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DECOMPOSABILITY OF GRAPH C^*-ALGEBRAS

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Abstract. We give conditions on an arbitrary directed graph E for the associated Cuntz-Krieger algebra $C^*(E)$ to be decomposable as a direct sum. We describe the direct summands as certain graph algebras.

0. Introduction

Recently various generalizations of Cuntz-Krieger algebras [2] have attracted a lot of attention. In this article we are concerned with generalized Cuntz-Krieger algebras based on directed graphs ([7] and references therein). One of the key advantages in the theory of graph algebras is that a directed graph E is used to conveniently represent generators and relations of the associated graph algebra $C^*(E)$. Thanks to the combined efforts of a number of researchers, it is now known how to read from the graph many of the basic properties and invariants of the algebra.

As with many a mathematical theory, classification of the objects in question presents itself as an important objective. A future classification of graph algebras might be very useful in paving the way for other classifications of more general classes of C^*-algebras, similarly to the way the classification of Cuntz-Krieger algebras was the starting point for the Kirchberg-Phillips classification of purely infinite simple algebras. In this context, the class of non-simple purely infinite graph algebras (in the sense of Kirchberg-Rørdam) appears to be of particular interest (cf. [6]). Certainly, the first necessary step towards a classification of non-simple algebras is good understanding of their ideal structure. For graph algebras this has been recently achieved (cf. [1, 5]). These results have already been successfully applied in solutions to some concrete problems in the classification of graph algebras as well as in quantum groups (cf. [4, 8, 3]).

In the present article we consider the question when an ideal of a graph algebra is a direct summand or, in other words, when a graph algebra decomposes as a direct sum. This very natural question turns out to be more complicated than it appears. Obviously, $C^*(E)$ splits as a direct sum when the graph E is disconnected. However, such a splitting also exists for many connected directed graphs. Especially in the context of infinite directed graphs this is a subtle problem requiring careful

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analysis, and even for finite graphs it is not a trivial one. The main result of this paper is a necessary and sufficient condition on an arbitrary infinite graph E that guarantees that the associated graph algebra $C^*(E)$ decomposes as a direct sum. Furthermore, we show that the summands are themselves isomorphic to certain graph algebras.

1. Preliminaries on graph C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph with countably many vertices E^0 and edges E^1, and range, source functions $r, s : E^1 \to E^0$, respectively. The graph C^*-algebra, or simply graph algebra $C^*(E)$ is defined as the universal C^*-algebra generated by families of projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$, subject to the following relations:

(GA1) $P_v P_w = 0$ for $v, w \in E^0$, $v \neq w$.
(GA2) $S_e^* S_f = 0$ for $e, f \in E^1$, $e \neq f$.
(GA3) $S_e^* S_e = P_{r(e)}$ for $e \in E^1$.
(GA4) $S_e S_e^* \leq P_{s(e)}$ for $e \in E^1$.
(GA5) $P_v = \displaystyle\sum_{e \in E^1:\, s(e)=v} S_e S_e^*$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

In this case, $\{P_v, S_e : v \in E^0, e \in E^1\}$ is called a Cuntz-Krieger E-family. Universality in the definition means that if $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ are families of projections and partial isometries, respectively, satisfying conditions (GA1–GA5), then there exists a C^*-algebra homomorphism from $C^*(E)$ to the C^*-algebra generated by $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ such that $P_v \mapsto Q_v$ and $S_e \mapsto T_e$ for $v \in E^0$, $e \in E^1$. It is also equivalent to the existence of a gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$, which is characterized by $\gamma_t(S_e) = tS_e$ and $\gamma_t(P_v) = P_v$ for $e \in E^1$, $v \in E^0$, $t \in \mathbb{T}$.

As usual we denote by E^* the set of all finite paths in E (vertices in E^0 are identified with paths of length 0), and by E^∞ the set of all infinite paths in E. By writing $v \geq w$ when there is a path from v to w, we say that a subset H of E^0 is *hereditary* if $v \in H$ and $v \geq w$ imply $w \in H$. A subset X of E^0 is said to be *saturated* if every vertex v that satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \Longrightarrow r(e) \in X$ itself belongs to X. The following definitions come from [1]. For a hereditary and saturated subset X of E^0, we denote $X_\infty^{\mathrm{fin}} = \{v \in E^0 \setminus X : |s^{-1}(v)| = \infty,\, 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus X)| < \infty\}$. If $w \in X_\infty^{\mathrm{fin}}$, $P_{w,X} = \sum_{e \in E^1, s(e)=w, r(e) \notin X} S_e S_e^*$ denotes the subprojection of P_w.

Our methods in this article are mainly based on the structure of gauge-invariant ideals of graph algebras (see [1] for details). Let X be a hereditary and saturated subset of E^0. For $B \subseteq X_\infty^{\mathrm{fin}}$, we denote by $J_{X,B}$ the ideal of $C^*(E)$ generated by $\{P_v : v \in X\}$ and $\{P_w - P_{w,X} : w \in B\}$. When $B = \emptyset$, we write $J_{X,\emptyset} = I_X$, the ideal generated by $\{P_v : v \in X\}$. We have

$$(1) \qquad J_{X,B} = \overline{\mathrm{span}}\{S_\alpha P_v S_\eta^*,\ S_\mu (P_w - P_{w,X}) S_\nu^* : \\ \alpha, \eta, \mu, \nu \in E^*,\ r(\alpha) = r(\eta) = v \in X,\ r(\mu) = r(\nu) = w \in B\}.$$

The ideal $J_{X,B}$ is gauge-invariant, i.e. $\gamma_t(J_{X,B}) = J_{X,B}$ for all $t \in \mathbb{T}$. It is now fully known that $J_{X,B}$ and the quotient $C^*(E)/J_{X,B}$ are isomorphic to graph algebras, associated with a directed graph ${}_XE_B$ and a quotient graph E/X. To form a directed graph ${}_XE_B$, let $\widetilde{F}_E(X,B)$ be the collection of all finite paths

