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A B.A.I. PROOF OF THE NON-ARENS REGULARITY OF $L^1(\mathcal{G})$

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Abstract. The non-Arens regularity of $L^1(\mathcal{G})$ is proved for all non-discrete groups and all amenable groups. The proof uses bounded approximate identities in an elementary way.

Introduction

A Banach algebra is *Arens regular* if the two multiplications $\Box, \Diamond$ of Arens [1] agree on $\mathcal{A}^{**}$. The L^1-algebra of a locally compact group is not Arens regular, as proved in [2] for abelian groups and amenable discrete groups, and [10] for the general case. See [5] and [4] for surveys.

Theorem. *Let $\mathcal{G}$ be a locally compact group. If $\mathcal{G}$ is either non-discrete or amenable, then $L^1(\mathcal{G})$ is not Arens regular and the radical of $L^1(\mathcal{G})^{**}$ has dimension at least c.*

Remarks. (i). The result is a slight improvement on [2]. The proof here is simpler than [2], the reason for this note. An advantage of this proof is that it constructs an orthogonal (almost orthogonal in the discrete case) sequence, even for non-metrizable $\mathcal{G}$. For a quick non-constructive proof using Rosenthal's theorem, see [7] or [6].

(ii). This proof was suggested by the idea of Day points in Granirer [7]. Day points use a generalized version of the notion of a bounded approximate identity containing more-or-less mutually singular elements to conclude much more than non-Arens regularity. We do not know how to apply Day points in the non-amenable, discrete case. For results on Day points, see [7, 6].

(iii). Ülger [9] shows that a weakly sequentially complete Banach algebra with a b.a.i. is not Arens regular.

The Arens multiplications

The multiplication $\Box$ on the second dual of the Banach algebra $\mathcal{A}$ is $x \Box y = \lim_\alpha \lim_\beta x_\alpha y_\beta$. The second Arens multiplication $\Diamond$ is defined identically, except that the order of the limits is swapped. Here, $x_\alpha, y_\beta \in \mathcal{A}$ are bounded nets converging to $x, y \in \mathcal{A}^{**}$, respectively. See [4, (1.5)] and [3, p. 249 ff] and their references.

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Proof of Theorem: non-discrete case. We find a sequence W_n of neighbourhoods of the identity of $\mathcal{G}$. Let $\mathcal{U} = \{U_\alpha\}$ be a neighbourhood base at the identity of G consisting of relatively compact open sets. Then $m(U_\alpha)^{-1}\chi_{U_\alpha}$ is a bounded approximate identity for $L^1(G)$. Here m denotes Haar measure.

Let $W_1, W_1' \in \mathcal{U}$ be such that $W_1 \supset (\overline{W_1'W_1'})$, but $W_1 \neq \overline{W_1'W_1'}$. Assume $n \geq 2$ and that W_j, W_j', $1 \leq j \leq n$, have been found such that for $V_j = W_j \backslash \overline{W_j'W_j'}$, $f_j = m(V_j)^{-1}\chi_{V_j}$, and $1 \leq j \leq n$:

$$W_j \supset \overline{W_j'W_j'}, \quad W_j \neq \overline{W_j'W_j'}, \text{ and}$$
$$\|f_j - f_j * f_k\| < 2^{-j-k+2}, \ 1 \leq j < k \leq n.$$

We now find a $U_\alpha \in \mathcal{U}$ such that $\overline{U_\alpha U_\alpha}$ is a proper subset of W_n. Then there exists a $\gamma > \beta \geq \alpha$ such that for $W_{n+1} = U_\beta$ and $W_{n+1}' = U_\gamma$ the above holds with n replaced by $n+1$.

