DECOMPOSABILITY OF GRAPH C*-ALGEBRAS

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Abstract. We give conditions on an arbitrary directed graph $E$ for the associated Cuntz-Krieger algebra $C^*(E)$ to be decomposable as a direct sum. We describe the direct summands as certain graph algebras.

0. Introduction

Recently various generalizations of Cuntz-Krieger algebras [2] have attracted a lot of attention. In this article we are concerned with generalized Cuntz-Krieger algebras based on directed graphs [7] and references therein). One of the key advantages in the theory of graph algebras is that a directed graph $E$ is used to conveniently represent generators and relations of the associated graph algebra $C^*(E)$. Thanks to the combined efforts of a number of researchers, it is now known how to read from the graph many of the basic properties and invariants of the algebra.

As with many a mathematical theory, classification of the objects in question presents itself as an important objective. A future classification of graph algebras might be very useful in paving the way for other classifications of more general classes of C*-algebras, similarly to the way the classification of Cuntz-Krieger algebras was the starting point for the Kirchberg-Phillips classification of purely infinite simple algebras. In this context, the class of non-simple purely infinite graph algebras (in the sense of Kirchberg-Rørdam) appears to be of particular interest (cf. [6]). Certainly, the first necessary step towards a classification of non-simple algebras is good understanding of their ideal structure. For graph algebras this has been recently achieved (cf. [1, 5]). These results have already been successfully applied in solutions to some concrete problems in the classification of graph algebras as well as in quantum groups (cf. [1, 8, 3]).

In the present article we consider the question when an ideal of a graph algebra is a direct summand or, in other words, when a graph algebra decomposes as a direct sum. This very natural question turns out to be more complicated than it appears. Obviously, $C^*(E)$ splits as a direct sum when the graph $E$ is disconnected. However, such a splitting also exists for many connected directed graphs. Especially in the context of infinite directed graphs this is a subtle problem requiring careful

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analysis, and even for finite graphs it is not a trivial one. The main result of this paper is a necessary and sufficient condition on an arbitrary infinite graph $E$ that guarantees that the associated graph algebra $C^*(E)$ decomposes as a direct sum. Furthermore, we show that the summands are themselves isomorphic to certain graph algebras.

1. Preliminaries on Graph $C^*$-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph with countably many vertices $E^0$ and edges $E^1$, and range, source functions $r, s : E^1 \to E^0$, respectively. The graph $C^*$-algebra, or simply graph algebra $C^*(E)$ is defined as the universal $C^*$-algebra generated by families of projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$, subject to the following relations:

- (GA1) $P_v^2 = P_v$ for all $v \in E^0$.
- (GA2) $S_e^*S_f = 0$ for $e, f \in E^1$, $e \neq f$.
- (GA3) $S_e^*S_{ef} = P_{r(e)}$ for $e \in E^1$.
- (GA4) $S_eS_{e}^* \leq P_{s(e)}$ for $e \in E^1$.
- (GA5) $P_v = \sum_{e \in E^1 : s(e) = v} S_eS_{e}^*$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

In this case, $\{P_v, S_e : v \in E^0, e \in E^1\}$ is called a Cuntz-Krieger $E$-family. Universality in the definition means that if $Q_v : v \in E^0$ and $T_e : e \in E^1$ are families of projections and partial isometries, respectively, satisfying conditions (GA1–GA5), then there exists a $C^*$-algebra homomorphism from $C^*(E)$ to the $C^*$-algebra generated by $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ such that $P_v \mapsto Q_v$ and $S_e \mapsto T_e$ for $v \in E^0$, $e \in E^1$. It is also equivalent to the existence of a gauge action $\gamma : \mathbb{T} \to \text{Aut}(C^*(E))$, which is characterized by $\gamma_t(S_e) = tS_e$ and $\gamma_t(P_v) = P_v$ for $e \in E^1$, $v \in E^0$, $t \in \mathbb{T}$.

As usual we denote by $E^*$ the set of all finite paths in $E$ (vertices in $E^0$ are identified with paths of length 0), and by $E^\infty$ the set of all infinite paths in $E$. By writing $v \geq w$ when there is a path from $v$ to $w$, we say that a subset $H$ of $E^0$ is hereditary if $v \in H$ and $v \geq w$ imply $w \in H$. A subset $X$ of $E^0$ is said to be saturated if every vertex $v$ that satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \implies r(e) \in X$ itself belongs to $X$. The following definitions come from [1]. For a hereditary and saturated subset $X$ of $E^0$, we denote $X^\text{fin} = \{v \in E^0 \setminus X : |s^{-1}(v)| = \infty, 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus X)| < \infty\}$. If $w \in X^\text{fin}$, $P_{w,X} = \sum_{e \in E^1, s(e) = w, t(e) \notin X} S_eS_{e}^*$ denotes the subprojection of $P_w$.

Our methods in this article are mainly based on the structure of gauge-invariant ideals of graph algebras (see [1] for details). Let $X$ be a hereditary and saturated subset of $E^0$. For $B \subseteq X^\text{fin}$, we denote by $J_{X,B}$ the ideal of $C^*(E)$ generated by $\{P_v : v \in X\}$ and $\{P_w - P_{w,X} : w \in B\}$. When $B = \emptyset$, we write $J_{X,\emptyset} = I_X$, the ideal generated by $\{P_v : v \in X\}$. We have

\begin{equation}
J_{X,B} = \oplus_{\alpha, \eta, \mu, \nu \in E^*, r(\alpha) = r(\eta) = v \in X, r(\mu) = r(\nu) = w \in B} \bigoplus_{S} S_{\alpha}P_{\nu,S_{\eta}}S_{\nu}(P_{w} - P_{w,X})S_{\nu}^*.
\end{equation}