$\alpha = (a_1, \ldots, a_{|\alpha|})$ of positive length such that $s(\alpha) \in E^0 \setminus X$, $r(\alpha) \in X \cup B$, and $r(a_j) \notin X \cup B$ for $j < |\alpha|$. Set $F_E(X,B) = \widetilde{F}_E(X,B) \setminus \{e \in E^1 : s(e) \in B \text{ and } r(e) \in X\}$. We denote by $\overline{F}_E(X,B)$ another copy of $F_E(X,B)$, and write $\overline{\alpha} \in \overline{F}_E(X,B)$ for the copy of $\alpha \in F_E(X,B)$. Then the graph ${}_XE_B$ is given as follows:

$$\begin{aligned}({}_XE_B)^0 &= {}_XE_B^0 = X \cup B \cup F_E(X,B),\\ ({}_XE_B)^1 &= {}_XE_B^1\\ &= \{e \in E^1 : s(e) \in X\} \cup \{e \in E^1 : s(e) \in B \text{ and } r(e) \in X\} \cup \overline{F}_E(X,B),\end{aligned}$$

with $s(\overline{\alpha}) = \alpha$ and $r(\overline{\alpha}) = r(\alpha)$ for $\alpha \in F_E(X,B)$, and the source and range as in E for the other edges of ${}_XE_B^1$. When $B = \emptyset$, we simply denote ${}_XE_\emptyset = {}_XE$. The quotient graph E/X is given by $(E/X)^0 = (E^0 \setminus X) \cup \{\beta(v) : v \in X_\infty^{\rm fin}\}$ and $(E/X)^1 = r^{-1}(E^0 \setminus X) \cup \{\beta(e) : e \in E^1, r(e) \in X_\infty^{\rm fin}\}$, where r, s are extended by $s(\beta(e)) = s(e)$ and $r(\beta(e)) = \beta(r(e))$. Here β is just a symbol helping to distinguish v and e from the extra $\beta(v)$ and $\beta(e)$ in E/X, respectively. See [3, Example 1.4] and [1, Example 3.3], which illustrate the graphs ${}_XE_B$ and E/X respectively.

Theorem 1.1. *Let E be a directed graph. Then there is a 1-1 correspondence between the set of gauge-invariant ideals of $C^*(E)$ and the set of ideals of the form $J_{X,B}$ where X is a hereditary and saturated subset of E^0 and $B \subseteq X_\infty^{\rm fin}$. Moreover,*

(i) ([3, Lemma 1.5]) *the ideal $J_{X,B}$ is isomorphic to $C^*({}_XE_B)$, and*
(ii) ([1, Corollary 3.5]) *its quotient $C^*(E)/J_{X,B}$ is isomorphic to $C^*((E/X) \setminus \beta(B))$.*

If $B = X_\infty^{\rm fin}$, then
$$C^*(E)/J_{X,X_\infty^{\rm fin}} \cong C^*(E \setminus X),$$
where $E \setminus X = (E^0 \setminus X, r^{-1}(E^0 \setminus X), r, s)$.

2. Direct sum decompositions of graph C^*-algebras

Definition 2.1. Let E be a directed graph. If there exist two non-zero C^*-algebras A, B such that $C^*(E) \cong A \oplus B$, then $C^*(E)$ is said to be *decomposable.* Otherwise $C^*(E)$ is *indecomposable.*

Our aim is to find conditions on E so that $C^*(E)$ is decomposable. We denote by $\mathrm{Prim}(\mathbf{A})$ the set of all primitive ideals in a C^*-algebra $\mathbf{A}$, equipped with the hull-kernel topology.

Lemma 2.2. *If $C^*(E) = A \oplus B$ with non-zero closed ideals A and B, then A and B are gauge-invariant.*

Proof. Since every ideal in $C^*(E)$ can be realized as the intersection of a family of primitive ideals and $\mathrm{Prim}(C^*(E))$ is the disjoint union of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$, it suffices to show that $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are invariant under the gauge action γ.

If J is a primitive ideal of $C^*(E)$ that is not gauge-invariant, then there exists a homeomorphic imbedding $\phi : \mathbb{T} \to \mathrm{Prim}(C^*(E))$ such that J belongs to $\phi(\mathbb{T})$, by combining Lemma 2.8 and Theorem 2.10 of [5]. Furthermore, $\phi(\mathbb{T})$ is invariant under the gauge action. Since $\mathbb{T}$ is connected, $\phi(\mathbb{T})$ is connected in $\mathrm{Prim}(C^*(E))$. However, both $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are closed and open, and therefore $\phi(\mathbb{T})$ is entirely contained either in $\mathrm{Prim}(A)$ or in $\mathrm{Prim}(B)$. Consequently, both $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are invariant under the gauge action. □

Note that in Definition 2.1 for $C^*(E)$ to be decomposable we do not a priori require that A and B be graph algebras. However, this turns out to be true by the following theorem.

Theorem 2.3. *If $C^*(E) = A \oplus B$ with non-zero closed ideals A and B, then there exist non-empty, disjoint, hereditary and saturated subsets X and Y of E^0 such that $A = J_{X,X_\infty^{\text{fin}}}$, $B = J_{Y,Y_\infty^{\text{fin}}}$, and $(X \cup X_\infty^{\text{fin}}) \cap (Y \cup Y_\infty^{\text{fin}}) = \emptyset$. Furthermore, $C^*(E)$ is decomposed into the direct sum of two graph algebras as $C^*(E \setminus Y) \oplus C^*(E \setminus X)$.*

Proof. By Lemma 2.2, A and B are gauge-invariant. Then by Theorem 1.1 there exist two hereditary and saturated subsets X and Y of E^0 such that $A = J_{X,C}$ and $B = J_{Y,D}$ where $C \subseteq X_\infty^{\text{fin}}$ and $D \subseteq Y_\infty^{\text{fin}}$. We have

$$X = \{v \in E^0 : P_v \in A\} \ , \quad C = \{v \in E^0 \setminus X : P_v - P_{v,X} \in A\} \ ,$$
$$Y = \{v \in E^0 : P_v \in B\} \ , \quad D = \{v \in E^0 \setminus Y : P_v - P_{v,Y} \in B\} \ .$$

It follows from the decomposability of $C^*(E)$ that X and Y are non-empty and disjoint. Then, by the definitions of X_∞^{fin} and Y_∞^{fin}, $X_\infty^{\text{fin}} \cap Y_\infty^{\text{fin}} = \emptyset$.

