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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR QUASILINEAR ELLIPTIC SYSTEMS

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ABSTRACT. We obtain necessary and sufficient conditions for the existence of positive solutions for a class of sublinear Dirichlet quasilinear elliptic systems.

1. Introduction

Consider the quasilinear elliptic system

$$\text{(I)} \qquad \begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_q v = \mu g(u) & \text{in } \Omega, \\ \ \ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f, g : [0,\infty) \to [0,\infty)$, Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $\Delta_q v = div(|\nabla v|^{q-2}\nabla v)$, $p, q > 1$, and λ, μ are positive parameters.

Recently, Dalmasso [2] studied existence and uniqueness of positive solutions to (I) when $p = q = 2$ and $f(cg(x))$ is sublinear at 0 and ∞ for every $c > 0$. Related results in the case when $f(0) < 0$ or $g(0) < 0$ are obtained in [4]. In this paper, we are interested in the existence and uniqueness of positive solutions for the quasilinear system (I) when $f^{1/(p-1)}(cg^{1/(q-1)}(x))$ is sublinear at 0 and ∞ for every $c > 0$. We also show under additional assumptions that these conditions are necessary for (I) to have a positive solution for all $\lambda, \mu > 0$. Our results complement and extend corresponding results in [2]. Our approach depends on fixed points arguments and maximum principles.

2. Main results

We make the following assumptions:

(H.1) $f, g : [0,\infty) \to [0,\infty)$ are continuous, nondecreasing, and $g(x) > 0$ for $x > 0$.

(H.2) For each $c > 0$,

$$\limsup_{x\to 0^+} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = \infty.$$

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(H.3) For each $c > 0$,

$$\liminf_{x\to\infty} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = 0.$$

Then we have

Theorem 1. *Let* (H.1)-(H.3) *hold. Then* (I) *has a positive solution* (u, v) *for all* $\lambda, \mu > 0$.

Theorem 2. *Let* f, g *satisfy* (H.1). *Suppose that there exist positive numbers* r, s *with* $rs < (p-1)(q-1)$ *such that*

$$\frac{f(x)}{x^r} \text{ and } \frac{g(x)}{x^s}$$

are nonincreasing for $x \geq 0$. *Then* (I) *has at most one positive solution.*

Theorem 3. *Let* (H.1) *hold, and suppose that there exist positive numbers* k_1, k_2 *such that*

$$\liminf_{x\to 0^+} \frac{f^{\frac{1}{p-1}}(k_1 g^{\frac{1}{q-1}}(x))}{x} > 0 \tag{*}$$

and

$$\limsup_{x\to\infty} \frac{f^{\frac{1}{p-1}}(k_2 g^{\frac{1}{q-1}}(x))}{x} < \infty. \tag{**}$$

Suppose that (I) *has a positive solution for all* $\lambda, \mu > 0$. *Then for each* $c > 0$,

$$\limsup_{x\to 0^+} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = \infty$$

and

$$\liminf_{x\to\infty} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = 0.$$

Remarks. 1. Theorems 1 and 2 were established in [2] in the case when $p = q = 2$.

2. The conclusion of Theorem 3 holds under the weaker assumption that (I) has a positive solution for λ, μ small and λ, μ large.

3. Let $f(x) = x^r + 1$, $g(x) = x^s$, where $r, s > 0$ and $rs < (p-1)(q-1)$. Then f, g satisfy the assumptions of Theorems 1 and 2, thus giving the existence and uniqueness of a positive solution of (I).

4. Let $f(x) = e^{\alpha x}, g(x) = (\ln(1+x))^\beta$ where $\alpha, \beta > 0$ and $\beta \leq q-1$. Then (*) is satisfied for all $k_1 > 0$ and (**) is satisfied for small $k_2 > 0$. It follows from Theorems 1 and 3 that (I) has a positive solution for all $\lambda, \mu > 0$ if and only if $\beta < q - 1$.

