EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. We obtain necessary and sufficient conditions for the existence of positive solutions for a class of sublinear Dirichlet quasilinear elliptic systems.

1. INTRODUCTION

Consider the quasilinear elliptic system

\[
\begin{cases}
-\Delta_p u = \lambda f(v) & \text{in } \Omega, \\
-\Delta_q v = \mu g(u) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f, g : [0, \infty) \to [0, \infty) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \), \( \Delta_q v = \text{div}(|\nabla v|^{q-2}\nabla v) \), \( p, q > 1 \), and \( \lambda, \mu \) are positive parameters.

Recently, Dalmasso [2] studied existence and uniqueness of positive solutions to (I) when \( p = q = 2 \) and \( f(cg(x)) \) is sublinear at 0 and \( \infty \) for every \( c > 0 \). Related results in the case when \( f(0) < 0 \) or \( g(0) < 0 \) are obtained in [1]. In this paper, we are interested in the existence and uniqueness of positive solutions for the quasilinear system (I) when \( f^{1/(p-1)}(cg^{1/(q-1)}(x)) \) is sublinear at 0 and \( \infty \) for every \( c > 0 \). We also show under additional assumptions that these conditions are necessary for (I) to have a positive solution for all \( \lambda, \mu > 0 \). Our results complement and extend corresponding results in [2]. Our approach depends on fixed points arguments and maximum principles.

2. MAIN RESULTS

We make the following assumptions:

(H.1) \( f, g : [0, \infty) \to [0, \infty) \) are continuous, nondecreasing, and \( g(x) > 0 \) for \( x > 0 \).

(H.2) For each \( c > 0 \),

\[
\limsup_{x \to 0^+} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = \infty.
\]
For each \( c > 0 \),
\[
\liminf_{x \to \infty} \frac{f_{\frac{1}{p-1}} \left( c \left( \frac{1}{q-1} \right) g \right)}{x} = 0.
\]

Then we have

**Theorem 1.** Let (H.1)-(H.3) hold. Then (I) has a positive solution \((u,v)\) for all \( \lambda, \mu > 0 \).

**Theorem 2.** Let \( f, g \) satisfy (H.1). Suppose that there exist positive numbers \( r, s \) with \( rs < (p-1)(q-1) \) such that
\[
\frac{f(x)}{x^r} \quad \text{and} \quad \frac{g(x)}{x^s}
\]
are nonincreasing for \( x \geq 0 \). Then (I) has at most one positive solution.

**Theorem 3.** Let (H.1) hold, and suppose that there exist positive numbers \( k_1, k_2 \) such that
\[
(*) \quad \liminf_{x \to 0^+} \frac{f_{\frac{1}{p-1}} (k_1 g_{\frac{1}{q-1}} (x))}{x} > 0
\]
and
\[
(**) \quad \limsup_{x \to \infty} \frac{f_{\frac{1}{p-1}} (k_2 g_{\frac{1}{q-1}} (x))}{x} < \infty.
\]
Suppose that (I) has a positive solution for all \( \lambda, \mu > 0 \). Then for each \( c > 0 \),
\[
\limsup_{x \to 0^+} \frac{f_{\frac{1}{p-1}} (c g_{\frac{1}{q-1}} (x))}{x} = \infty
\]
and
\[
\liminf_{x \to \infty} \frac{f_{\frac{1}{p-1}} (c g_{\frac{1}{q-1}} (x))}{x} = 0.
\]

**Remarks.**
1. Theorems 1 and 2 were established in [2] in the case when \( p = q = 2 \).
2. The conclusion of Theorem 3 holds under the weaker assumption that (I) has a positive solution for \( \lambda, \mu \) small and \( \lambda, \mu \) large.
3. Let \( f(x) = x^r + 1, \quad g(x) = x^s \), where \( r, s > 0 \) and \( rs < (p-1)(q-1) \). Then \( f, g \) satisfy the assumptions of Theorems 1 and 2, thus giving the existence and uniqueness of a positive solution of (I).
4. Let \( f(x) = e^{\alpha x}, \quad g(x) = (\ln(1 + x))^\beta \) where \( \alpha, \beta > 0 \) and \( \beta \leq q - 1 \). Then (*) is satisfied for all \( k_1 > 0 \) and (**) is satisfied for small \( k_2 > 0 \). It follows from Theorems 1 and 3 that (I) has a positive solution for all \( \lambda, \mu > 0 \) if and only if \( \beta < q - 1 \).