To show the fact $C = X_\infty^{\text{fin}}$, suppose that there is a vertex $v \in X_\infty^{\text{fin}} \setminus C$. Then the projection $P_v - P_{v,X} \notin J_{X,C}$. Since $C^*(E) = J_{X,C} \oplus J_{Y,D}$, we must have $P_v - P_{v,X} \in J_{Y,D}$, or $v \in D \subseteq Y_\infty^{\text{fin}}$, a contradiction to the fact $X_\infty^{\text{fin}} \cap Y_\infty^{\text{fin}} = \emptyset$. Thus we must have $C = X_\infty^{\text{fin}}$, and a similar argument yields $D = Y_\infty^{\text{fin}}$. The fact $(X \cup X_\infty^{\text{fin}}) \cap (Y \cup Y_\infty^{\text{fin}}) = \emptyset$ then follows easily from the hereditary and saturated properties of X and Y.

Moreover, if $C^*(E) = J_{X,X_\infty^{\text{fin}}} \oplus J_{Y,Y_\infty^{\text{fin}}}$, then

$$J_{X,X_\infty^{\text{fin}}} \cong C^*(E)/J_{Y,Y_\infty^{\text{fin}}} \cong C^*(E \setminus Y)$$

and

$$J_{Y,Y_\infty^{\text{fin}}} \cong C^*(E)/J_{X,X_\infty^{\text{fin}}} \cong C^*(E \setminus X),$$

i.e. $C^*(E) \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$. □

Remark 2.4. By Theorem 1.1, we know that $J_{X,X_\infty^{\text{fin}}} \cong C^*({}_XE_{X_\infty^{\text{fin}}})$. If $C^*(E)$ is decomposable as in Theorem 2.3, then $C^*({}_XE_{X_\infty^{\text{fin}}}) \cong J_{X,X_\infty^{\text{fin}}} \cong C^*(E \setminus Y)$. In general, the two graphs ${}_XE_{X_\infty^{\text{fin}}}$ and $E \setminus Y$ are different even though their associated graph algebras are isomorphic.

The following observation will be useful later in this article.

Lemma 2.5. *Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0 such that $C^*(E) = J_{X,X_\infty^{\text{fin}}} \oplus J_{Y,Y_\infty^{\text{fin}}}$. If $u \in E^0 \setminus (X \cup Y)$, then there exists a path in E from u into $X \cup Y$.*

Proof. Suppose that there exists no paths in E from u into $X \cup Y$. We have $u \notin X_\infty^{\text{fin}} \cup Y_\infty^{\text{fin}}$. Since P_u must be in one of the summands, it suffices to show $P_u J_{X,X_\infty^{\text{fin}}} = 0 = P_u J_{Y,Y_\infty^{\text{fin}}}$ to obtain a contradiction. We use the description of the ideals $J_{X,X_\infty^{\text{fin}}}$ and $J_{Y,Y_\infty^{\text{fin}}}$ given by the formula (1). If $w \in X$ (or Y) and α, η are paths in E with $r(\alpha) = r(\eta) = w$, then there must be no paths in E from u to both α and η by assumption. Hence $P_u(S_\alpha P_w S_\eta^*) = 0$. Let $w \in X_\infty^{\text{fin}}$ and μ, ν be paths in E with $r(\mu) = r(\nu) = w$. Again there must be no paths in E from u to both μ and ν. Hence we get $P_u\left(S_\mu(P_w - P_{w,X})S_\nu^*\right) = 0$. Similarly we obtain $P_u\left(S_\mu(P_w - P_{w,Y})S_\nu^*\right) = 0$ for the case of $w \in Y_\infty^{\text{fin}}$. □

3. Certain representations

We now focus on constructing two representations (Lemmas 3.2 and 3.4) of $C^*(E)$, which will play a crucial role in proving our main result. To this end, it is useful to consider a certain subgraph F of E. Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0. Then the subgraph $F = (F^0, F^1, r, s)$ of E is given by

$$F^0 = E^0, \tag{2}$$

(3)
$$F^1 = E^1 \setminus \left(\{e \in E^1 \colon s(e) \in X_\infty^{\text{fin}},\ r(e) \in X\} \cup \{f \in E^1 \colon s(f) \in Y_\infty^{\text{fin}},\ r(f) \in Y\}\right).$$

Let Ω be the collection of all finite paths in F beginning outside $X \cup Y$ and ending inside $X \cup Y$, upon the first entry into $X \cup Y$, i.e.,

$$\Omega := \{\omega = (e_1, \ldots, e_k) \in F^* \colon\ s(\omega) \notin X \cup Y,\ r(\omega) \in X \cup Y,\ r(e_i) \notin X \cup Y \quad \text{for } i < k\ (k \in \mathbb{N} \setminus \{0\})\},$$

and let $\mathcal{H}_\Omega$ be the Hilbert space with an orthonormal basis $\{\xi_\omega : \omega \in \Omega\}$ indexed by Ω. We define projections $\{Q_v : v \in E^0\}$ and partial isometries $\{T_e : e \in E^1\}$ on $\mathcal{H}_\Omega$ as follows:

$$Q_v(\xi_\omega) = \begin{cases} \xi_\omega & \text{if } v = s(\omega), \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$T_e(\xi_\omega) = \begin{cases} \xi_{(e,\omega)} & \text{if } r(e) = s(\omega), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that for $v \in E^0 \setminus (X \cup Y)$, the projection Q_v has finite rank if and only if there exist finitely many paths in F from v to $X \cup Y$. Indeed, the vertex $v \in E^0 \setminus (X \cup Y)$ corresponds to a projection Q_v that maps onto $\overline{\text{span}}\{\xi_\omega : s(\omega) = v, \omega \in \Omega\}$.

Lemma 3.1. *Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0. Let F be the subgraph of E as defined in* (2) *and* (3). *If for every $v \in E^0 \setminus (X \cup Y)$ there exist finitely many (and at least one) paths in F with source v to $X \cup Y$, then there exist only finitely many edges in F emitting from v.*

Proof. Let $v \in E^0 \setminus (X \cup Y)$ satisfy the above conditions. Suppose that there are infinitely many edges e in F with $s(e) = v$. By passing this into the graph E, we see that $v \notin X_\infty^{\text{fin}} \cup Y_\infty^{\text{fin}}$ in E. Hence only two cases are possible.