The ideal $J_{X,B}$ is gauge-invariant, i.e. $\gamma_t(J_{X,B}) = J_{X,B}$ for all $t \in \mathbb{T}$. It is now fully known that $J_{X,B}$ and the quotient $C^*(E)/J_{X,B}$ are isomorphic to graph algebras, associated with a directed graph $XE_B$ and a quotient graph $E/X$. To form a directed graph $XE_B$, let $\overline{F}_E(X,B)$ be the collection of all finite paths.
\[ \alpha = (a_1, \ldots, a_n) \] of positive length such that \( s(\alpha) \in E^0 \setminus X \), \( r(\alpha) \in X \cup B \), and \( r(a_j) \notin X \cup B \) for \( j < |\alpha| \). Set \( F_E(X, B) = \overline{F_E}(X, B) \setminus \{ e \in E^1 : s(e) \in B \text{ and } r(e) \in X \} \). We denote by \( \overline{F_E}(X, B) \) another copy of \( F_E(X, B) \), and write \( \overline{\alpha} \in \overline{F_E}(X, B) \) for the copy of \( \alpha \in F_E(X, B) \). Then the graph \( X \overline{E}_B \) is given as follows:

\[
\begin{align*}
(x \overline{E}_B)^0 &= x \overline{E}_B^0 = X \cup B \cup F_E(X, B), \\
(x \overline{E}_B)^1 &= x \overline{E}_B^1 \\
&= \{ e \in E^1 : s(e) \in X \} \cup \{ e \in E^1 : s(e) \in B \text{ and } r(e) \in X \} \cup \overline{F_E}(X, B),
\end{align*}
\]

with \( s(\overline{\alpha}) = \alpha \) and \( r(\overline{\alpha}) = r(\alpha) \) for \( \alpha \in F_E(X, B) \), and the source and range as in \( E \) for the other edges of \( X \overline{E}_B \). When \( B = \emptyset \), we simply denote \( X \overline{E}_B = X \overline{E} \).

The quotient graph \( E/X \) is given by \((E/X)^0 = (E^0 \setminus X) \cup \{ \beta(v) : v \in X^\text{fin}_\infty \}\) and \((E/X)^1 = r^{-1}(E^0 \setminus X) \cup \{ \beta(e) : e \in E^1, r(e) \in X^\text{fin}_\infty \}\), where \( r, s \) are extended by \( s(\beta(e)) = s(e) \) and \( r(\beta(e)) = \beta(r(e)) \). Here \( \beta \) is just a symbol helping to distinguish \( v \) and \( e \) from the extra \( \beta(v) \) and \( \beta(e) \) in \( E/X \), respectively. See \cite{3} Example 1.4 and \cite{1} Example 3.3, which illustrate the graphs \( X \overline{E}_B \) and \( E/X \) respectively.

**Theorem 1.1.** Let \( E \) be a directed graph. Then there is a 1-1 correspondence between the set of gauge-invariant ideals of \( C^*(E) \) and the set of ideals of the form \( J_{X, B} \) where \( X \) is a hereditary and saturated subset of \( E^0 \) and \( B \subseteq X^\text{fin}_\infty \). Moreover,

(i) \cite{3} Lemma 1.5)) the ideal \( J_{X, B} \) is isomorphic to \( C^*(X \overline{E}_B) \), and

(ii) \cite{1} Corollary 3.5) its quotient \( C^*(E)/J_{X, B} \overline{E}_B \) is isomorphic to \( C^*((E/X) \setminus \beta(B)) \).

If \( B = X^\text{fin}_\infty \), then

\[
C^*(E)/J_{X, X^\text{fin}_\infty} \cong C^*(E \setminus X),
\]

where \( E \setminus X = (E^0 \setminus X, r^{-1}(E^0 \setminus X), r, s) \).

2. **Direct sum decompositions of graph \( C^*-algebras**

**Definition 2.1.** Let \( E \) be a directed graph. If there exist two non-zero \( C^*-algebras \) \( A, B \) such that \( C^*(E) \cong A \oplus B \), then \( C^*(E) \) is said to be **decomposable**. Otherwise \( C^*(E) \) is **indecomposable**.

Our aim is to find conditions on \( E \) so that \( C^*(E) \) is decomposable. We denote by \( \text{Prim}(A) \) the set of all primitive ideals in a \( C^*-algebra \) \( A \), equipped with the hull-kernel topology.

**Lemma 2.2.** If \( C^*(E) = A \oplus B \) with non-zero closed ideals \( A \) and \( B \), then \( A \) and \( B \) are gauge-invariant.

**Proof.** Since every ideal in \( C^*(E) \) can be realized as the intersection of a family of primitive ideals and \( \text{Prim}(C^*(E)) \) is the disjoint union of \( \text{Prim}(A) \) and \( \text{Prim}(B) \), it suffices to show that \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are invariant under the gauge action \( \gamma \).

If \( J \) is a primitive ideal of \( C^*(E) \) that is not gauge-invariant, then there exists a homeomorphic imbedding \( \phi : T \to \text{Prim}(C^*(E)) \) such that \( J \) belongs to \( \phi(T) \), by combining Lemma 2.8 and Theorem 2.10 of \cite{5}. Furthermore, \( \phi(T) \) is invariant under the gauge action. Since \( T \) is connected, \( \phi(T) \) is connected in \( \text{Prim}(C^*(E)) \). However, both \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are closed and open, and therefore \( \phi(T) \) is entirely contained either in \( \text{Prim}(A) \) or in \( \text{Prim}(B) \). Consequently, both \( \text{Prim}(A) \) and \( \text{Prim}(B) \) are invariant under the gauge action. \(\Box\)
By Lemma 2.2, require that $A$ and $B$ be graph algebras. However, this turns out to be true by the following theorem.

**Theorem 2.3.** If $C^*(E) = A \oplus B$ with non-zero closed ideals $A$ and $B$, then there exist non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that $A = J_{X,X}^{\infty}$, $B = J_{Y,Y}^{\infty}$, and $(X \cup X^{\infty}) \cap (Y \cup Y^{\infty}) = \emptyset$. Furthermore, $C^*(E)$ is decomposed into the direct sum of two graph algebras as $C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

**Proof.** By Lemma 2.2 $A$ and $B$ are gauge-invariant. Then by Theorem 1.1 there exist two hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that $A = J_{X,C}$ and $B = J_{Y,D}$ where $C \subseteq X^{\infty}$ and $D \subseteq Y^{\infty}$. We have

$$X = \{ v \in E^0 : P_v \in A \}, \quad C = \{ v \in E^0 \setminus X : P_v - P_{v,X} \in A \}, \quad Y = \{ v \in E^0 : P_v \in B \}, \quad D = \{ v \in E^0 \setminus Y : P_v - P_{v,Y} \in B \}.$$

It follows from the decomposability of $C^*(E)$ that $X$ and $Y$ are non-empty and disjoint. Then, by the definitions of $X^{\infty}$ and $Y^{\infty}$, $X^{\infty} \cap Y^{\infty} = \emptyset$.

