

COMPACT OPERATORS ON HILBERT MODULES

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ABSTRACT. We prove that an adjointable contraction acting on a countably generated Hilbert module over a separable unital C^* -algebra is compact if and only if the set of its second contractive perturbations is separable.

1. INTRODUCTION AND PRELIMINARIES

Let E be a Banach space and $\mathcal{S} \subseteq E$ an arbitrary subset of the unit ball. In [2] the first-named author and Katsoulis defined the contractive perturbations of \mathcal{S} as

$$\text{cp}(\mathcal{S}) = \{x \in E : \|a \pm x\| \leq 1 \text{ for each } a \in \mathcal{S}\},$$

while the set of the second contractive perturbations of \mathcal{S} as $\text{cp}^2(\mathcal{S}) = \text{cp}(\text{cp}(\mathcal{S}))$.

They proved there that for an element x of a C^* -algebra \mathcal{A} the set $\text{cp}^2(x)$ is compact if and only if there exists a faithful representation π of \mathcal{A} such that $\pi(x)$ is a compact operator (here, and in the sequel, $\text{cp}^2(x) = \text{cp}^2(\{x\})$). In particular, a contraction T acting on a Hilbert space \mathcal{H} is compact if and only if $\text{cp}^2(T)$ is compact, cp^2 being computed within the Banach space of all bounded linear operators on \mathcal{H} .

In this work we are looking for a characterization of the compact operators acting on a Hilbert module in terms of the size of their second contractive perturbations.

The above result does not generalize in a straightforward manner if we replace \mathcal{H} by a Hilbert module. Indeed, let A be a unital C^* -algebra, and consider the Hilbert module $X = A$ over A with the canonical inner product. It is known that the set of all adjointable operators on X coincides with the set of the compact operators and equals A . But in general the elements with compact second perturbations form a proper subset of the C^* -algebra (consider for example the algebra $C([0, 1])$ of the continuous functions on the unit interval; see also [2]).

However under certain assumptions we obtain an analogous characterization if we replace compactness by separability. Our main result is the following:

If A is a separable unital C^ -algebra and X a countably generated Hilbert A -module, then an adjointable contraction T on X is compact if and only if $\text{cp}^2(T)$ is separable.*

As an immediate corollary of this fact we obtain that any surjective isometry between the algebras of the adjointable operators on two Hilbert modules within

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the considered class must map the subalgebra of the compact operators acting on the first Hilbert module onto the subalgebra of the compact operators acting on the second one.

Before proceeding to the notation, we would like to point out that the contractive perturbations have been useful in studying the geometry of operator algebras. As examples, we refer the reader to [1] and [3].

All Hilbert modules over a C^* -algebra A are assumed to be right modules over A . If X is a Hilbert module over A , $(\cdot, \cdot) : X \times X \rightarrow A$ will denote the A -valued inner product, linear on the second and anti-linear on the first variable. By $\mathcal{B}(X)$ we denote the C^* -algebra of adjointable operators, while by $\mathcal{K}(X)$ the C^* -algebra of compact operators on X . These are the operators in the closed linear span, within $\mathcal{B}(X)$, of the set of the rank-one operators $\Theta_{x,y}$, $x, y \in X$, where $\Theta_{x,y}(z) = x(y, z)$, $z \in X$.

We end this section with a useful result [2] which enables one to produce elements of cp^2 .

Theorem 1.1. *If A is a C^* -algebra and $a \in A$, then $axa \in \text{cp}^2(a)$ for each $x \in A$ with $\|x\| \leq \frac{1}{2}$.*

2. THE MAIN RESULT

Let A be a C^* -algebra and I an arbitrary set; we let

$$l^2(A, I) = \{(a_i)_{i \in I} : a_i \in A, \sum_{i \in I} a_i^* a_i < \infty\}.$$

When $l^2(A, I)$ is equipped with the canonical right A -module operation and the inner product $((a_i), (b_i)) = \sum_{i \in I} a_i^* b_i$, it becomes a Hilbert A -module. The **standard Hilbert A -module** H_A is by definition equal to $l^2(A, \mathbb{N})$.