(i) If all edges $e \in s^{-1}(v)$ satisfy $r(e) \in E^0 \setminus (X \cup Y)$ except finitely many edges, then there are finitely many paths in F from $r(e)$ to $X \cup Y$ by assumption. These would produce infinitely many paths in F from v to $X \cup Y$, a contradiction.

(ii) If there are infinitely many $e \in E^1$ with $s(e) = v$, $r(e) \in X$ and infinitely many $f \in E^1$ with $s(f) = v$, $r(f) \in Y$, then all these edges still remain in the graph F to yield infinitely many paths in F from v to $X \cup Y$, a contradiction. □

We now examine whether the family $\{Q_v, T_e : v \in E^0, e \in E^1\}$ satisfies the Cuntz-Krieger relations for E. Conditions (GA1) and (GA2) are obvious. Condition (GA3) follows from the fact that $T_e^* T_e(\xi_\omega) = Q_v(\xi_\omega)$ if and only if $s(\omega) = v = r(e)$. Similarly, condition (GA4) is fulfilled. Unfortunately this family may not satisfy (GA5). Indeed, suppose $v \in E^0$ has the property $0 < |s^{-1}(v)| < \infty$ in E. For any $e \in s^{-1}(v)$, $T_e T_e^*$ is a projection onto $\overline{\text{span}}\{\xi_\omega : \omega = (e, \omega')$ for some $\omega' \in$

Ω and $r(e) = s(\omega')\}$, while the range of the projection Q_v includes the vector $\xi_{(f,\omega'')}$ for some $\omega'' \in \Omega$ and $r(f) = s(\omega'')$, whenever there is an edge $f \in E^1$ such that $s(f) = s(e) = v$ and $e \neq f$. Bearing this in mind, let us define R_v as the projection onto $\overline{\text{span}}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X \cup Y\}$. Then R_v is a projection of finite rank in $\mathcal{B}(\mathcal{H}_\Omega)$ by the construction, and

$$Q_v = \sum_{e\in E^1, s(e)=v, r(e)\notin X\cup Y} T_e T_e^* + R_v. \tag{6}$$

By hint of this, we pass the generating family into $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$ so that the image forms a Cuntz-Krieger E-family. To this end let $\pi : \mathcal{B}(\mathcal{H}_\Omega) \longrightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$ be the canonical quotient map. Then $\{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\}$ forms a Cuntz-Krieger E-family in $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$. Indeed, it is enough to check (GA5). So let $v \in E^0$ be $0 < |s^{-1}(v)| < \infty$. Since the rank of R_v onto $\overline{\text{span}}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X\cup Y\}$ is finite in $\mathcal{B}(\mathcal{H}_\Omega)$, the formula (6) yields $\pi(Q_v) = \sum_{e\in E^1, s(e)=v} \pi(T_e)\pi(T_e)^*$ in $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$, and hence (GA5) holds.

Lemma 3.2. *Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0. Then there is a $*$-homomorphism*

$$\rho : C^*(E) \longrightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega),$$

satisfying $\rho(S_e) = \pi(T_e)$, $\rho(P_v) = \pi(Q_v)$, $e \in E^1$, $v \in E^0$, and $\rho(J_{X,X_\infty^{\text{fin}}}) = \rho(J_{Y,Y_\infty^{\text{fin}}}) = \{0\}$.

Proof. Since $\{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\}$ forms a Cuntz-Krieger E-family in $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$ by the above argument, universality of $C^*(E)$ implies that there exists a $*-$homomorphism $\rho : C^*(E) \longrightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$ such that $\rho(S_e) = \pi(T_e)$, $\rho(P_v) = \pi(Q_v)$, $e \in E^1$, $v \in E^0$.

For the remaining part $\rho(J_{X,X_\infty^{\text{fin}}}) = \{0\}$, we check on the generators P_v and $P_w - P_{w,X}$ of $J_{X,X_\infty^{\text{fin}}}$, when $v \in X$ and $w \in X_\infty^{\text{fin}}$. If $v \in X$, then $\rho(P_v) = \pi(Q_v) = 0$ because there is no $\alpha \in \Omega$ satisfying $s(\alpha) = v$. Let $w \in X_\infty^{\text{fin}}$. If $\alpha = (e_1, \ldots, e_k) \in \Omega$ with $s(\alpha) = w$, then $r(e_1) \notin X$. Hence we have $\pi(Q_w)\xi_\alpha = \pi(Q_{w,X})\xi_\alpha$ for all $\alpha \in \Omega$, and thus $\rho(P_w - P_{w,X}) = \pi(Q_w - Q_{w,X}) = 0$. The similar argument gives $\rho(J_{Y,Y_\infty^{\text{fin}}}) = \{0\}$. $\square$

Now let $v \in E^0 \setminus (X \cup Y)$ and suppose there is an infinite path, say $\alpha = (a_1, a_2, a_3, \cdots)$ in E such that $s(\alpha) = v$. We define a corresponding representation of $C^*(E)$. Let Λ be the collection of all infinite paths in E that are shift-tail equivalent to α (recall that an infinite path $(e_1, e_2, e_3, \cdots)$ is shift-tail equivalent to α if there exist $k, m \in \mathbb{N}$ such that $e_{k+i} = a_{m+i}$ for all $i \in \mathbb{N}$), i.e.,

$$\Lambda := \{\beta = (b_1, b_2, b_3, \cdots) \in E^\infty : \ \exists\ k, m \in \mathbb{N} \ \ s.t. \ \ b_{k+i} = a_{m+i} \ \text{ for all } \ i \in \mathbb{N}\},$$

and let $\mathcal{H}_\Lambda$ be the Hilbert space with an orthonormal basis $\{\zeta_\beta : \beta \in \Lambda\}$ indexed by Λ. For $w \in E^0$ and $e \in E^1$ we define a projection U_w and a partial isometry D_e on $\mathcal{H}_\Lambda$ as follows:

$$U_w(\zeta_\beta) = \begin{cases} \zeta_\beta & \text{if } w = s(\beta), \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

$$D_e(\zeta_\beta) = \begin{cases} \zeta_{(e,b_1,b_2,\ldots)} & \text{if } r(e) = s(\beta), \text{ where } \beta = (b_1, b_2, \ldots) \in \Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 3.3. *Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0. If a vertex $v \in E^0 \setminus (X \cup Y)$ has an infinite path $\alpha = (a_1, a_2, a_3, \cdots)$ in E with $s(\alpha) = v$, then there is a $*$-homomorphism*

$$\varrho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Lambda)$$

such that $\varrho(P_w) = U_w$ and $\varrho(S_e) = D_e$ for all $w \in E^0$, $e \in E^1$. Furthermore, if α never enters $X \cup Y$, then $\varrho(J_{X,X_\infty^{\mathrm{fin}}}) = \varrho(J_{Y,Y_\infty^{\mathrm{fin}}}) = \{0\}$.