To show the fact $C = X^{\infty}$, suppose that there is a vertex $v \in X^{\infty} \setminus C$. Then the projection $P_v - P_{v,C} \notin J_{X,C}$. Since $C^*(E) = J_{X,C} \oplus J_{Y,D}$, we must have $P_v - P_{v,X} \in J_{Y,D}$, or $v \in D \subseteq Y^{\infty}$, a contradiction to the fact $X^{\infty} \cap Y^{\infty} = \emptyset$. Thus we must have $C = X^{\infty}$, and a similar argument yields $D = Y^{\infty}$. The fact $(X \cup X^{\infty}) \cap (Y \cup Y^{\infty}) = \emptyset$ then follows easily from the hereditary and saturated properties of $X$ and $Y$.

Moreover, if $C^*(E) = J_{X,X}^{\infty} \oplus J_{Y,Y}^{\infty}$, then

$$J_{X,X}^{\infty} \cong C^*(E)/J_{Y,Y}^{\infty} \cong C^*(E \setminus Y)$$

and

$$J_{Y,Y}^{\infty} \cong C^*(E)/J_{X,X}^{\infty} \cong C^*(E \setminus X),$$

i.e. $C^*(E) \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

**Remark 2.4.** By Theorem 1.1 we know that $J_{X,X}^{\infty} \cong C^*(x E_x^{\infty})$. If $C^*(E)$ is decomposable as in Theorem 2.3 then $C^*(x E_x^{\infty}) \cong J_{X,X}^{\infty} \cong C^*(E \setminus Y)$. In general, the two graphs $x E_x^{\infty}$ and $E \setminus Y$ are different even though their associated graph algebras are isomorphic.

The following observation will be useful later in this article.

**Lemma 2.5.** Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$ such that $C^*(E) = J_{X,X}^{\infty} \oplus J_{Y,Y}^{\infty}$. If $u \in E^0 \setminus (X \cup Y)$, then there exists a path in $E$ from $u$ into $X \cup Y$.

**Proof.** Suppose that there exists no paths in $E$ from $u$ into $X \cup Y$. We have $u \notin X^{\infty} \cup Y^{\infty}$. Since $P_u$ must be in one of the summands, it suffices to show $P_u J_{X,X}^{\infty} = 0 = P_u J_{Y,Y}^{\infty}$ to obtain a contradiction. We use the description of the ideals $J_{X,X}^{\infty}$ and $J_{Y,Y}^{\infty}$ given by the formula (1). If $w \in X$ (or $Y$) and $\alpha, \eta$ are paths in $E$ with $\alpha(0) = \eta(0) = w$, then there must be no paths in $E$ from $u$ to both $\alpha$ and $\eta$ by assumption. Hence $P_u (S_{\alpha} P_w S_{\eta}^*) = 0$. Let $w \in X^{\infty} \text{ and } \mu, \nu$ be paths in $E$ with $\mu(0) = \nu(0) = w$. Again there must be no paths in $E$ from $u$ to both $\mu$ and $\nu$. Hence we get $P_u (S_{\mu} (P_w - P_{w,X}) S_{\nu}^*) = 0$. Similarly we obtain $P_u (S_{\mu} (P_w - P_{w,Y}) S_{\nu}^*) = 0$ for the case of $w \in Y^{\infty}$.
3. Certain representations

We now focus on constructing two representations (Lemmas 3.2 and 3.4) of $C^*(E)$, which will play a crucial role in proving our main result. To this end, it is useful to consider a certain subgraph $F$ of $E$. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. Then the subgraph $F = (F^0, F^1, r, s)$ of $E$ is given by

\begin{equation}
F^0 = E^0,
\end{equation}

\begin{equation}
F^1 = E^1 \setminus \{(e \in E^1: s(e) \in X, r(e) \in X) \cup \{f \in E^1: s(f) \in Y, r(f) \in Y}\}.
\end{equation}

Let $\Omega$ be the collection of all finite paths in $F$ beginning outside $X \cup Y$ and ending inside $X \cup Y$, upon the first entry into $X \cup Y$, i.e.,

$$\Omega := \{\omega = (e_1, \ldots, e_k) \in F^*: s(\omega) \notin X \cup Y, r(\omega) \in X \cup Y, r(e_i) \notin X \cup Y \quad \text{for } i < k \ (k \in \mathbb{N} \setminus \{0\})\},$$

and let $H_\Omega$ be the Hilbert space with an orthonormal basis $\{\xi_\omega : \omega \in \Omega\}$ indexed by $\Omega$. We define projections $\{Q_v : v \in E^0\}$ and partial isometries $\{T_e : e \in E^1\}$ on $H_\Omega$ as follows:

\begin{equation}
Q_v(\xi_\omega) = \begin{cases} 
\xi_\omega & \text{if } v = s(\omega), \\
0 & \text{otherwise},
\end{cases}
\end{equation}

\begin{equation}
T_e(\xi_\omega) = \begin{cases} 
\xi_{(e,\omega)} & \text{if } r(e) = s(\omega), \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Note that for $v \in E^0 \setminus (X \cup Y)$, the projection $Q_v$ has finite rank if and only if there exist finitely many paths in $F$ from $v$ to $X \cup Y$. Indeed, the vertex $v \in E^0 \setminus (X \cup Y)$ corresponds to a projection $Q_v$ that maps onto $\text{span}\{\xi_\omega : s(\omega) = v, \omega \in \Omega\}$.

**Lemma 3.1.** Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. Let $F$ be the subgraph of $E$ as defined in (2) and (3). If for every $v \in E^0 \setminus (X \cup Y)$ there exist finitely many (and at least one) paths in $F$ with source $v$ to $X \cup Y$, then there exist only finitely many edges in $F$ emitting from $v$.

**Proof.** Let $v \in E^0 \setminus (X \cup Y)$ satisfy the above conditions. Suppose that there are infinitely many edges $e$ in $F$ with $s(e) = v$. By passing this into the graph $E$, we see that $v \notin X_{\infty} \cup Y_{\infty}$ in $E$. Hence only two cases are possible.

(i) If all edges $e \in s^{-1}(v)$ satisfy $r(e) \in E^0 \setminus (X \cup Y)$ except finitely many edges, then there are finitely many paths in $F$ from $v$ to $X \cup Y$ by assumption. These would produce infinitely many paths in $F$ from $v$ to $X \cup Y$, a contradiction.

