

A B.A.I. PROOF OF THE NON-ARENS REGULARITY OF $L^1(\mathcal{G})$

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ABSTRACT. The non-Arens regularity of $L^1(\mathcal{G})$ is proved for all non-discrete groups and all amenable groups. The proof uses bounded approximate identities in an elementary way.

INTRODUCTION

A Banach algebra is *Arens regular* if the two multiplications \square, \diamond of Arens [1] agree on \mathcal{A}^{**} . The L^1 -algebra of a locally compact group is not Arens regular, as proved in [2] for abelian groups and amenable discrete groups, and [10] for the general case. See [5] and [4] for surveys.

Theorem. *Let \mathcal{G} be a locally compact group. If \mathcal{G} is either non-discrete or amenable, then $L^1(\mathcal{G})$ is not Arens regular and the radical of $L^1(\mathcal{G})^{**}$ has dimension at least c .*

Remarks. (i). The result is a slight improvement on [2]. The proof here is simpler than [2], the reason for this note. An advantage of this proof is that it constructs an orthogonal (almost orthogonal in the discrete case) sequence, even for non-metrizable \mathcal{G} . For a quick non-constructive proof using Rosenthal's theorem, see [7] or [6].

(ii). This proof was suggested by the idea of Day points in Granirer [7]. Day points use a generalized version of the notion of a bounded approximate identity containing more-or-less mutually singular elements to conclude much more than non-Arens regularity. We do not know how to apply Day points in the non-amenable, discrete case. For results on Day points, see [7, 6].

(iii). Ülger [9] shows that a weakly sequentially complete Banach algebra with a b.a.i. is not Arens regular.

THE ARENS MULTIPLICATIONS

The multiplication \square on the second dual of the Banach algebra \mathcal{A} is $x\square y = \lim_{\alpha} \lim_{\beta} x_{\alpha} y_{\beta}$. The second Arens multiplication \diamond is defined identically, except that the order of the limits is swapped. Here, $x_{\alpha}, y_{\beta} \in \mathcal{A}$ are bounded nets converging to $x, y \in \mathcal{A}^{**}$, respectively. See [4, (1.5)] and [3, p. 249ff] and their references.

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Proof of Theorem: non-discrete case. We find a sequence W_n of neighbourhoods of the identity of \mathcal{G} . Let $\mathcal{U} = \{U_\alpha\}$ be a neighbourhood base at the identity of G consisting of relatively compact open sets. Then $m(U_\alpha)^{-1}\chi_{U_\alpha}$ is a bounded approximate identity for $L^1(G)$. Here m denotes Haar measure.

Let $W_1, W'_1 \in \mathcal{U}$ be such that $W_1 \supset \overline{(W'_1 W'_1)}$, but $W_1 \neq \overline{W'_1 W'_1}$. Assume $n \geq 2$ and that W_j, W'_j , $1 \leq j \leq n$, have been found such that for $V_j = W_j \setminus \overline{W'_j W'_j}$, $f_j = m(V_j)^{-1}\chi_{V_j}$, and $1 \leq j \leq n$:

$$W_j \supset \overline{W'_j W'_j}, \quad W_j \neq \overline{W'_j W'_j}, \text{ and}$$

$$\|f_j - f_j * f_k\| < 2^{-j-k+2}, \quad 1 \leq j < k \leq n.$$

We now find a $U_\alpha \in \mathcal{U}$ such that $\overline{U_\alpha U_\alpha}$ is a proper subset of W_n . Then there exists a $\gamma > \beta \geq \alpha$ such that for $W_{n+1} = U_\beta$ and $W'_{n+1} = U_\gamma$ the above holds with n replaced by $n + 1$.

Now consider the sequence $g_j = f_{2j-1} - f_{2j}$, $j \geq 1$. Because the supports of the f_j are pairwise disjoint, there exists $S \in L^\infty(\mathcal{G})$ such that $\langle S, g_j \rangle = 2$ for all j and $\|S\|_\infty = 1$. Let G be any weak* cluster point of the g_j in $L^1(G)^{**}$. Then $\|G\| = 2$, since $\|G\| \leq 2 \sup_j \|f_j\|_1 = 2$, and $\|G\| \geq |\langle S, g_j \rangle| = 2$.

Let F be any cluster point of the f_j . Then $\|F\| = 1$, by a similar argument. Now, $\lim_j \lim_n g_j * f_n = \lim_j g_j$, because of the limit properties of the f_j . On the other hand, $\lim_n \lim_j g_j * f_n = \lim_n f_n - f_n = 0$, because of the limit properties of the f_j . That proves that $G \square F = G$, while $G \diamond F = F \square G = 0$. Similarly, $G \square G = 0$, so G is in the radical of $L^1(\mathcal{G})^{**}$.

The dimension of the radical is at least c . This is because the weak* closures of $\{f_{2j}\}$ and $\{f_{2j-1}\}$ in $L^1(\mathcal{G})^{**}$ are homeomorphic to the Stone-Ćech compactification of \mathbb{N} , and so each closure has c distinct elements. \square

Proof of Theorem: discrete amenable case. The proof here uses Følner conditions. $A \Delta B$ will denote the symmetric difference of two sets A, B . For $n < k < \infty$ let W_n be a finite subset of G such that each contains the identity and

$$(0.1) \quad \frac{\#[(W_k W_n) \Delta W_k]}{\#W_k} + \frac{\#[(W_n W_k) \Delta W_k]}{\#W_k} < 10^{-n-k}.$$

This is possible for discrete amenable G , by [8, p. 62 ff]. (If $\#W_{n+1}$ is large enough, adding the identity to W_{n+1} won't affect the inequality.) Let $f_n = (\#W_n)^{-1}\chi_{W_n}$. It then follows that

$$\lim_{k \rightarrow \infty} \|f_n * f_k - f_k\|_1 = 0 \quad \text{for all } n.$$

Indeed, let $n < k < \infty$. Then $f_n * f_k(x)$ is an average of $\#W_k$ terms each of which is either 0 or $1/\#W_n$. This means that

$$\|f_n * f_k - f_k\|_1 = \sum_{x \in W_n \Delta W_k} f_k(x) + \sum_{x \in W_k} (f_k(x) - f_n * f_k(x))$$

$$\leq \#(W_k \Delta W_n) / \#W_k + \#(W_k \Delta W_n) / \#W_k \leq 10^{-n-k},$$

since the mass on W_k ‘‘removed’’ by the convolution is exactly the mass on the complement of W_k . We used equation (0.1) for the last inequality. Similarly, $\lim_{n \rightarrow \infty} \|f_n * f_k - f_n\|_1 = 0$.

Let $g_k = f_{2k-1} - f_{2k}$, $k \geq 1$. By passing to a subsequence of the f_j we find $S \in L^\infty(G)$ with $\|S\|_\infty = 1$ and $\lim \langle S, g_j \rangle = 2$.

Let G be any weak* cluster point of the sequence g_j in $L^1(G)^{**}$ and F any weak* cluster point of the f_j . Then

$$\begin{aligned}\lim_n \lim_k \langle S, f_n * g_k \rangle &= \lim_n \lim_k \langle S, f_n * f_{2k-1} - f_n * f_{2k} \rangle \\ &= \lim_k \langle S, f_{2k-1} - f_{2k} \rangle = 2,\end{aligned}$$

but

$$\lim_k \lim_n f_n * g_k = \lim_k \lim_n f_n * f_{2k-1} - f_n * f_{2k} = 0.$$

Hence $F \square G \neq 0$ but $F \diamond G = 0$ and $G \square G = 0$. The radical has dimension c , just as in the non-discrete case. \square

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