Proof. It is easy to show that the family $\{U_w, D_e : w \in E^0, e \in E^1\}$ as defined in (7) and (8) is a Cuntz-Krieger E-family, and thus universality of $C^*(E)$ implies that there is a $*$-homomorphism $\varrho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Lambda)$ such that $\varrho(P_w) = U_w$ and $\varrho(S_e) = D_e$ for all $w \in E^0$, $e \in E^1$.

Now suppose that α never enters $X \cup Y$, that is, $r(a_j) \notin X \cup Y$ for all j. It suffices to check that ϱ kills all the generating projections of the ideals $J_{X,X_\infty^{\mathrm{fin}}}$ and $J_{Y,Y_\infty^{\mathrm{fin}}}$. First we show that ϱ kills all generating projections P_w and $P_u - P_{u,X}$ of $J_{X,X_\infty^{\mathrm{fin}}}$ for $w \in X$ and $u \in X_\infty^{\mathrm{fin}}$. If $w \in X$ satisfies $\varrho(P_w) \neq 0$, then there exists an infinite path $\beta = (b_1, b_2, \cdots)$ such that $s(\beta) = w$ and $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$. By the hereditary property of X, $r(b_k) = r(a_m) \in X$, a contradiction. Now let $u \in X_\infty^{\mathrm{fin}}$ satisfy $\varrho(P_{u,X}) \neq 0$. Since $P_{u,X} = P_u - \sum_{e \in E^1, s(e)=u, r(e) \notin X} S_e S_e^*$, there exists an infinite path $\beta = (b_1, b_2, \ldots)$ such that $s(\beta) = u$, $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$, and $r(b_1) \in X$. This is a contradiction, since the hereditary property of X would then imply that α enters X.

By replacing X with Y in the above argument we can easily show that ϱ kills the generating projections P_w and $P_u - P_{u,Y}$ of $J_{Y,Y_\infty^{\mathrm{fin}}}$ for $w \in Y$ and $u \in Y_\infty^{\mathrm{fin}}$. □

Lemma 3.4. *Let X and Y be non-empty, disjoint, hereditary and saturated subsets of E^0 such that $C^*(E) = J_{X,X_\infty^{\mathrm{fin}}} \oplus J_{Y,Y_\infty^{\mathrm{fin}}}$, and let F be the subgraph of E as defined in* (2) *and* (3)*. If $v \in E^0 \setminus (X \cup Y)$ and there are no paths in F from v to $X \cup Y$, then there exists a representation ϱ of $C^*(E)$ such that $\varrho(J_{X,X_\infty^{\mathrm{fin}}}) = \varrho(J_{Y,Y_\infty^{\mathrm{fin}}}) = \{0\}$ but $\varrho(P_v) \neq 0$.*

Proof. Let $v \in E^0 \setminus (X \cup Y)$. Assume that there are no paths in F from v into $X \cup Y$. We first show that there exists an infinite path in E which begins at v and never enters $X \cup Y$. By Lemma 2.5 there exists a path in E from v to $X \cup Y$. Suppose that it enters, say X. Let w be the last vertex on this path which is not in X. Because of our assumption on v, the vertex w must satisfy the following:

(i) w belongs to X_∞^{fin},

(ii) there is no edge in E which begins at w and ends inside Y.

Condition (i) is obvious. Indeed, if $w \notin X_\infty^{\mathrm{fin}}$, then it would yield a path in F from v to X, a contradiction. Condition (ii) also holds obviously by the same reason as the case of (i). (Note that if the path enters Y, then the conditions become (i) $w \in Y_\infty^{\mathrm{fin}}$ and (ii) there is no edge in E which begins at w and ends inside X.) This means that there are finitely many edges $e \in E^1$ such that $s(e) = w$ and $r(e) \notin X \cup Y$. Then, by applying Lemma 2.5 again to the vertex $r(e)$, we see that there exists a path in E from $r(e)$ to $X \cup Y$. Now the same argument with $r(e)$ as before yields a vertex u satisfying the conditions (i) and (ii). Continuously we can produce an infinite path in E from v that never enters $X \cup Y$. Call it $\alpha = (a_1, a_2, a_3, \cdots)$.

Now consider the corresponding representation ϱ on the Hilbert space $\mathcal{H}_\Lambda$ as defined in (7) and (8). Since the path α begins at v, it follows that $\varrho(P_v) \neq 0$.

Moreover, since α never enters $X \cup Y$, ϱ is zero on both $J_{X,X_\infty^{\rm fin}}$ and $J_{Y,Y_\infty^{\rm fin}}$ by Lemma 3.3. ☐

4. The main results

We now have all the ingredients to prove our main theorem.

Theorem 4.1. *Let E be a directed graph. Then $C^*(E)$ is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets X and Y of E^0 such that for every $v \in E^0 \setminus (X \cup Y)$ there exist finitely many (and at least one) paths in F from v to $X \cup Y$, where F is the subgraph of E as defined in* (2) *and* (3)*.*

If this is the case, $C^*(E)$ is decomposed into two graph algebras as $C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

Proof. ($\Longrightarrow$) Let $C^*(E)$ be decomposable. Then by Theorem 2.3, there exist two non-empty, disjoint, hereditary and saturated subsets X and Y of E^0 such that $C^*(E) = J_{X,X_\infty^{\rm fin}} \oplus J_{Y,Y_\infty^{\rm fin}}$ and $(X \cup X_\infty^{\rm fin}) \cap (Y \cup Y_\infty^{\rm fin}) = \emptyset$. We now form the subgraph F of E as defined in (2) and (3), and let $v \in E^0 \setminus (X \cup Y)$. By Lemma 3.2, there is a representation ρ of $C^*(E)$ on the Hilbert space $\mathcal{H}_\Omega$ such that $\rho(J_{X,X_\infty^{\rm fin}}) = \rho(J_{Y,Y_\infty^{\rm fin}}) = \{0\}$.