(ii) If there are infinitely many $e \in E^1$ with $s(e) = v, r(e) \in X$ and infinitely many $f \in E^1$ with $s(f) = v, r(f) \in Y$, then all these edges still remain in the graph $F$ to yield infinitely many paths in $F$ from $v$ to $X \cup Y$, a contradiction. \hfill $\square$

We now examine whether the family $\{Q_v, T_e : v \in E^0, e \in E^1\}$ satisfies the Cuntz-Krieger relations for $E$. Conditions (GA1) and (GA2) are obvious. Condition (GA3) follows from the fact that $T_e^* T_e(\xi_\omega) = Q_v(\xi_\omega)$ if and only if $s(\omega) = v = r(e)$. Similarly, condition (GA4) is fulfilled. Unfortunately this family may not satisfy (GA5). Indeed, suppose $v \in E^0$ has the property $0 < |s^{-1}(v)| < \infty$ in $E$. For any $e \in s^{-1}(v)$, $T_e^* T_e$ is a projection onto $\text{span}\{\xi_\omega : \omega = (e, \omega')\}$ for some $\omega' \in
\(\Omega\) and \(r(e) = s(\omega')\), while the range of the projection \(Q_v\) includes the vector \(\xi_{(f,\omega')}\) for some \(\omega' \in \Omega\) and \(r(f) = s(\omega')\), whenever there is an edge \(f \in E^1\) such that \(s(f) = s(e) = v\) and \(e \neq f\). Bearing this in mind, let us define \(R_v\) as the projection onto \(\overline{\operatorname{span}}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X \cup Y\}\). Then \(R_v\) is a projection of finite rank in \(\mathcal{B}(\mathcal{H}_\Omega)\) by the construction, and

\[
Q_v = \sum_{e \in E^1, s(e) = v, r(e) \in X \cup Y} T_e T_e^* + R_v.
\]

By hint of this, we pass the generating family into \(\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\) so that the image forms a Cuntz-Krieger \(E\)-family. To this end let \(\pi : \mathcal{B}(\mathcal{H}_\Omega) \rightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\) be the canonical quotient map. Then \(\{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\}\) forms a Cuntz-Krieger \(E\)-family in \(\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\). Indeed, it is enough to check \((GA5)\). So let \(v \in E^0\) be \(0 < |s^{-1}(v)| < \infty\). Since the rank of \(R_v\) onto \(\overline{\operatorname{span}}\{\xi_e \mid e \in E^1, s(e) = v, r(e) \in X \cup Y\}\) is finite in \(\mathcal{B}(\mathcal{H}_\Omega)\), the formula (6) yields \(\pi(Q_v) = \sum_{e \in E^1, s(e) = v} \pi(T_e)\pi(T_e)^*\) in \(\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\), and hence \((GA5)\) holds.

**Lemma 3.2.** Let \(X\) and \(Y\) be non-empty, disjoint, hereditary and saturated subsets of \(E^0\). Then there is a \(*\)-homomorphism

\[
\rho : C^*(E) \rightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega),
\]

satisfying \(\rho(S_{\alpha}) = \pi(T_{\alpha}), \rho(P_{\alpha}) = \pi(Q_{\alpha}), e \in E^1, v \in E^0, \) and \(\rho(J_{X,X^\infty}) = \rho(J_{Y,Y^\infty}) = \{0\}\).

**Proof.** Since \(\{\pi(Q_v), \pi(T_e) : v \in E^0, e \in E^1\}\) forms a Cuntz-Krieger \(E\)-family in \(\mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\) by the above argument, universality of \(C^*(E)\) implies that there exists a \(*\)-homomorphism \(\rho : C^*(E) \rightarrow \mathcal{B}(\mathcal{H}_\Omega)/\mathcal{K}(\mathcal{H}_\Omega)\) such that \(\rho(S_{\alpha}) = \pi(T_{\alpha}), \rho(P_{\alpha}) = \pi(Q_{\alpha}), e \in E^1, v \in E^0, \) and \(\rho(J_{X,X^\infty}) = \rho(J_{Y,Y^\infty}) = \{0\}\).

For the remaining part \(\rho(J_{X,X^\infty}) = \{0\}\), we check on the generators \(P_v\) and \(P_w - P_{w,X}\) of \(J_{X,X^\infty}\), when \(v \in X\) and \(w \in X^\infty\). If \(v \in X\), then \(\rho(P_v) = \pi(Q_v) = 0\) because there is no \(\alpha \in \Omega\) satisfying \(s(\alpha) = v\). Let \(w \in X^\infty\). If \(\alpha = (e_1, \ldots, e_k) \in \Omega\) with \(s(\alpha) = w\), then \(r(e_i) \notin X\). Hence we have \(\pi(Q_w)\xi_{\alpha} = \pi(Q_{w,X})\xi_{\alpha}\) for all \(\alpha \in \Omega\), and thus \(\rho(P_w - P_{w,X}) = \pi(Q_w - Q_{w,X}) = 0\). The similar argument gives \(\rho(J_{Y,Y^\infty}) = \{0\}\). \([\square]\)

Now let \(v \in E^0 \setminus (X \cup Y)\) and suppose there is an infinite path, say \(\alpha = (a_1, a_2, a_3, \ldots)\) in \(E\) such that \(s(\alpha) = v\). We define a corresponding representation of \(C^*(E)\). Let \(\Lambda\) be the collection of all infinite paths in \(E\) that are shift-tail equivalent to \(\alpha\) (recall that an infinite path \((e_1, e_2, e_3, \ldots)\) is shift-tail equivalent to \(\alpha\) if there exist \(k, m \in \mathbb{N}\) such that \(e_{k+i} = a_{m+i}\) for all \(i \in \mathbb{N}\), i.e.,

\[
\Lambda := \{ \beta = (b_1, b_2, b_3, \ldots) \in E^\infty : \exists k, m \in \mathbb{N} \text{ s.t. } b_{k+i} = a_{m+i} \text{ for all } i \in \mathbb{N} \},
\]

and let \(\mathcal{H}_{\Lambda}\) be the Hilbert space with an orthonormal basis \(\{\xi_\beta : \beta \in \Lambda\}\) indexed by \(\Lambda\). For \(w \in E^0\) and \(e \in E^1\) we define a projection \(U_w\) and a partial isometry \(D_e\) on \(\mathcal{H}_{\Lambda}\) as follows:

