A B.A.I. PROOF OF THE NON-ARENS REGULARITY OF $L^1(G)$

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Abstract. The non-Arens regularity of $L^1(G)$ is proved for all non-discrete groups and all amenable groups. The proof uses bounded approximate identities in an elementary way.

Introduction


Theorem. Let $G$ be a locally compact group. If $G$ is either non-discrete or amenable, then $L^1(G)$ is not Arens regular and the radical of $L^1(G)^{**}$ has dimension at least $c$.

Remarks. (i). The result is a slight improvement on [2]. The proof here is simpler than [2], the reason for this note. An advantage of this proof is that it constructs an orthogonal (almost orthogonal in the discrete case) sequence, even for nonmetrizable $G$. For a quick non-constructive proof using Rosenthal’s theorem, see [7] or [6].

(ii). This proof was suggested by the idea of Day points in Granirer [7]. Day points use a generalized version of the notion of a bounded approximate identity containing more-or-less mutually singular elements to conclude much more than non-Arens regularity. We do not know how to apply Day points in the non-amenable, discrete case. For results on Day points, see [7, 6].

(iii). Ulger [9] shows that a weakly sequentially complete Banach algebra with a b.a.i. is not Arens regular.

The Arens multiplications

The multiplication $\Box$ on the second dual of the Banach algebra $A$ is $x \Box y = \lim \alpha \lim \beta x_\alpha y_\beta$. The second Arens multiplication $\Diamond$ is defined identically, except that the order of the limits is swapped. Here, $x_\alpha, y_\beta \in A$ are bounded nets converging to $x, y \in A^{**}$, respectively. See [4, (1.5)] and [5, p. 249 ff] and their references.
Proof of Theorem: non-discrete case. We find a sequence $W_n$ of neighbourhoods of the identity of $G$. Let $\mathcal{U} = \{U_{\alpha}\}$ be a neighbourhood base at the identity of $G$ consisting of relatively compact open sets. Then $m(U_{\alpha})^{-1}1_{U_{\alpha}}$ is a bounded approximate identity for $L^1(G)$. Here $m$ denotes Haar measure.

Let $W_1, W_2 \in \mathcal{U}$ be such that $W_1 \supset (W_1^2W_2^2)$, but $W_1 \neq W_1^2W_2^2$. Assume $n \geq 2$ and that $W_j, W'_{j}$, $1 \leq j \leq n$, have been found such that for $V_j = W_j^2W_j'$, $f_j = m(V_j)^{-1}1_{V_j}$, and $1 \leq j \leq n$:

$$W_j \supset W_j^2W_j', \quad W_j \neq W_j^2W_j', \quad \|f_j - f_j * f_k\| < 2^{\beta - j + k + 2}, \quad 1 \leq j < k \leq n.$$ 

We now find a $U_{\alpha} \in \mathcal{U}$ such that $U_{\alpha}U_{\alpha}$ is a proper subset of $W_n$. Then there exists a $\gamma > \beta > \alpha$ such that for $W_{n+1} = U_{\beta}$ and $W'_{n+1} = U_{\gamma}$ the above holds with $n$ replaced by $n + 1$.

Now consider the sequence $g_j = f_{2j-1} - f_{2j}$, $j \geq 1$. Because the supports of the $f_j$ are pairwise disjoint, there exists $S \in L^\infty(G)$ such that $\langle S, g_j \rangle = 2$ for all $j$ and $\|S\|_\infty = 1$. Let $G$ be any weak* cluster point of the $g_j$ in $L^1(G)^\ast$. Then $\|G\| = 2$, since $\|G\| \leq 2\sup_j \|f_j\| = 2$, and $\|G\| \geq \|\langle S, g_j \rangle\| = 2$.

Let $F$ be any cluster point of the $f_j$. Then $\|F\| = 1$, by a similar argument. Now, $\lim_{j} \lim_{k} \langle g_j, f_n \rangle = \lim_{k} \langle g_j, f_n \rangle$, because of the limit properties of the $f_j$. On the other hand, $\lim_{k} \lim_{j} \langle g_j, f_n \rangle = \lim_{n} f_n - f_n = 0$, because of the limit properties of the $f_j$. That proves that $G \square F = G$, while $G \pentangle F = F \square G = 0$. Similarly, $G \square G = 0$, so $G$ is in the radical of $L^1(G)^\ast$.

The dimension of the radical is at least $c$. This is because the weak* closures of $\{f_{2j}\}$ and $\{f_{2j-1}\}$ in $L^1(G)^\ast$ are homeomorphic to the Stone-Cech compactification of $\mathbb{N}$, and so each closure has $c$ distinct elements.

Proof of Theorem: discrete amenable case. The proof here uses Følner conditions. $A \Delta B$ will denote the symmetric difference of two sets $A, B$. For $n < k < \infty$ let $W_n$ be a finite subset of $G$ such that each contains the identity and

$$\frac{\#([W_nW_n] \Delta W_k)}{\#W_k} + \frac{\#([W_nW_k] \Delta W_k)}{\#W_k} < 10^{-n-k}. \tag{0.1}$$

This is possible for discrete amenable $G$, by [3] p. 62 ff.]. (If $\#W_{n+1}$ is large enough, adding the identity to $W_{n+1}$ won’t affect the inequality.) Let $f_n = (\#W_n)^{-1}1_{W_n}$. It then follows that

$$\lim_{k \to \infty} \|f_n * f_k - f_k\|_1 = 0 \quad \text{for all } n.$$ 

Indeed, let $n < k < \infty$. Then $f_n * f_k(x)$ is an average of $\#W_n$ terms each of which is either 0 or 1/$\#W_n$. This means that

$$\|f_n * f_k - f_k\|_1 = \sum_{x \in W_n \Delta W_k} f_k(x) + \sum_{x \in W_n} (f_k(x) - f_n * f_k(x)) \leq \#(W_k \Delta W_n)/\#W_k + \#(W_k \Delta W_n)/\#W_k \leq 10^{-n-k},$$

since the mass on $W_k$ “removed” by the convolution is exactly the mass on the complement of $W_k$. We used equation (0.1) for the last inequality. Similarly, $\lim_{n \to \infty} \|f_n * f_k - f_n\|_1 = 0$.

Let $g_k = f_{2k-1} - f_{2k}$, $k \geq 1$. By passing to a subsequence of the $f_j$ we find $S \in L^\infty(G)$ with $\|S\|_\infty = 1$ and $\langle S, g_j \rangle = 2$. 

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Let $G$ be any weak* cluster point of the sequence $g_j$ in $L^1(G)^{**}$ and $F$ any weak* cluster point of the $f_j$. Then
\[
\lim_{n} \lim_{k} \langle S, f_n * g_k \rangle = \lim_{n} \lim_{k} \langle S, f_n * f_{2k-1} - f_n * f_{2k} \rangle = \lim_{k} \langle S, f_{2k-1} - f_{2k} \rangle = 2,
\]
but
\[
\lim_{n} \lim_{k} f_n * g_k = \lim_{k} f_n * f_{2k-1} - f_n * f_{2k} = 0.
\]
Hence $F \square G \neq 0$ but $F \circ G = 0$ and $G \square G = 0$. The radical has dimension $c$, just as in the non-discrete case.

References


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