Let ϕ, ψ satisfy

$$-\Delta_p \phi = 1 \text{ in } \Omega,\ \phi = 0 \text{ on } \partial\Omega, \tag{1}$$

$$-\Delta_q \psi = 1 \text{ in } \Omega,\ \psi = 0 \text{ on } \partial\Omega. \tag{2}$$

Let D be a sub-domain of Ω with $\bar{D} \subset \Omega$. Let

$$h(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{if } x \notin D, \end{cases}$$

and let $\tilde{\phi}$, $\tilde{\psi}$ be the solution of

$$\begin{cases} -\Delta_p \tilde{\phi} = h & \text{in } \Omega, \\ \tilde{\phi} = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\Delta_q \tilde{\psi} = h & \text{in } \Omega, \\ \tilde{\psi} = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. By the strong maximum principle (see [7]), there exist positive numbers M, m such that $\tilde{\phi} \geq M\phi$ in Ω and $\phi, \psi, \tilde{\phi}, \tilde{\psi} \geq m$ in $\bar{D}$.

Without loss of generality, we assume that $\lambda = \mu = 1$ in the proof of Theorems 1 and 2.

Proof of Theorem 1. By (H.2), there exists $\varepsilon \in (0,1)$ such that

$$M\, f^{\frac{1}{p-1}}\left(mg^{\frac{1}{q-1}}(\varepsilon m)\right) \geq \varepsilon.$$

For each $w \in C(\bar{\Omega})$, let $u = Tw$ be the solution of

$$\begin{cases} -\Delta_p u = f(v) & \text{in } \Omega, \\ -\Delta_q v = g(\max(w, \varepsilon\phi)) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $T : C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous (see e.g. [3], [5]). By (H.3), there exists a number $R > |\phi|_\infty$ such that

$$f^{\frac{1}{p-1}}\left(|\psi|_\infty g^{\frac{1}{q-1}}(R)\right)|\phi|_\infty \leq R.$$

We claim that $T : \bar{B}(0,R) \to \bar{B}(0,R)$, where $\bar{B}(0,R)$ denotes the closed ball centered at 0 with radius R in $C(\bar{\Omega})$. Indeed, let $w \in C(\bar{\Omega})$ with $|w|_\infty \leq R$. Then we have

$$-\Delta_q v = g\left(\max(w, \varepsilon\phi)\right) \leq g(R) \quad \text{in } \Omega,$$

which implies by the maximum principle that

$$v \leq g^{\frac{1}{q-1}}(R)\,\psi.$$

Thus

$$-\Delta_p u = f(v) \leq f\left(g^{\frac{1}{q-1}}(R)\,\psi\right) \leq f\left(|\psi|_\infty g^{\frac{1}{q-1}}(R)\right),$$

and therefore

$$u \leq f^{\frac{1}{p-1}}\left(|\psi|_\infty g^{\frac{1}{q-1}}(R)\right)\phi.$$

Consequently,

$$|u|_\infty \leq f^{\frac{1}{p-1}}\left(|\psi|_\infty g^{\frac{1}{q-1}}(R)\right)|\phi|_\infty \leq R,$$

proving the claim.

By the Schauder fixed point theorem, T has a fixed point u with $|u|_\infty \leq R$. Next, we verify that $u \geq \varepsilon\phi$. Since

$$-\Delta_q v = g(\max(u, \varepsilon\phi)) \geq \begin{cases} g(\varepsilon m) & \text{in } D, \\ 0 & \text{in } \Omega | D, \end{cases}$$

it follows from the maximum principle that

$$v \geq g^{\frac{1}{q-1}}(\varepsilon m)\,\tilde{\psi}.$$

Using this in the equation for u gives

$$-\Delta_p u = f(v) \ \geq \begin{cases} f\left(mg^{\frac{1}{q-1}}(\varepsilon m)\right) & \text{in } D, \\ 0 & \text{in } \Omega | D, \end{cases}$$

and therefore

$$u \geq f^{\frac{1}{p-1}}\left(mg^{\frac{1}{q-1}}(\varepsilon m)\right)\tilde{\phi} \geq M\ f^{\frac{1}{p-1}}\left(mg^{\frac{1}{q-1}}(\varepsilon m)\right)\phi \geq \varepsilon\phi.$$

