A B.A.I. PROOF OF THE NON-ARENS REGULARITY OF $L^1(G)$

COLIN C. GRAHAM

(Communicated by Andreas Seeger)

Abstract. The non-Arens regularity of $L^1(G)$ is proved for all non-discrete groups and all amenable groups. The proof uses bounded approximate identities in an elementary way.

Introduction


Theorem. Let $G$ be a locally compact group. If $G$ is either non-discrete or amenable, then $L^1(G)$ is not Arens regular and the radical of $L^1(G)^{**}$ has dimension at least $c$.

Remarks. (i). The result is a slight improvement on [2]. The proof here is simpler than [2], the reason for this note. An advantage of this proof is that it constructs an orthogonal (almost orthogonal in the discrete case) sequence, even for nonmetrizable $G$. For a quick non-constructive proof using Rosenthal’s theorem, see [7] or [6].

(ii). This proof was suggested by the idea of Day points in Granirer [7]. Day points use a generalized version of the notion of a bounded approximate identity containing more-or-less mutually singular elements to conclude much more than non-Arens regularity. We do not know how to apply Day points in the non-amenable, discrete case. For results on Day points, see [7, 8].

(iii). Ulger [9] shows that a weakly sequentially complete Banach algebra with a b.a.i. is not Arens regular.

The Arens Multiplications

The multiplication $\boxtimes$ on the second dual of the Banach algebra $A$ is $x \boxtimes y = \lim_\alpha \lim_{\beta} x_\alpha y_\beta$. The second Arens multiplication $\lozenge$ is defined identically, except that the order of the limits is swapped. Here, $x_\alpha, y_\beta \in A$ are bounded nets converging to $x, y \in A^{**}$, respectively. See [4, (1.5)] and [5, p. 249 ff] and their references.
Proof of Theorem: non-discrete case. We find a sequence \(W_n\) of neighbourhoods of the identity of \(G\). Let \(\mathcal{U} = \{U_\alpha\}\) be a neighbourhood base at the identity of \(G\) consisting of relatively compact open sets. Then \(m(U_\alpha)^{-1}\chi_{U_\alpha}\) is a bounded approximate identity for \(L^1(G)\). Here \(m\) denotes Haar measure.

Let \(W_1, W'_1 \in \mathcal{U}\) be such that \(W_1 \supset (W'_1)^c\), but \(W_1 \neq (W'_1)^c\). Assume \(n \geq 2\) and that \(W_j, W'_j, 1 \leq j \leq n\), have been found such that for \(V_j = W_j \setminus W'_j\), \(f_j = m(V_j)^{-1}\chi_{V_j}\), and \(1 \leq j \leq n\):

\[
W_j \supset W'_j, \quad W_j \neq W'_j, \quad \text{and} \quad \|f_j - f_j * f_k\| < 2^{-j-k+2}, \quad 1 \leq j < k \leq n.
\]

We now find a \(U_\alpha \in \mathcal{U}\) such that \(U_\alpha \cup \alpha\) is a proper subset of \(W_n\). Then there exists a \(\gamma > \beta \geq \alpha\) such that for \(W_{n+1} = U_\beta\) and \(W'_{n+1} = U_\gamma\) the above holds with \(n\) replaced by \(n + 1\).

Now consider the sequence \(g_j = f_{2j-1} - f_{2j}, j \geq 1\). Because the supports of the \(f_j\) are pairwise disjoint, there exists \(S \in L^\infty(G)\) such that \(|S, g_j| = 2\) for all \(j\) and \(\|S\|_\infty = 1\). Let \(G\) be any weak* cluster point of the \(g_j\) in \(L^1(G)^*\). Then \(|G| = 2\), since \(|G| \leq 2\sup_j \|f_j\|_1 = 2\), and \(|G| \geq |(S, g_j)| = 2\).

Let \(F\) be any cluster point of the \(f_j\). Then \(|F| = 1\), by a similar argument. Now, \(\lim_n \lim_n g_j * f_n = \lim_n g_j\), because of the limit properties of the \(f_j\). On the other hand, \(\lim_n \lim_n g_j * f_n = \lim_n f_n - f_n = 0\), because of the limit properties of the \(f_j\). That proves that \(G \square F = G\), while \(G \square F = F \square G = 0\). Similarly, \(G \square G = 0\), so \(G\) is in the radical of \(L^1(G)^*\).

The dimension of the radical is at least \(c\). This is because the weak* closures of \(\{f_{2j}\}\) and \(\{f_{2j-1}\}\) in \(L^1(G)^*\) are homeomorphic to the Stone-Cech compactification of \(\mathbb{N}\), and so each closure has \(c\) distinct elements.

Proof of Theorem: discrete amenable case. The proof here uses Følner conditions. \(A\Delta B\) will denote the symmetric difference of two sets \(A, B\). For \(n < k < \infty\), let \(W_n\) be a finite subset of \(G\) such that each contains the identity and

\[
\frac{\#[(W_k W_n) \Delta W_k]}{\#W_k} + \frac{\#[(W_n W_k) \Delta W_k]}{\#W_k} < 10^{-n-k}.
\]

This is possible for discrete amenable \(G\), by [3] p. 62 ff]. (If \(\#W_{n+1}\) is large enough, adding the identity to \(W_{n+1}\) won’t affect the inequality.) Let \(f_n = (\#W_n)^{-1}\chi_{W_n}\). It then follows that

\[
\lim_{k \to \infty} \|f_n * f_k - f_k\|_1 = 0 \quad \text{for all } n.
\]

Indeed, let \(n < k < \infty\). Then \(f_n * f_k(x)\) is an average of \(#W_k\) terms each of which is either 0 or 1/\(#W_k\). This means that

\[
\|f_n * f_k - f_k\|_1 = \sum_{x \in W_k \setminus W_n} f_k(x) + \sum_{x \in W_n} (f_k(x) - f_n * f_k(x))
\]

\[
\leq \#(W_k \setminus W_n)/\#W_k + \#(W_k \setminus W_n)/\#W_k \leq 10^{-n-k},
\]

since the mass on \(W_k\) “removed” by the convolution is exactly the mass on the complement of \(W_k\). We used equation (3.1) for the last inequality. Similarly, \(\lim_{n \to \infty} \|f_n * f_k - f_n\|_1 = 0\).

Let \(g_k = f_{2k-1} - f_{2k}, k \geq 1\). By passing to a subsequence of the \(f_j\) we find \(S \in L^\infty(G)\) with \(|S|_\infty = 1\) and \(\langle S, g_j \rangle = 2\).
Let $G$ be any weak* cluster point of the sequence $g_j$ in $L^1(G)^{**}$ and $F$ any weak* cluster point of the $f_j$. Then
\[
\lim_{n} \lim_{k} (S, f_n * g_k) = \lim_{k} (S, f_{2k-1} f_{2k} - f_n * f_{2k}) = \lim_{k} (S, f_{2k-1} f_{2k}) = 2,
\]
but
\[
\lim_{n} \lim_{k} f_n * g_k = \lim_{k} f_{2k-1} f_{2k} - f_n * f_{2k} = 0.
\]
Hence $F \square G \neq 0$ but $F \circ G = 0$ and $G \square G = 0$. The radical has dimension $c$, just as in the non-discrete case. \qed

References


Department of Mathematics, University of British Columbia (Mailing address: RR#1–H-46, Bowen Island, British Columbia, Canada VON 1G0)

E-mail address: ccgraham@alum.mit.edu

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use