\[
U_w(\xi_\beta) = \begin{cases} 
\xi_\beta & \text{if } w = s(\beta), \\
0 & \text{otherwise},
\end{cases}
\]

\[
D_e(\xi_\beta) = \begin{cases} 
\xi_{(e,b_1,b_2,\ldots)} & \text{if } r(e) = s(\beta), \text{ where } \beta = (b_1, b_2, \ldots) \in \Lambda, \\
0 & \text{otherwise}.
\end{cases}
\]
Lemma 3.3. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$. If a vertex $v \in E^0 \setminus (X \cup Y)$ has an infinite path $\alpha = (a_1, a_2, a_3, \cdots)$ in $E$ with $s(\alpha) = v$, then there is a $*$-homomorphism

$$\varrho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Lambda)$$

such that $\varrho(P_w) = U_w$ and $\varrho(S_v) = D_v$ for all $w \in E^0$, $e \in E^1$. Furthermore, if $\alpha$ never enters $X \cup Y$, then $\varrho(J_{X,Y}^{\infty}) = \varrho(J_{X,Y}^{\infty}) = \{0\}$.

Proof. It is easy to show that the family $\{U_w, D_v : w \in E^0, e \in E^1\}$ as defined in (7) and (8) is a Cuntz-Krieger $E$-family, and thus universality of $C^*(E)$ implies that there is a $*$-homomorphism $\varrho : C^*(E) \to \mathcal{B}(\mathcal{H}_\Lambda)$ such that $\varrho(P_w) = U_w$ and $\varrho(S_v) = D_v$ for all $w \in E^0$, $e \in E^1$.

Now suppose that $\alpha$ never enters $X \cup Y$, that is, $r(a_j) \not\in X \cup Y$ for all $j$. It suffices to check that $\varrho$ kills all the generating projections of the ideals $J_{X,X}^{\infty}$ and $J_{Y,Y}^{\infty}$. First we show that $\varrho$ kills all generating projections $P_w$ and $P_u - P_{u,X}$ of $J_{X,X}^{\infty}$ for $w \in X$ and $u \in X^{\infty}$. If $w \in X$ satisfies $\varrho(P_w) \neq 0$, then there exists an infinite path $\beta = (b_1, b_2, \cdots)$ such that $s(\beta) = w$ and $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$. By the hereditary property of $X$, $r(b_k) = r(a_m) \in X$, a contradiction. Now let $u \in X^{\infty}$ satisfy $\varrho(P_u,X) \neq 0$. Since $P_{u,X} = P_u - \sum_{s(c) = u, r(c) \not\in X} S_c E_c$, there exists an infinite path $\beta = (b_1, b_2, \cdots)$ such that $s(\beta) = u$, $b_{k+i} = a_{m+i}$ for some $k, m \in \mathbb{N}$, and $r(b_1) \in X$. This is a contradiction, since the hereditary property of $X$ would then imply that $\alpha$ enters $X$.

By replacing $X$ with $Y$ in the above argument we can easily show that $\varrho$ kills the generating projections $P_w$ and $P_u - P_{u,Y}$ of $J_{Y,Y}^{\infty}$ for $w \in Y$ and $u \in Y^{\infty}$.

Lemma 3.4. Let $X$ and $Y$ be non-empty, disjoint, hereditary and saturated subsets of $E^0$ such that $C^*(E) = J_{X,X}^{\infty} \oplus J_{Y,Y}^{\infty}$, and let $F$ be the subgraph of $E$ as defined in (2) and (3). If $v \in E^0 \setminus (X \cup Y)$ and there are no paths in $F$ from $v$ to $X \cup Y$, then there exists a representation $\varrho$ of $C^*(E)$ such that $\varrho(J_{X,X}^{\infty}) = \varrho(J_{Y,Y}^{\infty}) = \{0\}$ but $\varrho(P_v) \neq 0$.

Proof. Let $v \in E^0 \setminus (X \cup Y)$. Assume that there are no paths in $F$ from $v$ into $X \cup Y$. We first show that there exists an infinite path in $E$ which begins at $v$ and never enters $X \cup Y$. By Lemma 2.3 there exists a path in $E$ from $v$ to $X \cup Y$. Suppose that it enters, say $X$. Let $w$ be the last vertex on this path which is not in $X$. Because of our assumption on $v$, the vertex $w$ must satisfy the following:

(i) $w$ belongs to $X^{\infty}$,
(ii) there is no edge in $E$ which begins at $w$ and ends inside $Y$.

Condition (i) is obvious. Indeed, if $w \not\in X^{\infty}$, then it would yield a path in $F$ from $v$ to $X$, a contradiction. Condition (ii) also holds obviously by the same reason as the case of (i). (Note that if the path enters $Y$, then the conditions become (i) $w \in Y^{\infty}$ and (ii) there is no edge in $E$ which begins at $w$ and ends inside $X$.) This means that there are finitely many edges $e \in E^1$ such that $s(e) = w$ and $r(e) \not\in X \cup Y$. Then, by applying Lemma 2.3 again to the vertex $r(e)$, we see that there exists a path in $E$ from $r(e)$ to $X \cup Y$. Now the same argument with $r(e)$ as before yields a vertex $u$ satisfying the conditions (i) and (ii). Continuously we can produce an infinite path in $E$ from $v$ that never enters $X \cup Y$. Call it $\alpha = (a_1, a_2, a_3, \cdots)$.

Now consider the corresponding representation $\varrho$ on the Hilbert space $\mathcal{H}_\Lambda$ as defined in (7) and (8). Since the path $\alpha$ begins at $v$, it follows that $\varrho(P_\alpha) \neq 0$. 

Moreover, since \( \alpha \) never enters \( X \cup Y \), \( \rho \) is zero on both \( J_{X, X_{\infty}} \) and \( J_{Y, Y_{\infty}} \) by Lemma 3.3.

4. THE MAIN RESULTS

We now have all the ingredients to prove our main theorem.

**Theorem 4.1.** Let \( E \) be a directed graph. Then \( C^*(E) \) is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets \( X \) and \( Y \) of \( E^0 \) such that for every \( v \in E^0 \setminus (X \cup Y) \) there exist finitely many (and at least one) paths in \( F \) from \( v \) to \( X \cup Y \), where \( F \) is the subgraph of \( E \) as defined in (2) and (3).