Since $g(x) > 0$ for $x > 0$, we have $v > 0$ in Ω by the strong maximum principle. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let (u, v) and (u_1, v_1) be positive solutions of (I). As in [1], [6], we define $\delta = \sup\{\varepsilon > 0 : v \geq \varepsilon v_1 \text{ in } \Omega\}$. Then $v \geq \delta v_1$. If $\delta < 1$, then we have

$$-\Delta_p u = f(v) \geq f(\delta v_1) \geq \delta^r f(v_1).$$

Since

$$-\Delta_p(\delta^{\frac{r}{p-1}} u_1) = \delta^r f(v_1),$$

it follows that

$$u \geq \delta^{\frac{r}{p-1}} u_1.$$

Using this in the equation for v gives

$$-\Delta_q v = g(u) \geq g(\delta^{\frac{r}{p-1}} u_1) \geq \delta^{\frac{rs}{p-1}} g(u_1),$$

which implies

$$v \geq \delta^{\frac{rs}{(p-1)(q-1)}} v_1,$$

a contradiction with the definition of δ. Thus $\delta = 1$, i.e., $v \geq v_1$, and so $v = v_1, u = u_1$, proving Theorem 2. □

Before proving Theorem 3, we first establish

Lemma 4. *Suppose that* (H.1) *and* (*) *hold, and let* $\sigma, \eta > 0$ *be such that*

$$f^{\frac{1}{p-1}}\left(k_1 g^{\frac{1}{q-1}}(x)\right) \geq \sigma x \quad \textit{for} \ \ x \in [0, \eta]. \tag{3}$$

Then there exists a positive number $K > 0$ *such that for* $\lambda, \mu > K$, *any positive solution* (u, v) *of* (I) *satisfies*

$$u \geq C\phi,$$

where $C = \dfrac{\eta}{|\phi|_\infty}$.

Proof. Let (u, v) be a positive solution of (I), and let $\alpha = \sup\{\varepsilon > 0 : u \geq \varepsilon\phi$ in $\Omega\}$. Then $\alpha > 0$ and $u \geq \alpha\phi$. We claim that $\alpha \geq C$ if λ, μ are sufficiently large. Suppose to the contrary that $\alpha < C$. We have

$$-\Delta_q v = \mu g(u) \geq \mu g(\alpha\phi) \geq \begin{cases} \mu g(\alpha m) & \text{in } D, \\ 0 & \text{in } \Omega | D, \end{cases}$$

and so

$$v \geq (\mu g(\alpha m))^{\frac{1}{q-1}}\,\tilde{\psi}.$$

This, in turn, implies

$$-\Delta_p u = \lambda f(v) \geq \lambda f\left((\mu g(\alpha m))^{\frac{1}{q-1}}\,\tilde{\psi}\right)$$

and

$$u \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left((\mu g(\alpha m))^{\frac{1}{q-1}}\, m\right)\tilde{\phi} \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left((\mu g(\alpha m))^{\frac{1}{q-1}}\, m\right) M\phi \tag{4}$$

follows. If $m\mu^{1/(q-1)} > k_1$ and $\lambda^{\frac{1}{p-1}}\sigma M m > 2$, then since $\alpha m < C|\phi|_\infty = \eta$, it follows from (3) and (4) that

$$u \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left(k_1 g^{\frac{1}{q-1}}(\alpha m)\right) M\phi \geq \lambda^{\frac{1}{p-1}}\sigma\alpha m M\phi > 2\alpha\phi,$$

a contradiction with the definition of α. This completes the proof of Lemma 4. □

Proof of Theorem 3. Let $c > 0$, and suppose that (I) has a positive solution (u, v) for every $\lambda, \mu > 0$. Then we have

$$|u|_\infty \leq (\lambda f(|v|_\infty))^{\frac{1}{p-1}}|\phi|_\infty$$

and

$$|v|_\infty \leq (\mu g(|u|_\infty))^{\frac{1}{q-1}}|\psi|_\infty.$$

Hence

$$|u|_\infty \leq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left(\mu^{\frac{1}{p-1}}|\psi|_\infty g^{\frac{1}{q-1}}(|u|_\infty)\right)|\phi|_\infty.$$