Let \( \phi, \psi \) satisfy
\[
(1) \quad -\Delta_p \phi = 1 \quad \text{in} \quad \Omega, \quad \phi = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
(2) \quad -\Delta_q \psi = 1 \quad \text{in} \quad \Omega, \quad \psi = 0 \quad \text{on} \quad \partial \Omega.
\]
Let \( D \) be a sub-domain of \( \Omega \) with \( \bar{D} \subset \Omega \). Let
\[
h(x) = \begin{cases} 
1 & \text{if} \ x \in D, \\
0 & \text{if} \ x \notin D.
\end{cases}
\]
and let \( \bar{\phi}, \bar{\psi} \) be the solution of
\[
\begin{aligned}
-\Delta_p \bar{\phi} &= h \quad \text{in } \Omega, \\
\bar{\phi} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
and
\[
\begin{aligned}
-\Delta_q \bar{\psi} &= h \quad \text{in } \Omega, \\
\bar{\psi} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
respectively. By the strong maximum principle (see [7]), there exist positive numbers \( M, m \) such that \( \bar{\phi} \geq M \phi \) in \( \Omega \) and \( \phi, \bar{\psi}, \bar{\phi}, \bar{\psi} \geq m \) in \( \bar{D} \).

Without loss of generality, we assume that \( \lambda = \mu = 1 \) in the proof of Theorems 1 and 2.

**Proof of Theorem 1.** By (H.2), there exists \( \varepsilon \in (0, 1) \) such that
\[
M \int_{\Omega} \left( mg^{\frac{1}{q'}}(\varepsilon m) \right) \geq \varepsilon.
\]
For each \( w \in C(\bar{\Omega}) \), let \( u = Tw \) be the solution of
\[
\begin{aligned}
-\Delta_p u &= f(v) \quad \text{in } \Omega, \\
-\Delta_q v &= g(\max(w, \varepsilon \phi)) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
Then \( T : C(\bar{\Omega}) \rightarrow C(\bar{\Omega}) \) is completely continuous (see e.g. [3], [5]). By (H.3), there exists a number \( R > \|\phi\| \) such that
\[
f^{\frac{1}{p'}} \left( \|\psi\| g^{\frac{1}{q'}}(R) \right) \|\phi\| \leq R.
\]
We claim that \( T : \bar{B}(0, R) \rightarrow \bar{B}(0, R) \), where \( \bar{B}(0, R) \) denotes the closed ball centered at 0 with radius \( R \) in \( C(\bar{\Omega}) \). Indeed, let \( w \in C(\bar{\Omega}) \) with \( \|w\| \leq R \). Then we have
\[
-\Delta_q v = g(\max(w, \varepsilon \phi)) \leq g(R) \quad \text{in } \Omega,
\]
which implies by the maximum principle that
\[
v \leq g^{\frac{1}{q'}}(R) \psi.
\]
Thus
\[
-\Delta_p u = f(v) \leq f \left( g^{\frac{1}{q'}}(R) \psi \right) \leq f \left( \|\psi\| g^{\frac{1}{q'}}(R) \right),
\]
and therefore
\[
u \leq f^{\frac{1}{p'}} \left( \|\psi\| g^{\frac{1}{q'}}(R) \right) \phi.
\]
Consequently,
\[
\|u\| \leq f^{\frac{1}{p'}} \left( \|\psi\| g^{\frac{1}{q'}}(R) \right) \|\phi\| \leq R,
\]
proving the claim.

By the Schauder fixed point theorem, \( T \) has a fixed point \( u \) with \( \|u\| \leq R \). Next, we verify that \( u \geq \varepsilon \phi \). Since
\[
-\Delta_q v = g(\max(u, \varepsilon \phi)) \geq \begin{cases} g(\varepsilon m) & \text{in } D, \\
0 & \text{in } \Omega \setminus D,
\end{cases}
\]
it follows from the maximum principle that
\[
v \geq g^{\frac{1}{q'}}(\varepsilon m) \bar{\psi}.
\]
Using this in the equation for $u$ gives
\[-\Delta_p u = f(v) \geq \begin{cases} f \left( mg^{\frac{1}{p-1}}(\varepsilon m) \right) & \text{in } D, \\ 0 & \text{in } \Omega \setminus D, \end{cases}\]
and therefore
\[u \geq f^{\frac{1}{p-1}} \left( mg^{\frac{1}{p-1}}(\varepsilon m) \right) \phi \geq M f^{\frac{1}{p-1}} \left( mg^{\frac{1}{p-1}}(\varepsilon m) \right) \phi \geq \varepsilon \phi.\]
Since $g(x) > 0$ for $x > 0$, we have $v > 0$ in $\Omega$ by the strong maximum principle. This completes the proof of Theorem 1.

