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A NEW GEOMETRIC PROOF OF JUNG'S THEOREM ON FACTORISATION OF AUTOMORPHISMS OF $\mathbb{C}^2$

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Abstract. Building up on the classical theory of algebraic surfaces and their birational transformations we prove Jung's theorem on factorisation of automorphisms of $\mathbb{C}^2$ reducing it to a simple combinatorial argument.

Let $Aut(\mathbb{C}^2)$ be the group of algebraic automorphisms of $\mathbb{C}^2$. Let $\mathbb{G}$ be the subgroup of automorphisms fixing the origin and whose differential at it is the identity. In this article we fix a coordinate system (x, y) of $\mathbb{C}^2$. We say that $\phi \in Aut(\mathbb{C}^2)$ is triangular if it is of the form $\phi(x, y) = (x, y + \sum_{i=2}^{n} a_i x^i)$.

Theorem 1 (Jung [1]). *The group $Aut(\mathbb{C}^2)$ is generated by affine and triangular automorphisms.*

Nagata gave another proof of this result, based also on geometric ideas (see [2]). Yoshihara applied techniques similar to ours in [3]. Our proof uses factorisation of birational maps of surfaces as compositions of blowing ups and blowing downs to reduce the proof to a simple combinatorial argument. Our point of view raises the question of whether the present knowledge on birational geometry of threefolds can help to find generators of the automorphism group of $\mathbb{C}^3$. Connected with this is the question of whether the famous Nagata's automorphism can be factorised in affine and De Jonquieres automorphisms (see [2]).

Consider $\mathbb{P}^2$ together with a projective reference (X_0, Y_0, Z_0). We embed $\mathbb{C}^2$ into $\mathbb{P}^2$ declaring that the image of the embedding is the open subset U_{Z_0} defined by $Z_0 \neq 0$ and that $(x, y) = (X_0/Z_0, Y_0/Z_0)$. This allows us to view any automorphism of $\mathbb{C}^2$ as a birational transformation of $\mathbb{P}^2$. Consider $L := \mathbb{P}^2 \setminus \mathbb{C}^2$; by *blowing up process* we will mean a composition of blowing ups of points infinitely near L. Consider $\phi \in Aut(\mathbb{C}^2)$, let $\pi : X \to \mathbb{P}^2$ be a blowing up process. The map $\psi := \phi \circ \pi$ takes the points of $\pi^* L$ in which it is defined into L. A component E of $\pi^* L$ is called *dicritical* if $\psi_{|E} : E \to L$ is dominant.

Lemma 1. *Let ϕ, π and ψ be as above. If ψ has no indetermination, there is a unique dicritical component of $\pi^* L$. If ψ has indetermination, then it has a unique indetermination point and no component of $\pi^* L$ is dicritical.*

Proof. Let $\sigma : X' \to X$ be the minimal composition of blowing ups, such that $\psi' := \psi \circ \sigma$ has no indetermination. Define $\pi' := \pi \circ \sigma$. If $x \in X$ is an indetermination point

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of ψ, then there exists a component of $\sigma^{-1}(x)$ that is dicritical for ψ'. Otherwise, using Riemann's extension theorem, we could extend ψ to x.

The image by ψ' of the nondicritical components of π' is a finite set Z included in L. The restriction φ of ψ' to $X' \setminus {\psi'}^{-1}(Z)$ is a finite mapping of degree 1 (its restriction to $\mathbb{C}^2 \subset \mathbb{P}^2 \setminus Z$ is the automorphism ϕ). The cardinality of $\varphi^{-1}(z)$, when $z \in L \setminus Z$, is at least the number of dicritical components of ψ', which is at least the sum of the number of dicritical components plus the number of indeterminacy points of ψ. As the degree of φ is 1, our lemma follows. $\square$

We associate a graph to any blowing up process π as follows: draw a vertex for each component of π^*L, weighted with its self-intersection; connect two vertices if and only if the divisors that they represent meet. We denote by $\mathcal{A}_n$ the graph

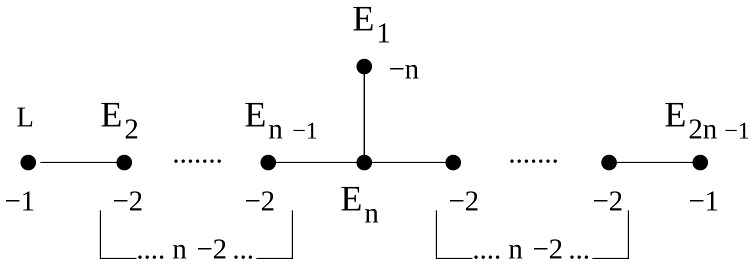

FIGURE 1.