Suppose that there are infinitely many paths in F from v into $X \cup Y$. Then the projection Q_v, as defined in (4), has infinite rank in $\mathcal{B}(\mathcal{H}_\Omega)$, and hence $\pi(Q_v) \neq 0$ in $\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)$. Since $\rho(P_v) = \pi(Q_v)$, it follows from Lemma 3.2 that $P_v \neq 0$ in $C^*(E)/(J_{X,X_\infty^{\rm fin}} \oplus J_{Y,Y_\infty^{\rm fin}})$, or $P_v \notin J_{X,X_\infty^{\rm fin}} \oplus J_{Y,Y_\infty^{\rm fin}} = C^*(E)$, a contradiction. Thus there exist only finitely many paths in F from v to $X \cup Y$.

Now suppose that there are no paths in F from v to $X \cup Y$. Then by Lemma 3.4 there exists a representation ϱ of $C^*(E)$ on the Hilbert space $\mathcal{H}_\Lambda$ such that $\varrho(J_{X,X_\infty^{\rm fin}}) = \varrho(J_{Y,Y_\infty^{\rm fin}}) = \{0\}$, but $\varrho(P_v) \neq 0$. This means $P_v \notin J_{X,X_\infty^{\rm fin}} \oplus J_{Y,Y_\infty^{\rm fin}} = C^*(E)$, a contradiction. Hence there must exist at least one path in F from v to $X \cup Y$.

($\Longleftarrow$) Let X and Y be non-empty subsets of E^0 satisfying the above conditions. We show that $J_{X,X_\infty^{\rm fin}} \cap J_{Y,Y_\infty^{\rm fin}} = \{0\}$ and $J_{X,X_\infty^{\rm fin}} \cup J_{Y,Y_\infty^{\rm fin}}$ generates $C^*(E)$.

Step 1. For $J_{X,X_\infty^{\rm fin}} \cap J_{Y,Y_\infty^{\rm fin}} = \{0\}$, it is enough to check the following cases by virtue of (1).

(i) We show $(S_\alpha P_w S_\beta^*)(S_\mu P_v S_\nu^*) = 0$ for paths α, β, μ, ν such that $r(\alpha) = r(\beta) = w \in Y$, $r(\mu) = r(\nu) = v \in X$. Suppose that either β is an initial subpath of μ or μ is an initial subpath of β. This implies that either there is a path from Y to X or vice versa, a contradiction since X and Y are hereditary and disjoint. Thus $S_\beta^* S_\mu = 0$.

(ii) We show $(S_\alpha P_w S_\beta^*)(S_\mu (P_v - P_{v,X}) S_\nu^*) = 0$ for paths α, β, μ, ν such that $r(\alpha) = r(\beta) = w \in Y$ and $r(\mu) = r(\nu) = v \in X_\infty^{\rm fin}$. Note that β cannot be an initial subpath of μ. Otherwise this creates a path from $w \in Y$ to v, and the hereditary property of Y implies $v \in Y$; but now $v \in X_\infty^{\rm fin} \cap Y$ and $v \geq X$ implies that $X \cap Y \neq \emptyset$ since X and Y are hereditary. This contradicts the fact $X_\infty^{\rm fin} \cap Y = \emptyset$. Next, assume that μ is an initial subpath of β. We write $\beta = (\mu, \beta_1)$ and $\beta_1 = (e, e_1, \cdots, e_k)$. Note that $r(e) \notin X$ because X is hereditary and $r(\beta) = w \in Y$. Thus $S_e^* P_{v,X} = S_e^* \sum_{f \in E^1, s(f) = v, r(f) \notin X} S_f S_f^* = S_e^*$. This implies $S_e^*(P_v - P_{v,X}) = S_e^* P_v - S_e^* = 0$, and hence

$$S_\beta^* S_\mu (P_v - P_{v,X}) = S_{\beta_1}^* (P_v - P_{v,X}) = S_{e_k}^* \dots S_{e_1}^* S_e^* (P_v - P_{v,X}) = 0.$$

(iii) The same argument as (ii) gives $(S_\alpha(P_w-P_{w,Y})S_\beta^*)(S_\mu P_v S_\nu^*)=0$ if α,β,μ,ν are paths such that $r(\alpha)=r(\beta)=w\in Y_\infty^{\mathrm{fin}}$ and $r(\mu)=r(\nu)=v\in X$.

(iv) We show $(S_\alpha(P_w-P_{w,Y})S_\beta^*)(S_\mu(P_v-P_{v,X})S_\nu^*)=0$ for paths α,β,μ,ν such that $r(\alpha)=r(\beta)=w\in Y_\infty^{\mathrm{fin}}$ and $r(\mu)=r(\nu)=v\in X_\infty^{\mathrm{fin}}$. It is enough to show this when there would be a path from v to w or vice versa. If $v\geq w$, then write $\beta=(\mu,\beta_1)$ with a subpath β_1 (if $w\geq v$, then write $\mu=(\beta,\mu_1)$ with a subpath μ_1). The same argument as (ii) yields the result.

Step 2. To show that the ideal J generated by $J_{X,X_\infty^{\mathrm{fin}}}\cup J_{Y,Y_\infty^{\mathrm{fin}}}$ equals $C^*(E)$, it suffices to observe that J contains all projections P_v, $v\in E^0$. Clearly, we only need to examine vertices $v\in E^0\setminus(X\cup Y)$. To this end, we use the induction on the number of finite paths in F from $v\in E^0\setminus(X\cup Y)$ to $X\cup Y$.