Proof of Theorem 2. Let $(u, v)$ and $(u_1, v_1)$ be positive solutions of (I). As in [1], [6], we define $\delta = \sup \{ \varepsilon > 0 : v \geq \varepsilon v_1 \text{ in } \Omega \}$. Then $v \geq \delta v_1$. If $\delta < 1$, then we have
\[-\Delta_p u = f(v) \geq f(\delta v_1) \geq \delta^\sigma f(v_1).\]
Since
\[-\Delta_p (\delta^{\frac{1}{p-1}} u_1) = \delta^\sigma f(v_1),\]
it follows that
\[u \geq \delta^{\frac{1}{p-1}} u_1.\]
Using this in the equation for $v$ gives
\[-\Delta_q v = g(u) \geq g(\delta^{\frac{1}{p-1}} u_1) \geq \delta^{\frac{1}{p-1}} g(u_1),\]
which implies
\[v \geq \delta^{\frac{1}{(p-1)(q-1)}} v_1,\]
a contradiction with the definition of $\delta$. Thus $\delta = 1$, i.e., $v \geq v_1$, and so $v = v_1, u = u_1$, proving Theorem 2. \( \square \)

Before proving Theorem 3, we first establish

Lemma 4. Suppose that (H.1) and (*) hold, and let $\sigma, \eta > 0$ be such that
\[(3) \quad f^{\frac{1}{p-1}} \left( k_1 g^{\frac{1}{p-1}}(x) \right) \geq \sigma x \quad \text{for} \quad x \in [0, \eta].\]
Then there exists a positive number $K > 0$ such that for $\lambda, \mu > K$, any positive solution $(u, v)$ of (I) satisfies
\[u \geq C \phi,\]
where $C = \frac{\eta}{\| \phi \|_\infty}$.

Proof. Let $(u, v)$ be a positive solution of (I), and let $\alpha = \sup \{ \varepsilon > 0 : u \geq \varepsilon \phi \text{ in } \Omega \}$. Then $\alpha > 0$ and $u \geq \alpha \phi$. We claim that $\alpha \geq C$ if $\lambda, \mu$ are sufficiently large. Suppose to the contrary that $\alpha < C$. We have
\[-\Delta_q v = \mu g(u) \geq \mu g(\alpha \phi) \geq \begin{cases} \mu g(\alpha m) & \text{in } D, \\ 0 & \text{in } \Omega \setminus D, \end{cases}\]
and so
\[v \geq (\mu g(\alpha m))^{\frac{1}{p-1}} \tilde{\psi}.\]
This, in turn, implies
\[-\Delta_p u = \lambda f(v) \geq \lambda f \left( (\mu g(\alpha m))^{\frac{1}{p-1}} \tilde{\psi} \right)\]
and
\[(4) \quad u \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}} \left( (\mu g(\alpha m))^{\frac{1}{p-1}} m \right) \tilde{\phi} \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}} \left( (\mu g(\alpha m))^{\frac{1}{p-1}} m \right) M \phi.\]
follows. If \( m\mu^{1/(q-1)} > k_1 \) and \( \lambda^{\frac{1}{\tau-1}}\sigma M m > 2 \), then since \( \alpha m < C|\phi|_\infty = \eta \), it follows from (3) and (4) that

\[
u \geq \lambda^{\frac{1}{\tau-1}} f^{\frac{1}{\tau-1}} \left( k_1 g^{\frac{1}{\tau-1}} (\alpha m) \right) M\phi \geq \lambda^{\frac{1}{\tau-1}}\sigma \alpha M \phi > 2\alpha \phi,
\]
a contradiction with the definition of \( \alpha \). This completes the proof of Lemma 4. \( \square \)

Proof of Theorem 3. Let \( c > 0 \), and suppose that (I) has a positive solution \((u, v)\) for every \( \lambda, \mu > 0 \). Then we have

\[
|u|_\infty \leq (\lambda f(|v|_\infty))^{\frac{1}{\tau-1}} |\phi|_\infty
\]
and

\[
|v|_\infty \leq (\mu g(|u|_\infty))^{\frac{1}{\tau-1}} |\psi|_\infty.
\]