If E is a Banach space, $E_1 \subseteq E$ is a subspace and $x \in E_1$, then by $\text{cp}_{E_1}^2(x)$ we denote the set of the second contractive perturbations of x computed with respect to E_1 . We first prove the following:

Proposition 2.1. *Let A be a C^* -algebra, $p \in A$ a projection and $A_1 = pAp$. If $a = pap \in A$, then $\text{cp}_{A_1}^2(a) \subseteq \text{cp}_A^2(a)$.*

Proof. First we note that if E is a Banach space and $b \in E$ has the property $\|x + b\| \leq 1$ for each $x \in E$ with $\|x\| \leq 1$, then $b = 0$. Indeed, if the above property holds, then taking $x = b$ we have $\|2b\| \leq 1$. If we have shown that $\|kb\| \leq 1$, then by taking $x = kb$ in the above relation we obtain that $\|(k+1)b\| \leq 1$. Thus we have that $\|nb\| \leq 1$ for each $n \in \mathbb{N}$, which means that $b = 0$.

Suppose that A is represented faithfully and non-degenerately on a Hilbert space H , and set $H_1 = pH$ and $H_2 = (1-p)H$. Each element of A has a two by two matrix representation with respect to the decomposition $H = H_1 \oplus H_2$. If a_1 is a viewed as an element of the C^* -algebra A_1 , then the matrix of a has zero entries except at the $(1, 1)$ -place. Thus, it suffices to show that

$$\text{cp}^2(a) = \text{cp}^2 \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} \text{cp}^2(a_1) & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \text{cp}^2(a).$$

Since

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \text{cp}(a)$$

for every $x \in \text{cp}(a_1)$ and $y \in A_4 = (1 - p)A(1 - p)$ with $\|y\| \leq 1$, it follows directly that $b_1 \in \text{cp}_{A_1}^2(a_1)$ while from the remark at the beginning of the proof it follows that $b_4 = 0$. We have that $\begin{pmatrix} b_1 & b_2 \\ b_3 & y \end{pmatrix}$ is a contraction for every $y \in (1 - p)A(1 - p)$ with $\|y\| \leq 1$. Thus, if $\eta \in \mathcal{H}_2$, we have that $\|b_2\eta\|^2 + \|y\eta\|^2 \leq \|\eta\|^2$, for each $y \in (1 - p)A(1 - p)$ with $\|y\| \leq 1$. Take an approximate identity $\{y_\alpha\}$ for A_4 . It follows that $y_\alpha\eta \rightarrow \eta$, and we conclude that $b_2\eta = 0$. Since η was arbitrary, we have $b_2 = 0$. By symmetry, we obtain $b_3 = 0$. Thus we showed that each element of $\text{cp}^2(a)$ is necessarily of the form $b = pbp$ and so $\text{cp}_A^2(a) = p\text{cp}_A^2(a)p$. Now the fact that $\text{cp}_A^2(a) \subseteq \text{cp}_{A_1}^2(a)$ is immediate. \diamond

Note that the containment in the last proposition may be strict. Indeed, let v be an isometry on a Hilbert space H that is not unitary. Let $A = B(H \oplus H)$, $p \in A$ be the projection onto the first coordinate, $a = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}$ and e be the orthogonal complement of vv^* . Then $A_1 = B(H)$, and since v is an extreme point of A_1 , we have that $\text{cp}_{A_1}^2(v)$ is equal to the unit ball of $B(H)$. On the other hand, $\begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \in \text{cp}_A^2(a)$ and so if $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \text{cp}_A^2(a)$, then $xx^* \leq 1 - e$.

The following lemma, valid for arbitrary C^* -algebras, will be useful for us.

Lemma 2.2. *Let A be a C^* -algebra and $B = M(A)$ its multiplier algebra. If $a \in A$, then $\text{cp}_B^2(a) \subset A$.*

Proof. Theorem 3 of [5] implies that if $x \in B$ is a contraction, then

$$(1 - |a^*|)^{\frac{1}{2}}x(1 - |a|)^{\frac{1}{2}} \in \text{cp}(a).$$

Thus, if $b \in \text{cp}^2(a)$, then

$$\|b \pm (1 - |a^*|)^{\frac{1}{2}}(1 - |a|)^{\frac{1}{2}}\| \leq 1.$$

This implies that

$$b^*b \leq 1 - (1 - |a|)^{\frac{1}{2}}(1 - |a^*|)(1 - |a|)^{\frac{1}{2}}.$$

But

$$\begin{aligned} 1 - (1 - |a|)^{\frac{1}{2}}(1 - |a^*|)(1 - |a|)^{\frac{1}{2}} &= 1 - (1 - |a| - (1 - |a|)^{\frac{1}{2}}|a^*|(1 - |a|)^{\frac{1}{2}}) \\ &= |a| + (1 - |a|)^{\frac{1}{2}}|a^*|(1 - |a|)^{\frac{1}{2}} =: c. \end{aligned}$$

It is clear that $c \in A$ because $|a^*| \in A$ and A is an ideal in B . It is also clear that $c \geq 0$. Since A is hereditary in B , we have that $b^*b \in A$; so $|b|^{\frac{1}{2}} \in A$. It follows from Proposition 1.4.5 of [6] that $b = u|b|^{\frac{1}{2}}$, for some element $u \in B$. Since A is an ideal, it follows that $b \in A$. \diamond

The following proposition is central in our investigation.