Let $\pi : X \to \mathbb{P}^2$ be any blowing up process with graph $\mathcal{A}_n$, we associate to it an automorphism of $\mathbb{P}^2$; let $L, E_1, ..., E_{2n-1}$ be the components of π^*L by order of appearance. By Castelnuovo's contractibility criterion we can find morphisms of smooth algebraic surfaces that successively contract $L, E_2, ..., E_{2n-2}, E_1$. Let $\pi' : X \to Y$ be their composition and $\psi := \pi' \circ \pi^{-1}$. The divisor E_{2n-1} has self-intersection 1 in Y. As we contract the same number of curves of X to get $\mathbb{P}^2$ as to get Y, the Euler characteristics of $\mathbb{P}^2$ and Y are equal. As $\mathbb{P}^2$ is the only complete rational smooth surface with Euler characteristic 3 we deduce that Y must be isomorphic to $\mathbb{P}^2$. Let (X_0', Y_0', Z_0') be the unique projective coordinate system of Y such that the divisor E_{2n+1} is defined by $Z_0' = 0$, and, if we consider the affine charts $(U_{Z_0}, (X_0/Z_0, Y_0/Z_0))$ and $(U_{Z_0'}, (X_0'/Z_0', Y_0'/Z_0'))$ of $\mathbb{P}^2$ and Y respectively, then the restriction $\psi : U_{Z_0} \to U_{Z_0'}$ takes the origin of U_{Z_0} to the origin $U_{Z_0'}$ having the identity as differential. Identifying each of the affine charts with $(\mathbb{C}^2, (x, y))$, we can view ψ as an element of $\mathbb{G}$, which is the *automorphism associated to* π. Observe that, if $\phi \in \mathbb{G}$ is such that the graph of the minimal blowing up process π that resolves its indetermination is $\mathcal{A}_n$, then ϕ must be the automorphism associated to π. Define $\mathbb{T}_n \subset \mathbb{G}$ to be formed by the automorphisms associated to any blowing up process π with graph $\mathcal{A}_n$ and whose first blowing up is centered at $(0:1:0)$.

Lemma 2. *Any automorphism of* $\mathbb{T}_n$ *is triangular.*

Proof. Consider the family $h_\lambda(x, y) = (x, y + \lambda x)$ where $\lambda \in \mathbb{C}$. Let $\phi(x, y) = (f(x, y), g(x, y))$ be an automorphism of $\mathbb{G}$ that commutes with the whole family. Clearly $f(x, y) = f(x, y + \lambda x)$ for any $\lambda \in \mathbb{C}$, and hence f should be a polynomial involving only the variable x. Using that the jacobian of any automorphism should be a nonzero constant, we show easily that ϕ must be triangular.

We will finish showing that for any $\phi \in \mathbb{T}_n$ and any λ we have that $\phi' := h_\lambda \circ \phi \circ h_\lambda^{-1} = \phi$. Observe that if ϕ is associated to a blowing up process π, then π is the minimal blowing up process that resolves the indetermination of ϕ. We claim that π is also the minimal resolution of the indetermination of ϕ'. Then, as $\phi' \in \mathbb{G}$, it must be the automorphism associated to π, and hence $\phi = \phi'$.

Now we show our claim. Denote by x_i and E_i the center and the exceptional divisor of $\pi_i : X^i \to X^{i-1}$, the i-th blowing up of π (where $X^0 = \mathbb{P}^2$). As $\phi' = h_\lambda \circ \phi \circ h_\lambda^{-1}$, the first indetermination point of ϕ' is

$$h_\lambda(x_1) = h_\lambda(0{:}1{:}0) = (0{:}1{:}0) = x_1.$$

Lift h_λ to an automorphism H_1 of X^1. Then the indetermination point of $\phi' \circ \pi_1$ is

$$H_1(x_2) = H_1(E_1 \cap L) = E_1 \cap L = x_2.$$

Iterating this we deduce that the n first indetermination points of ϕ' are $x_1, ..., x_n$. Let π' be the composition of the blowing ups at these points. Lift h_λ to an automorphism H_n of X^n, then the indetermination point of $\phi' \circ \pi'$ is $H_n(x_{n+1})$. The point x_{n+1} belongs to E_n. In the next paragraph we show that the restriction of H_n to E_n is the identity, and hence that $H_n(x_{n+1}) = x_{n+1}$.