Hence

\[
|u|_\infty \leq \lambda^{\frac{1}{\tau-1}} f^{\frac{1}{\tau-1}} \left( \mu^{\frac{1}{\tau-1}} |\psi|_\infty g^{\frac{1}{\tau-1}} (|u|_\infty) \right) |\phi|_\infty.
\]

For \( \mu^{\frac{1}{\tau-1}} |\psi|_\infty < c_0 \equiv \min(c, k_2) \), this implies

\[
\frac{f^{\frac{1}{\tau-1}} (c_0 g^{\frac{1}{\tau-1}} (|u|_\infty))}{|u|_\infty} \geq \frac{f^{\frac{1}{\tau-1}} (\mu^{\frac{1}{\tau-1}} |\psi|_\infty g^{\frac{1}{\tau-1}} (|u|_\infty))}{|u|_\infty} \geq \frac{1}{\lambda^{\frac{1}{\tau-1}} |\phi|_\infty}.
\]

Using (***) and (5), we deduce that \( |u|_\infty \) is bounded for \( \lambda \) small, and hence

\[
\lim_{\lambda \to 0^+} |u|_\infty = 0.
\]
Combining (5) and (6) gives

\[
\lim_{x \to 0^+} \frac{f^{\frac{1}{\tau-1}} (c_0 g^{\frac{1}{\tau-1}} (x))}{x} = \infty.
\]

Next, let \( \lambda, \mu > 0 \) be large enough so that \( \lambda, \mu > K, \lambda^{\frac{1}{\tau-1}} m \min(a_1, k) > 1 \), and \( m\mu^{\frac{1}{\tau-1}} > \max(1, c) \), where

\[
a_1 = f^{\frac{1}{\tau-1}} \left( g^{\frac{1}{\tau-1}} (Cm) \right),
\]
\[
k = f^{\frac{1}{\tau-1}} \left( g^{\frac{1}{\tau-1}} (1) \right).
\]

By Lemma 4,

\[-\Delta u v = \mu g(u) \geq \mu g(C\phi)
\]
and, using the arguments as in the proof of Lemma 4, we obtain

\[
v \geq \mu^{\frac{1}{\tau-1}} g^{\frac{1}{\tau-1}} (Cm) \tilde{\psi}.
\]
Similarly, using (7) in the equation for \( u \) gives

\[
u \geq \lambda^{\frac{1}{\tau-1}} f^{\frac{1}{\tau-1}} \left( \mu^{\frac{1}{\tau-1}} m g^{\frac{1}{\tau-1}} (Cm) \right) \tilde{\phi} \geq \lambda^{\frac{1}{\tau-1}} a_1 \tilde{\phi}.
\]
By repeating the arguments, we obtain a sequence of positive numbers \((a_n)\) such that for \( n = 1, 2, \ldots \),

\[
u \geq \lambda^{\frac{1}{\tau-1}} a_n \tilde{\phi},
\]
\[
a_{n+1} = f^{\frac{1}{\tau-1}} \left( m\mu^{\frac{1}{\tau-1}} g^{\frac{1}{\tau-1}} \left( \lambda^{\frac{1}{\tau-1}} m a_n \right) \right).
\]
Clearly, \((a_n)\) is monotone and bounded, thus converging to a limit \(a_\lambda\). An induction argument shows that
\[ a_n \geq k \quad \text{for} \quad n \geq 2. \]
Hence \(a_\lambda \geq k > 0\) and
\[ a_\lambda = f^{\frac{1}{p-1}} \left( m \mu^{\frac{1}{p-1}} g^{\frac{1}{q-1}} \left( \lambda^{\frac{1}{p-1}} ma_\lambda \right) \right). \]
This implies
\[ \frac{f^{\frac{1}{p-1}} \left( cg^{\frac{1}{q-1}} (z_\lambda) \right)}{z_\lambda} \leq \frac{f^{\frac{1}{p-1}} \left( m \mu^{\frac{1}{p-1}} g^{\frac{1}{q-1}} \left( \lambda^{\frac{1}{p-1}} ma_\lambda \right) \right)}{\lambda^{\frac{1}{p-1}} ma_\lambda} = \frac{1}{\lambda^{\frac{1}{p-1}} m}, \]
where \(z_\lambda = \lambda^{\frac{1}{p-1}} ma_\lambda\). Since \(z_\lambda \to \infty\) as \(\lambda \to \infty\),
\[ \liminf_{x \to -\infty} \frac{f^{\frac{1}{p-1}} (cg^{\frac{1}{q-1}} (x))}{x} = 0. \]
This completes the proof of Theorem 3. \(\square\)

References


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