Proposition 2.3. *Let A be a unital separable C^* -algebra, I an arbitrary set and $T \in \mathcal{B}(l^2(A, I))$. Then $T \in \mathcal{K}(l^2(A, I))$ if and only if $\text{cp}^2(T/\|T\|)$ is separable.*

Proof. Suppose that $T \in \mathcal{K}(l^2(A, I))$, and assume first that I is countable; then $l^2(A, I) = H_A$ is the standard Hilbert A -module. A well-known theorem of Kasparov [4] identifies the algebra of adjointable maps on a Hilbert module with the multiplier algebra of the algebra of compact operators on the module; in

our case $M(\mathcal{K}(H_A)) = \mathcal{B}(H_A)$. If $T \in \mathcal{K}(H_A)$, then by Lemma 2.2 we have $\text{cp}^2(T/\|T\|) \subset \mathcal{K}(H_A)$. Since A is separable, it follows that $\mathcal{K}(H_A)$ is separable and $\text{cp}^2(T/\|T\|)$ is separable as well.

Now relax the assumption that I is countable. If $x = (a_i) \in l^2(A, I)$, let $\text{supp } x = \{i \in I : a_i \neq 0\}$. Clearly, $\text{supp } x$ is countable for each $x \in l^2(A, I)$. Since T is compact, there exists a sequence $\{F_n\}$ of “finite rank” operators, say $F_n = \sum_{k=1}^{m_n} \Theta_{x_k^n, y_k^n}$, that converges to T . Let $I_0 = \bigcup_{n \in \mathbb{N}} \bigcup_{k=1}^{m_n} (\text{supp } x_k^n \cup \text{supp } y_k^n)$; clearly, I_0 is countable. Let P be the operator on $l^2(A, I)$ defined by $P((a_i)) = (b_i)$, where $b_i = a_i$ if $i \in I_0$ and $b_i = 0$ if $i \notin I_0$. It is easy to see that P is a projection in $\mathcal{B}(l^2(A, I))$ and that $PF_nP = F_n$, for each $n \in \mathbb{N}$. It follows that $PTP = T$. By Proposition 2.1 and the first paragraph, $\text{cp}^2(T/\|T\|)$ is separable.

Conversely, suppose that T is a non-compact contraction such that $\text{cp}^2(T)$ is separable. Then TT^* is non-compact as well. (Indeed, if $TT^* \in \mathcal{K}(l^2(A, I))$, then $|T^*| \in \mathcal{K}(l^2(A, I))$ and thus $T^* \in \mathcal{K}(l^2(A, I))$ by Proposition 1.4.5 of [6].) By Theorem 1.1 we have that the operators of the form TXT , where $X \in \mathcal{B}(l^2(A, I))$ has norm less than or equal to $\frac{1}{2}$, belong to $\text{cp}^2(T)$. Thus the operators of the form TT^*XT with $X \in \mathcal{B}(l^2(A, I))$, $\|X\| \leq \frac{1}{2}$ all belong to $\text{cp}^2(T)$. If, under the present assumptions, we are able to find uncountably many operators $X_\alpha \in \mathcal{B}(l^2(A, I))$, $\|X_\alpha\| \leq \frac{1}{2}$ ($\alpha \in I$) such that $\|TT^*(X_\alpha - X_\beta)TT^*\| \geq \delta$ for some $\delta > 0$ and all $\alpha, \beta \in I$, $\alpha \neq \beta$, then since $\|T\| \leq 1$ we would have also $\|TT^*(X_\alpha - X_\beta)T\| \geq \delta$ for all $\alpha, \beta \in I$ which would be a contradiction. Thus it suffices to prove the claim in the case when T is positive.