If this is the case, \( C^*(E) \) is decomposed into two graph algebras as \( C^*(E \setminus Y) \oplus C^*(E \setminus X) \).

*Proof. (\( \Rightarrow \))* Let \( C^*(E) \) be decomposable. Then by Theorem 2.3, there exist two non-empty, disjoint, hereditary and saturated subsets \( X \) and \( Y \) of \( E^0 \) such that \( C^*(E) = J_{X, X_{\infty}} \oplus J_{Y, Y_{\infty}} \) and \( (X \cup X_{\infty}) \cap (Y \cup Y_{\infty}) = \emptyset \). We now form the subgraph \( F \) of \( E \) as defined in (2) and (3), and let \( v \in E^0 \setminus (X \cup Y) \). By Lemma 3.2, there is a representation \( \rho \) of \( C^*(E) \) on the Hilbert space \( \mathcal{H}_A \) such that \( \rho(J_{X, X_{\infty}}) = \rho(J_{Y, Y_{\infty}}) = \{0\} \).

Suppose that there are infinitely many paths in \( F \) from \( v \) into \( X \cup Y \). Then the projection \( Q_v \), as defined in (4), has infinite rank in \( B(\mathcal{H}_A) \), and hence \( \pi(Q_v) \neq 0 \) in \( B(\mathcal{H}_A)/K(\mathcal{H}_A) \). Since \( \rho(P_v) = \pi(Q_v) \), it follows from Lemma 3.2 that \( P_v \neq 0 \) in \( C^*(E)/(J_{X, X_{\infty}} + J_{Y, Y_{\infty}}) \), or \( P_v \notin J_{X, X_{\infty}} \oplus J_{Y, Y_{\infty}} = C^*(E) \), a contradiction. Hence there exist only finitely many paths in \( F \) from \( v \) to \( X \cup Y \).

Now suppose that there are no paths in \( F \) from \( v \) to \( X \cup Y \). Then by Lemma 3.4, there exists a representation \( \rho \) of \( C^*(E) \) on the Hilbert space \( \mathcal{H}_A \) such that \( \rho(J_{X, X_{\infty}}) = \rho(J_{Y, Y_{\infty}}) = \{0\} \), but \( \rho(P_v) \neq 0 \). This means \( P_v \notin J_{X, X_{\infty}} + J_{Y, Y_{\infty}} = C^*(E) \), a contradiction. Hence there must exist at least one path in \( F \) from \( v \) to \( X \cup Y \).

*Proof. (\( \Leftarrow \)) Let \( X \) and \( Y \) be non-empty subsets of \( E^0 \) satisfying the above conditions. We show that \( J_{X, X_{\infty}} \cap J_{Y, Y_{\infty}} = \{0\} \) and \( J_{X, X_{\infty}} \cup J_{Y, Y_{\infty}} \) generates \( C^*(E) \).

**Step 1.** For \( J_{X, X_{\infty}} \cap J_{Y, Y_{\infty}} = \{0\} \), it is enough to check the following cases by virtue of (1).

(i) We show \( (S_\alpha P_v S_\beta)(S_\mu P_v S_\nu) = 0 \) for paths \( \alpha, \beta, \mu, \nu \) such that \( r(\alpha) = r(\beta) = w \in X \), \( r(\mu) = r(\nu) = v \in X \). Suppose that either \( \beta \) is an initial subpath of \( \mu \) or \( \mu \) is an initial subpath of \( \beta \). This implies that either there is a path from \( Y \) to \( X \) or vice versa, a contradiction since \( X \) and \( Y \) are hereditary and disjoint. Thus \( S_\beta S_\mu = 0 \).

(ii) We show \( (S_{\alpha} P_w S_\beta)(S_{\mu}(P_v - P_{v, X})S_{\nu}) = 0 \) for paths \( \alpha, \beta, \mu, \nu \) such that \( r(\alpha) = r(\beta) = w \in Y \) and \( r(\mu) = r(\nu) = v \in X_{\infty} \). Note that \( \beta \) cannot be an initial subpath of \( \mu \). Otherwise this creates a path from \( w \in Y \) to \( v \), and the hereditary property of \( Y \) implies \( v \in X_{\infty} \cap Y \), but now \( v \in X_{\infty} \cap Y \) implies \( Y \cap X_{\infty} = 0 \) since \( X \) and \( Y \) are hereditary. This contradicts the fact \( X_{\infty} \cap Y = \emptyset \). Next, assume that \( \mu \) is an initial subpath of \( \beta \). We write \( \beta = (\mu, \beta_1) \) and \( \beta_1 = (e, e_1, \cdots, e_k) \). Note that \( r(e) \notin X \) because \( X \) is hereditary and \( r(\beta) = w \in Y \). Thus \( S^*_{\epsilon_k} P_{v, X} = S^*_{\epsilon_k} \sum f \in E, r(f) = w S_f S^*_f = S^*_{\epsilon_k} \). This implies \( S^*_{\epsilon_k}(P_v - P_{v, X}) = S^*_{\epsilon_k} P_v - S^*_{\epsilon_k} = 0 \), and hence

\[
S^*_{\beta_1}(P_v - P_{v, X}) = S^*_{\beta_1}(P_v - P_{v, X}) = S^*_{\epsilon_k} \cdots S^*_{\epsilon_1} S^*_{\epsilon_1}(P_v - P_{v, X}) = 0.
\]
(iii) The same argument as (ii) gives \((S_n(P_w - P_w, Y)S_{\beta, Y}^*)(S_{\mu, P_v, S_{\mu, Y}^*}) = 0\) if \(\alpha, \beta, \mu, \nu\) are paths such that \(r(\alpha) = r(\beta) = w \in Y_{\infty}^{\text{fin}}\) and \(r(\mu) = r(\nu) = v \in X\).

(iv) We show \((S_n(P_w - P_w, Y)S_{\beta, Y}^*)(S_{\mu, P_v, S_{\mu, Y}^*}) = 0\) for paths \(\alpha, \beta, \mu, \nu\) such that \(r(\alpha) = r(\beta) = w \in Y_{\infty}^{\text{fin}}\) and \(r(\mu) = r(\nu) = v \in X_{\infty}^{\text{fin}}\). It is enough to show this when there would be a path from \(v\) to \(w\) or vice versa. If \(v \geq w\), then write \(\beta = (\mu, \beta_1)\) with a subpath \(\beta_1\) (if \(w \geq v\), then write \(\mu = (\beta, \mu_1)\) with a subpath \(\mu_1\)). The same argument as (ii) yields the result.