Now consider the sequence $g_j = f_{2j-1} - f_{2j}$, $j \geq 1$. Because the supports of the f_j are pairwise disjoint, there exists $S \in L^\infty(\mathcal{G})$ such that $\langle S, g_j\rangle = 2$ for all j and $\|S\|_\infty = 1$. Let G be any weak* cluster point of the g_j in $L^1(G)^{**}$. Then $\|G\| = 2$, since $\|G\| \leq 2\sup_j \|f_j\|_1 = 2$, and $\|G\| \geq |\langle S, g_j\rangle| = 2$.

Let F be any cluster point of the f_j. Then $\|F\| = 1$, by a similar argument. Now, $\lim_j \lim_n g_j * f_n = \lim_j g_j$, because of the limit properties of the f_j. On the other hand, $\lim_n \lim_j g_j * f_n = \lim_n f_n - f_n = 0$, because of the limit properties of the f_j. That proves that $G\Box F = G$, while $G\Diamond F = F\Box G = 0$. Similarly, $G\Box G = 0$, so G is in the radical of $L^1(\mathcal{G})^{**}$.

The dimension of the radical is at least c. This is because the weak* closures of $\{f_{2j}\}$ and $\{f_{2j-1}\}$ in $L^1(\mathcal{G})^{**}$ are homeomophic to the Stone-Čech compactification of $\mathbb{N}$, and so each closure has c distinct elements. $\Box$

Proof of Theorem: discrete amenable case. The proof here uses Følner conditions. $A\Delta B$ will denote the symmetric difference of two sets A, B. For $n < k < \infty$ let W_n be a finite subset of G such that each contains the identity and

$$\frac{\#[(W_kW_n)\Delta W_k]}{\#W_k} + \frac{\#[(W_nW_k)\Delta W_k]}{\#W_k} < 10^{-n-k}. \tag{0.1}$$

This is possible for discrete amenable G, by [8, p. 62 ff]. (If $\#W_{n+1}$ is large enough, adding the identity to W_{n+1} won't affect the inequality.) Let $f_n = (\#W_n)^{-1}\chi_{W_n}$. It then follows that

$$\lim_{k\to\infty} \|f_n * f_k - f_k\|_1 = 0 \quad \text{for all } n.$$

Indeed, let $n < k < \infty$. Then $f_n * f_k(x)$ is an average of $\#W_k$ terms each of which is either 0 or $1/\#W_n$. This means that

$$\begin{aligned}\|f_n * f_k - f_k\|_1 &= \sum_{x\in W_n\Delta W_k} f_k(x) + \sum_{x\in W_k} (f_k(x) - f_n * f_k(x)) \\ &\leq \#(W_k\Delta W_n)/\#W_k + \#(W_k\Delta W_n)/\#W_k \leq 10^{-n-k},\end{aligned}$$

since the mass on W_k "removed" by the convolution is exactly the mass on the complement of W_k. We used equation (0.1) for the last inequality. Similarly, $\lim_{n\to\infty} \|f_n * f_k - f_n\|_1 = 0$.

Let $g_k = f_{2k-1} - f_{2k}$, $k \geq 1$. By passing to a subsequence of the f_j we find $S \in L^\infty(G)$ with $\|S\|_\infty = 1$ and $\lim\langle S, g_j\rangle = 2$.

Let G be any weak* cluster point of the sequence g_j in $L^1(G)^{**}$ and F any weak* cluster point of the f_j. Then

$$\begin{aligned}\lim_n \lim_k \langle S,\, f_n * g_k\rangle &= \lim_n \lim_k \langle S,\, f_n * f_{2k-1} - f_n * f_{2k}\rangle \\ &= \lim_k \langle S,\, f_{2k-1} - f_{2k}\rangle = 2,\end{aligned}$$

but

$$\lim_k \lim_n f_n * g_k = \lim_k \lim_n f_n * f_{2k-1} - f_n * f_{2k} = 0.$$

Hence $F \Box G \neq 0$ but $F \diamond G = 0$ and $G \Box G = 0$. The radical has dimension c, just as in the non-discrete case. $\Box$

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