(i) Suppose that there is only one path $(e_1,\dots,e_k)$ in F such that $v=s(e_1)$, $r(e_i)\notin X\cup Y$ for $i<k$, and $r(e_k)\in X\cup Y$. We show by reverse induction on $i=1,\dots,k$ that $P_{s(e_i)}$ belongs to the ideal J generated by $J_{X,X_\infty^{\mathrm{fin}}}$ and $J_{Y,Y_\infty^{\mathrm{fin}}}$. (To simplify notation, we agree that e_{k+1} is the vertex $r(e_k)$.) Indeed, suppose that $P_{s(e_{i+1})}\in J$. If $s(e_i)$ does not belong to $X_\infty^{\mathrm{fin}}\cup Y_\infty^{\mathrm{fin}}$, then e_i is the only edge in E emitted by $s(e_i)$, and hence $r(e_i)=s(e_{i+1})$. Now $S_{e_i}=S_{e_i}P_{r(e_i)}=S_{e_i}P_{s(e_{i+1})}\in J$, and it follows that $P_{s(e_i)}=S_{e_i}S_{e_i}^*\in J$, and we are done. If $s(e_i)\in X_\infty^{\mathrm{fin}}$, then e_i is the only edge emitted by $s(e_i)$ whose range lies outside X. For otherwise there existed more than one path in F from v to $X\cup Y$. Thus $S_{e_i}S_{e_i}^*\in J$ by the previous observation. Hence we have $P_{s(e_i)}=(P_{s(e_i)}-P_{s(e_i),X})+S_{e_i}S_{e_i}^*\in J$, and the claim follows.

(ii) Now suppose that $P_w\in J$ for all vertices $w\in E^0\setminus(X\cup Y)$ for which there are at most n paths in F from w to $X\cup Y$. Let v be a vertex in $E^0\setminus(X\cup Y)$ with $n+1$ paths in F from v to $X\cup Y$. Let $(e_1,\dots,e_k)$ be one of these paths. Since there are at least two paths in F from v to $X\cup Y$, it follows that there exists an index i such that $s(e_i)$ emits at least two edges in F. Let m be the smallest such an index. Then for every edge f in F^1 with $s(f)=s(e_m)$ we have $P_{r(f)}\in J$, by the inductive hypothesis. Indeed, for each such f the number of paths in F from $r(f)$ to $X\cup Y$ is not greater than n.

If $s(e_m)$ does not belong to $X_\infty^{\mathrm{fin}}\cup Y_\infty^{\mathrm{fin}}$, then every edge emitted by $s(e_m)$ is in F, and there are only finitely many such edges by Lemma 3.1. Thus $P_{s(e_m)}=\sum_{s(f)=s(e_m)}S_fS_f^*$ belongs to J. If $s(e_m)\in X_\infty^{\mathrm{fin}}$, then every edge emitted by $s(e_m)$ with range outside X is in F. Thus in this case we again see that $P_{s(e_m)}=(P_{s(e_m)}-P_{s(e_m),X})+\sum_{s(f)=s(e_m)}S_fS_f^*$ belongs to J. In either case, by the choice of m, there exists a unique path $(e_1,\dots,e_{m-1})$ in F from $v=s(e_1)$ to $s(e_m)$. Thus, a reasoning as in part (i) above shows that $P_{s(e_j)}\in J$ for all $j=1,\dots,m$. Consequently $P_v\in J$. This ends the proof of the inductive step and the proof of Step 2.

Finally, if $C^*(E)=J_{X,X_\infty^{\mathrm{fin}}}\oplus J_{Y,Y_\infty^{\mathrm{fin}}}$, then it is clear from Theorem 1.1 that

$$J_{X,X_\infty^{\mathrm{fin}}}\cong C^*(E)/J_{Y,Y_\infty^{\mathrm{fin}}}\cong C^*(E\setminus Y)$$

and

$$J_{Y,Y_\infty^{\mathrm{fin}}}\cong C^*(E)/J_{X,X_\infty^{\mathrm{fin}}}\cong C^*(E\setminus X).$$

□

Corollary 4.2. *Let E be a directed graph with finitely many vertices. Then there exist finitely many subgraphs $F_1, \dots, F_n$ of E such that $C^*(E) \cong C^*(F_1) \oplus \cdots \oplus C^*(F_n)$ and each $C^*(F_k)$ is indecomposable.*

Proof. The decomposition of $C^*(E)$ depends on the existence of two non-empty, disjoint, hereditary and saturated subsets X and Y of E^0 satisfying conditions in Theorem 4.1. If no such subsets X and Y exist, then the algebra $C^*(E)$ is itself indecomposable. Otherwise, $C^*(E) \cong C^*(F_1) \oplus C^*(F_2)$ for some graphs F_1 and F_2. If both $C^*(F_1)$ and $C^*(F_2)$ are indecomposable, we are done. Otherwise, find a direct sum decomposition of the algebras $C^*(F_i)$ whenever they are decomposable. Continuing this process yields the result after a finite number of steps. □

5. C^*-algebras of finite graphs

This section is devoted to giving a more feasible criterion for C^*-algebras of finite directed graphs, even though it can be covered by Theorem 4.1.

Theorem 5.1. *Let E be a finite directed graph. Then $C^*(E)$ is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets X and Y of E^0 such that for every vertex $v \in E^0 \setminus (X \cup Y)$ (i) there exist paths from v to both X and Y, and (ii) there is no loop passing through v. If this is the case, $C^*(E) = I_X \oplus I_Y \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$.*

Proof. ($\Longrightarrow$) By taking $B = \emptyset$ in Theorem 2.3, we see that $C^*(E) = I_X \oplus I_Y$ with two non-empty, disjoint, hereditary and saturated subsets X and Y of E^0.

(i) Consider a vertex v in $E^0 \setminus (X \cup Y)$ and suppose for a moment that there is no path in E from v to a vertex in Y. Since $I_Y = \overline{\operatorname{span}}\{S_\alpha P_w S_\beta^* : \alpha, \beta \in E^*,\ w \in Y,\ r(\alpha) = r(\beta) = w\}$, it is clear that $P_v I_Y = \{0\}$. Since $C^*(E) = I_X \oplus I_Y$ we must have $P_v \in I_X$ and consequently $v \in X$, a contradiction. Hence there is a path from v to Y. Similarly, there must be a path from v to X.