**Step 2.** To show that the ideal \(J\) generated by \(J_{X, X_{\infty}^{\text{fin}}} \cup J_{Y, Y_{\infty}^{\text{fin}}}\) equals \(C^*(E)\), it suffices to observe that \(J\) contains all projections \(P_v, v \in E^0\). Clearly, we only need to examine vertices \(v \in E^0 \setminus (X \cup Y)\). To this end, we use the induction on the number of finite paths in \(F\) from \(v \in E^0 \setminus (X \cup Y)\) to \(X \cup Y\).

(i) Suppose that there is only one path \((e_1, \ldots, e_k)\) in \(F\) such that \(v = s(e_1), r(e_i) \notin X \cup Y\) for \(i < k\), and \(r(e_k) \in X \cup Y\). We show by reverse induction on \(i = 1, \ldots, k\) that \(P_{s(e_i)}\) belongs to the ideal \(J\) generated by \(J_{X, X_{\infty}^{\text{fin}}} \cup J_{Y, Y_{\infty}^{\text{fin}}}\). (To simplify notation, we agree that \(e_{k+1}\) is the vertex \(r(e_k)\).) Indeed, suppose that \(P_{s(e_{i+1})} \in J\). If \(s(e_i)\) does not belong to \(X_{\infty}^{\text{fin}} \cup Y_{\infty}^{\text{fin}}\), then \(e_i\) is the only edge in \(E\) emitted by \(s(e_i)\), and hence \(r(e_i) = s(e_{i+1})\). Now \(S_{e_i} = S_{e_i}, P_{r(e_i)} = S_{e_i} P_{s(e_i)} \in J\), and it follows that \(P_{s(e_i)} = S_{e_i} S_{e_i}^* \in J\), and we are done. If \(s(e_i) \in X_{\infty}^{\text{fin}}\), then \(e_i\) is the only edge emitted by \(s(e_i)\) whose range lies outside \(X\). For otherwise there existed more than one path in \(F\) from \(v\) to \(X \cup Y\). Thus \(S_{e_i} S_{e_i}^* \in J\) by the previous observation. Hence we have \(P_{s(e_i)} = (P_{s(e_i)} - P_{s(e_i), X}) + S_{e_i} S_{e_i}^* \in J\), and the claim follows.

(ii) Now suppose that \(P_w \in J\) for all vertices \(w \in E^0 \setminus (X \cup Y)\) for which there are at most \(n\) paths in \(F\) from \(w\) to \(X \cup Y\). Let \(v\) be a vertex in \(E^0 \setminus (X \cup Y)\) with \(n + 1\) paths in \(F\) from \(v\) to \(X \cup Y\). Let \((e_1, \ldots, e_k)\) be one of these paths. Since there are at least two paths in \(F\) from \(v\) to \(X \cup Y\), it follows that there exists an index \(i\) such that \(s(e_i)\) emits at least two edges in \(F\). Let \(m\) be the smallest such an index. Then for every edge \(f\) in \(F^1\) with \(s(f) = s(e_m)\) we have \(P_{r(f)} \in J\), by the inductive hypothesis. Indeed, for each such \(f\) the number of paths in \(F\) from \(r(f)\) to \(X \cup Y\) is not greater than \(n\).

If \(s(e_m)\) does not belong to \(X_{\infty}^{\text{fin}} \cup Y_{\infty}^{\text{fin}}\), then every edge emitted by \(s(e_m)\) is in \(F\), and there are only finitely many such edges by Lemma 5.1. Thus \(P_{s(e_m)} = \sum_{s(f) = s(e_m)} S_f S_f^*\) belongs to \(J\). If \(s(e_m) \in X_{\infty}^{\text{fin}}\), then every edge emitted by \(s(e_m)\) with range outside \(X\) is in \(F\). Thus in this case we again see that \(P_{s(e_m)} = (P_{s(e_m)} - P_{s(e_m), X}) + \sum_{s(f) = s(e_m)} S_f S_f^*\) belongs to \(J\). In either case, by the choice of \(m\), there exists a unique path \((e_1, \ldots, e_{m-1})\) in \(F\) from \(v = s(e_1)\) to \(s(e_m)\). Thus, a reasoning as in part (i) above shows that \(P_{s(e_j)} \in J\) for all \(j = 1, \ldots, m\). Consequently \(P_v \in J\). This ends the proof of the inductive step and the proof of Step 2.

Finally, if \(C^*(E) = J_{X, X_{\infty}^{\text{fin}}} \oplus J_{Y, Y_{\infty}^{\text{fin}}}\), then it is clear from Theorem 4.1 that \(J_{X, X_{\infty}^{\text{fin}}} \cong C^*(E)/J_{Y, Y_{\infty}^{\text{fin}}} \cong C^*(E \setminus Y)\) and \(J_{Y, Y_{\infty}^{\text{fin}}} \cong C^*(E)/J_{X, X_{\infty}^{\text{fin}}} \cong C^*(E \setminus X)\).
Corollary 4.2. Let $E$ be a directed graph with finitely many vertices. Then there exist finitely many subgraphs $F_1, \ldots, F_n$ of $E$ such that $C^*(E) \cong C^*(F_1) \oplus \cdots \oplus C^*(F_n)$ and each $C^*(F_k)$ is indecomposable.

Proof. The decomposition of $C^*(E)$ depends on the existence of two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ satisfying conditions in Theorem 4.1. If no such subsets $X$ and $Y$ exist, then the algebra $C^*(E)$ is itself indecomposable. Otherwise, $C^*(E) \cong C^*(F_1) \oplus C^*(F_2)$ for some graphs $F_1$ and $F_2$. If both $C^*(F_1)$ and $C^*(F_2)$ are indecomposable, we are done. Otherwise, find a direct sum decomposition of the algebras $C^*(F_i)$ whenever they are decomposable. Continuing this process yields the result after a finite number of steps. \hfill \Box

5. $C^*$-Algebras of Finite Graphs

This section is devoted to giving a more feasible criterion for $C^*$-algebras of finite directed graphs, even though it can be covered by Theorem 4.1.