Suppose that T is a positive contraction and $\text{cp}^2(T)$ is separable. Assume that, contrary to the desired conclusion, T is not compact. Consider the net Λ of all finite subsets α of I ordered by inclusion. For each $\alpha \in \Lambda$, let P_α be the projection with range consisting of the elements (a_i) of $l^2(A, I)$ such that $a_i = 0$ if $i \notin \alpha$. Since the operators P_α are compact, we have that the net $\{TP_\alpha^\perp\}$ does not tend to zero. Thus there exists $\epsilon > 0$ such that, for each $\alpha \in \Lambda$, there exists $\beta \in \Lambda$, $\alpha \subset \beta$, such that $\|TP_\beta^\perp\| > 2\epsilon$. It follows that there exists a countable set $I_0 \subseteq I$, say $I_0 = \{i_1, i_2, \dots\}$, a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of natural numbers and elements $x_k = (P_{n_{k+1}} - P_{n_k})x_k$ of $l^2(A, I)$, $k \in \mathbb{N}$ (where we have denoted $P_n = P_{\{i_1, i_2, \dots, i_n\}}$) such that $\|x_k\| \leq \frac{1}{2}$ and $\|Tx_k\| \geq \epsilon$ for each $k \in \mathbb{N}$. For each subset $J \subseteq \mathbb{N}$, let $S_J = \sum_{k \in J} \Theta_{x_k, x_k}$, the sum being understood in the weak sense. Then clearly $\|S_J\| \leq \frac{1}{2}$. We want to estimate the norm of S_J . We have that TS_JT is greater than $T\Theta_{x, x}T = \Theta_{y, y}$ where x is an arbitrary element belonging to the set $\{x_k : k \in J\}$ and $y = Tx$. But $\|\Theta_{y, y}(y)\| = \|y(y, y)\| \geq \epsilon^3$. We conclude that $\|\Theta_{y, y}\| \geq \epsilon^3$ and hence that $\|TS_JT\| \geq \epsilon^3$.

By Theorem 1.1, $TS_JT \in \text{cp}^2(T)$. Now, the differences of the elements of the form TS_JT are again elements of this form. It follows that $\text{cp}^2(T)$ contains uncountably many elements each two of which have difference of norm greater than ϵ^3 , a contradiction. It follows that T is compact. The proof is complete. \diamond

Theorem 2.4. *Let A be a unital separable C^* -algebra, X a countably generated Hilbert A -module and $T \in \mathcal{B}(X)$. Then $T \in \mathcal{K}(X)$ if and only if $\text{cp}^2(T/\|T\|)$ is separable.*

Proof. By Kasparov’s Stabilization Theorem [4], there exists a Hilbert module Y (which can be taken to be isomorphic to H_A , but we will not need this fact) such that $X \oplus Y = H_A$. Since A is separable, $\mathcal{K}(H_A)$ is separable and hence $\mathcal{K}(X)$ is

separable as well. Suppose that $T \in \mathcal{K}(X)$. By Lemma 2.2, $\text{cp}^2(\frac{T}{\|T\|}) \subseteq \mathcal{K}(X)$ and since every closed subset of a normed separable space is separable, we conclude that $\text{cp}^2(\frac{T}{\|T\|})$ is separable.

Conversely, suppose that $\text{cp}^2(T/\|T\|)$ is separable. If $\tilde{T} = T \oplus 0$, then by Proposition 2.1 we have that $\text{cp}^2(\tilde{T}/\|\tilde{T}\|)$ is separable. By Proposition 2.3 it follows that \tilde{T} , and thus T , is compact. \diamond

We should note that the compactness of $\text{cp}^2(a)$ is a condition, strictly stronger than the separability of its linear span $[\text{cp}^2(a)]$. Indeed, if $A = C([0, 1])$ and $\mathbf{1}$ is the constant function taking value 1, then $\text{cp}^2(\mathbf{1})$ equals the unit ball of A and thus is not compact. However, $[\text{cp}^2(\mathbf{1})] = A$ is obviously separable.

The main obstruction for characterizing the compact operators on arbitrary Hilbert modules as we did in Theorem 2.4 for countably generated ones is that, in our proof, we used Kasparov's Stabilization Theorem in an essential way. A version of this theorem for arbitrary Hilbert modules is not known at present.

From Theorem 2.4 we obtain the following immediate corollary.

Corollary 2.5. *Let A be a unital separable C^* -algebra and X and Y countably generated Hilbert modules over A . If $\Phi : \mathcal{B}(X) \longrightarrow \mathcal{B}(Y)$ is a surjective isometry, then $\Phi(\mathcal{K}(X)) = \mathcal{K}(Y)$.*

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