(ii) Suppose $v \in E^0 \setminus (X \cup Y)$ and there is a loop $\mu = (\mu_1, \dots, \mu_k)$ in E through v. Since the ideal $I_{X\cup Y}$, generated by I_X and I_Y, equals $C^*(E)$ we have $I_{X\cup Y} = I_{E^0}$. But $X \cup Y$ is a hereditary set and $I_{X\cup Y}$ equals $I_{\Sigma(X\cup Y)}$, where $\Sigma(X \cup Y)$ denotes the saturation of $X \cup Y$. Since a gauge-invariant ideal determines the corresponding hereditary and saturated subset of E^0, we must have $\Sigma(X \cup Y) = E^0$ and, in particular, v belongs to the saturation of $X \cup Y$. Now $\Sigma(X \cup Y) = E^0$ is the union of the sequence $\Sigma_n(X \cup Y)$, defined inductively by $\Sigma_0(X \cup Y) = X \cup Y$ and $\Sigma_{n+1} = \Sigma_n(X\cup Y) \cup \{w \in E^0 : 0 < |s^{-1}(w)| \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(X\cup Y)\}$ (cf. [1, Remark 3.1]). Due to $v \notin \Sigma_0(X \cup Y)$ we can choose the smallest integer $n > 0$ such that $v \in \Sigma_n(X\cup Y)$. Since $s(\mu_1) = v$ we have $r(\mu_1) \in \Sigma_{n-1}(X\cup Y)$. But it is easy to see that $\Sigma_{n-1}(X\cup Y)$ is hereditary. Therefore $v = r(\mu_k) \in \Sigma_{n-1}(X\cup Y)$, a contradiction to our choice of n.

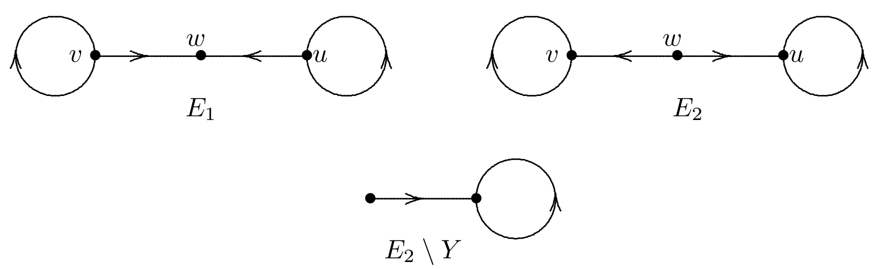
($\Longleftarrow$) Let X and Y be subsets of E^0 satisfying the above conditions. The fact that $I_X \cap I_Y = \{0\}$ follows by an argument similar to the proof of Theorem 4.1. To show that the ideal generated by $I_X \cup I_Y$ equals $C^*(E)$, i.e. this ideal contains all projections P_v, $v \in E^0$, we show that the saturation $\Sigma(X \cup Y)$ equals E^0. Suppose, by way of contradiction, that $\Sigma(X \cup Y) \neq E^0$. We define a partial order $\preceq$ for vertices in $E^0 \setminus \Sigma(X \cup Y)$ so that $v \preceq w$ if and only if there is a path from v to w. Note that since there are no loops through vertices in $E^0 \setminus \Sigma(X \cup Y)$ it follows that $v \preceq w$ implies $w \not\preceq v$. Since the set $E^0 \setminus \Sigma(X \cup Y)$ is finite, there exists a maximal element v_0 with respect to $\preceq$. This vertex v_0 is not a sink by assumption. Also,

v_0 does not emit any edges into $E^0 \setminus \Sigma(X \cup Y)$, and hence it emits all its finitely many edges into $\Sigma(X \cup Y)$. Consequently $v_0 \in \Sigma(X \cup Y)$, a contradiction. It is then clear that $C^*(E) = I_X \oplus I_Y \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$. □

6. Examples

Example 6.1. Let E_1, E_2 be the following finite directed graphs. The algebra $C^*(E_1)$ is indecomposable by Theorem 5.1, because $X = \{v, w\}$ and $Y = \{u, w\}$ are the only hereditary and saturated subsets of E_1^0, and they are not disjoint.

In the graph E_2, $X = \{v\}$ and $Y = \{u\}$ are disjoint hereditary and saturated subsets of E_2^0. Note that $C^*(E_2)$ is decomposable by Theorem 5.1 because $w \in E_2^0 \setminus (X \cup Y)$ has paths to both X and Y, and there is no loop based at w. The decomposition depends on the graph $E_2 \setminus Y = E_2 \setminus X$, and hence $C^*(E_2) \cong (M_2(\mathbb{C}) \otimes C(\mathbb{T})) \oplus (M_2(\mathbb{C}) \otimes C(\mathbb{T}))$.



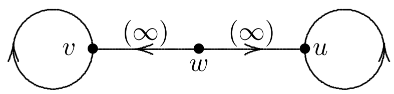
Example 6.2. Here (∞) denotes that there are infinitely many edges from w to u. $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of E^0. From the corresponding subgraph F of E as defined in (2) and (3), we see that $C^*(E)$ is decomposable by Theorem 4.1.

Hence $C^*(E \setminus X) \cong \mathcal{K} \otimes C(\mathbb{T})$ and $C^*(E \setminus Y) \cong M_2(\mathbb{C}) \otimes C(\mathbb{T})$, and both are indecomposable.

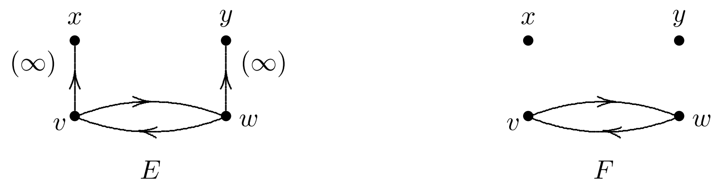
That is, $C^*(E) \cong (\mathcal{K} \otimes C(\mathbb{T})) \oplus (M_2(\mathbb{C}) \otimes C(\mathbb{T}))$.

Example 6.3. The sets $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of E^0. Here the corresponding subgraph F of E is the same as E. Since there are infinitely many path in F from $w \in E^0 \setminus (X \cup Y)$ entering $X \cup Y$, $C^*(E)$ is indecomposable by Theorem 4.1.

$E = F$



Example 6.4. The sets $X = \{x\}$ and $Y = \{y\}$ are disjoint, hereditary and saturated subsets of E^0. Since there exist no paths in F from $v \in E^0 \setminus (X \cup Y)$ (and from $w \in E^0 \setminus (X \cup Y)$) entering $X \cup Y$, $C^*(E)$ is indecomposable by Theorem 4.1.



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