Theorem 5.1. Let $E$ be a finite directed graph. Then $C^*(E)$ is decomposable if and only if there exist two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$ such that for every vertex $v \in E^0 \setminus (X \cup Y)$ (i) there exist paths from $v$ to both $X$ and $Y$, and (ii) there is no loop passing through $v$. If this is the case, $C^*(E) = I_X \oplus I_Y \cong C^*(E \setminus Y) \oplus C^*(E \setminus X)$.

Proof. ($\Rightarrow$) By taking $B = \emptyset$ in Theorem 2.3 we see that $C^*(E) = I_X \oplus I_Y$ with two non-empty, disjoint, hereditary and saturated subsets $X$ and $Y$ of $E^0$.

(i) Consider a vertex $v \in E^0 \setminus (X \cup Y)$ and suppose for a moment that there is no path in $E$ from $v$ to a vertex in $Y$. Since $I_Y = \sum_{\alpha \beta \in E^*, w \in Y} \{ r(\alpha) = r(\beta) = w \}$, it is clear that $P_v I_Y = \{ 0 \}$. Since $C^*(E) = I_X \oplus I_Y$ we must have $P_v \in I_X$ and consequently $v \in X$, a contradiction. Hence there is a path from $v$ to $X$. Similarly, there must be a path from $v$ to $X$.

(ii) Suppose $v \in E^0 \setminus (X \cup Y)$ and there is a loop $\mu = (\mu_1, \ldots, \mu_k)$ in $E$ through $v$. Since the ideal $I_{X \cup Y}$, generated by $I_X$ and $I_Y$, equals $C^*(E)$ we have $I_{X \cup Y} = I_{E^0}$. But $X \cup Y$ is a hereditary set and $I_{X \cup Y}$ equals $I_{\Sigma (X \cup Y)}$, where $\Sigma (X \cup Y)$ denotes the saturation of $X \cup Y$. Since a gauge-invariant ideal determines the corresponding hereditary and saturated subset of $E^0$, we must have $\Sigma (X \cup Y) = E^0$ and, in particular, $v$ belongs to the saturation of $X \cup Y$. Now $\Sigma (X \cup Y) = E^0$ is the union of the sequence $\Sigma_n(X \cup Y)$, defined inductively by $\Sigma_0(X \cup Y) = X \cup Y$ and $\Sigma_{n+1} = \Sigma_n(X \cup Y) \cup \{ w \in E^0 : 0 < |s^{-1}(w)| \text{ and } s(e) = w \text{ imply } r(e) \in \Sigma_n(X \cup Y) \}$ (cf. [11 Remark 3.1]). Due to $v \notin \Sigma_0(X \cup Y)$ we can choose the smallest integer $n > 0$ such that $v \in \Sigma_n(X \cup Y)$. Since $s(\mu_1) = v$ we have $r(\mu_1) \in \Sigma_{n-1}(X \cup Y)$. But it is easy to see that $\Sigma_{n-1}(X \cup Y)$ is hereditary. Therefore $v = r(\mu_k) \in \Sigma_{n-1}(X \cup Y)$, a contradiction to our choice of $n$.

($\Leftarrow$) Let $X$ and $Y$ be subsets of $E^0$ satisfying the above conditions. The fact that $I_X \cap I_Y = \{ 0 \}$ follows by an argument similar to the proof of Theorem 4.1. To show that the ideal generated by $I_X \cup I_Y$ equals $C^*(E)$, i.e. this ideal contains all projections $P_v$, $v \in E^0$, we show that the saturation $\Sigma (X \cup Y)$ equals $E^0$. Suppose, by way of contradiction, that $\Sigma (X \cup Y) \neq E^0$. We define a partial order $\preceq$ for vertices in $E^0 \setminus (X \cup Y)$ so that $v \preceq w$ if and only if there is a path from $v$ to $w$. Note that since there are no loops through vertices in $E^0 \setminus (X \cup Y)$ it follows that $v \preceq w$ implies $w \not\preceq v$. Since the set $E^0 \setminus (X \cup Y)$ is finite, there exists a maximal element $v_0$ with respect to $\preceq$. This vertex $v_0$ is not a sink by assumption. Also,
Example 6.1. Let $E_1, E_2$ be the following finite directed graphs. The algebra $C^*(E_1)$ is indecomposable by Theorem 5.1 because $X = \{v, w\}$ and $Y = \{u, w\}$ are the only hereditary and saturated subsets of $E_1^0$, and they are not disjoint.

In the graph $E_2$, $X = \{v\}$ and $Y = \{u\}$ are disjoint hereditary and saturated subsets of $E_2^0$. Note that $C^*(E_2)$ is decomposable by Theorem 5.1 because $w \in E_2^0 \setminus (X \cup Y)$ has paths to both $X$ and $Y$, and there is no loop based at $w$. The decomposition depends on the graph $E_2 \setminus Y = E_2 \setminus X$, and hence $C^*(E_2) \cong (M_2(\mathbb{C}) \otimes C(T)) \oplus (M_2(\mathbb{C}) \otimes C(T))$.

Example 6.2. Here $(\infty)$ denotes that there are infinitely many edges from $w$ to $u$. $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of $E^0$. From the corresponding subgraph $F$ of $E$ as defined in (2) and (3), we see that $C^*(E)$ is decomposable by Theorem 4.1.

Hence $C^*(E \setminus X) \cong K \otimes C(T)$ and $C^*(E \setminus Y) \cong M_2(\mathbb{C}) \otimes C(T)$, and both are indecomposable.

Example 6.3. The sets $X = \{v\}$ and $Y = \{u\}$ are disjoint, hereditary and saturated subsets of $E^0$. Here the corresponding subgraph $F$ of $E$ is the same as $E$. Since there are infinitely many path in $F$ from $w \in E^0 \setminus (X \cup Y)$ entering $X \cup Y$, $C^*(E)$ is indecomposable by Theorem 4.1.

$E = F$
Example 6.4. The sets \( X = \{x\} \) and \( Y = \{y\} \) are disjoint, hereditary and saturated subsets of \( E^0 \). Since there exist no paths in \( F \) from \( v \in E^0 \setminus (X \cup Y) \) (and from \( w \in E^0 \setminus (X \cup Y) \)) entering \( X \cup Y \), \( C^*(E) \) is indecomposable by Theorem 4.1.

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