** (1984), 370-372. MR87c:43004

[15] C. E. Rickart, *General theory of Banach algebra*, Princeton, N.J., Van Nostrand, 1960. MR22:5903

[16] N. Spronk, *Operator weak amenability of the Fourier algebra*, Proc. Amer. Math. Soc. **130** (2002), 3609-3617. MR2003f:46091

[17] M. Takesaki and N. Tatsuuma, *Duality and subgroups.* II, J. Functional Analysis **11** (1972), 184-190. MR52:5865

[18] J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J. (**2**) **12** (1960), 143-154. MR22:5910

Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2

E-mail address: umsameie@cc.umanitoba.ca