For $\mu^{\frac{1}{p-1}}|\psi|_\infty < c_0 \equiv \min(c, k_2)$, this implies

$$\frac{f^{\frac{1}{p-1}}\left(c_0 g^{\frac{1}{q-1}}(|u|_\infty)\right)}{|u|_\infty} \geq \frac{f^{\frac{1}{p-1}}\left(\mu^{\frac{1}{p-1}}|\psi|_\infty g^{\frac{1}{q-1}}(|u|_\infty)\right)}{|u|_\infty} \geq \frac{1}{\lambda^{\frac{1}{p-1}}|\phi|_\infty}. \tag{5}$$

Using (**) and (5), we deduce that $|u_\lambda|_\infty \equiv |u|_\infty$ is bounded for λ small, and hence

$$\lim_{\lambda\to 0^+}|u_\lambda|_\infty = 0. \tag{6}$$

Combining (5) and (6) gives

$$\limsup_{x\to 0^+}\frac{f^{\frac{1}{p-1}}(c g^{\frac{1}{q-1}}(x))}{x} = \infty.$$

Next, let λ, μ be large enough so that $\lambda, \mu > K$, $\lambda^{\frac{1}{p-1}} m \min(a_1, k) > 1$, and $m\mu^{\frac{1}{q-1}} > \max(1, c)$, where

$$\begin{aligned} a_1 &= f^{\frac{1}{p-1}}\left(g^{\frac{1}{q-1}}(Cm)\right), \\ k &= f^{\frac{1}{p-1}}\left(g^{\frac{1}{q-1}}(1)\right). \end{aligned}$$

By Lemma 4,

$$-\Delta_q v = \mu g(u) \geq \mu g(C\phi)$$

and, using the arguments as in the proof of Lemma 4, we obtain

$$v \geq \mu^{\frac{1}{q-1}} g^{\frac{1}{q-1}}(Cm)\ \tilde{\psi}. \tag{7}$$

Similarly, using (7) in the equation for u gives

$$u \geq \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left(\mu^{\frac{1}{q-1}} m g^{\frac{1}{q-1}}(Cm)\right)\tilde{\phi} \geq \lambda^{\frac{1}{p-1}} a_1\tilde{\phi}.$$

By repeating the arguments, we obtain a sequence of positive numbers (a_n) such that for $n = 1, 2, \ldots,$

$$\begin{aligned} u &\geq \lambda^{\frac{1}{p-1}} a_n\tilde{\phi}, \\ a_{n+1} &= f^{\frac{1}{p-1}}\left(m\mu^{\frac{1}{q-1}} g^{\frac{1}{q-1}}\left(\lambda^{\frac{1}{p-1}} m a_n\right)\right). \end{aligned}$$

Clearly, (a_n) is monotone and bounded, thus converging to a limit a_λ. An induction argument shows that

$$a_n \geq k \quad \text{for} \quad n \geq 2.$$

Hence $a_\lambda \geq k > 0$ and

$$a_\lambda = f^{\frac{1}{p-1}}\left(m\mu^{\frac{1}{q-1}} g^{\frac{1}{q-1}}\left(\lambda^{\frac{1}{p-1}} m a_\lambda\right)\right).$$

This implies

$$\frac{f^{\frac{1}{p-1}}\left(cg^{\frac{1}{q-1}}(z_\lambda)\right)}{z_\lambda} \leq \frac{f^{\frac{1}{p-1}}\left(m\mu^{\frac{1}{q-1}} g^{\frac{1}{q-1}}\left(\lambda^{\frac{1}{p-1}} m a_\lambda\right)\right)}{\lambda^{\frac{1}{p-1}} m a_\lambda} = \frac{1}{\lambda^{\frac{1}{p-1}} m},$$

where $z_\lambda = \lambda^{\frac{1}{p-1}} m a_\lambda$. Since $z_\lambda \to \infty$ as $\lambda \to \infty$,

$$\liminf_{x\to\infty} \frac{f^{\frac{1}{p-1}}(cg^{\frac{1}{q-1}}(x))}{x} = 0.$$

This completes the proof of Theorem 3. □

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