Consider the affine chart of $\mathbb{P}^2$ with domain U_{Y_0} (defined by $Y_0 \neq 0$) and coordinates $(u_0, v_0) := (X_0/Y_0, Z_0/Y_0)$. The expression of h_λ with respect to (u_0, v_0) is $h_\lambda(u_0, v_0) := (u_0/(1+\lambda u_0), v_0/(1+\lambda u_0))$. The blowing up at x_1 is the blowing up at the origin of the affine chart; therefore $\pi_1^{-1}(U_{Y_0})$ is covered by two standard blowing up charts, both of them with domain isomorphic to $\mathbb{C}^2$, and with coordinates $(u_0, u_0/v_0)$ and $(u_0/v_0, v_0)$ respectively. Let U_1 be the domain of the first of these charts and rename its coordinates as $(u_1, v_1) := (u_0, u_0/v_0)$. The expression of H_1 with respect to (u_1, v_1) is $H_1(u_1, v_1) = (u_1/(1+\lambda u_1), v_1)$, and x_2 is the origin of the chart. After repeating this computation for the blowing ups π_2,...,π_n, picking up always the *second* standard chart, we obtain a chart of X^n with domain U_n isomorphic to $\mathbb{C}^2$ and coordinates (u_n, v_n) such that $E_n \cap U_n$ is defined by $v_n = 0$ and the expression of H_n with respect to (u_n, v_n) is

$$H_n(u_n, v_n) = (\frac{u_n}{1 + \lambda u_n v_n^{n-1}}, v_n).$$

Hence the restriction of H_n to E_n is the identity.

The point x_{n+1} belongs to $\dot{E}_n$, which is contained in U_n; let $(a, 0)$ be its coordinates in the chart; change coordinates to $(u'_n, v'_n) := (u_n - a, v_n)$ so that x_{n+1} becomes the origin of the affine chart, and π_{n+1} the blowing up at the origin of the chart. The expression of H_{n+1} with respect to the coordinates $(u_{n+1}, v_{n+1}) := (u'_n/v'_n, v'_n)$ of the second standard chart of the blowing up is

$$H_{n+1}(u_{n+1}, v_{n+1}) = (\frac{u_{n+1} - a\lambda(u_{n+1}v_{n+1} + a)v_{n+1}^{n-2}}{1 + \lambda(u_{n+1}v_{n+1} + a)v_{n+1}^{n-1}}, v_{n+1}),$$

and the divisor E_{n+1} is defined by $v_{n+1} = 0$. Therefore, if $n > 2$, the restriction $H_{n+1|E_{n+1}}$ is the identity, and $H_{n+1}(x_{n+2}) = x_{n+2}$. Iterating this procedure we show that at each step the lifting of h_λ to X^n does not move the next blowing up center x_{n+1}. This finishes the proof of the claim. □

Proof of Theorem 1. Consider any $\phi \in Aut(\mathbb{C}^2)$. Let $\pi = \pi_1 \circ ... \circ \pi_n$ be the minimal resolution of the indetermination of ϕ as a birational transformation of $\mathbb{P}^2$. Let E_i be the exceptional divisor of π_i, define $\dot{E}_i := E_i \setminus \bigcup_{j<i} E_j$ and $\sigma_i := \pi_1 \circ ... \circ \pi_i$.

Lemma 1 implies that π_{i+1} is the blowing up at the unique indetermination point x_{i+1} of $\phi\circ\sigma_i$, for any $i \leq n-1$, and that the unique dicritical component of π^*L is the last exceptional divisor E_n. Moreover, x_i should meet E_{i-1} if $i \geq 2$ because, otherwise, $\phi\circ\pi_{i-2}$ would have two indetermination points. Conjugating with a linear automorphism we can assume that $x_1 = (0:1:0)$. The theory of birational transformations of smooth surfaces implies that $\phi\circ\pi$ equals $H\circ\pi'$ where $H : Y \to \mathbb{P}^2$ is an isomorphism and $\pi' : X \to Y$ is the successive contraction of the nondicritical components of π^*L with self-intersection -1. We claim that there exists r such that the graph of σ_{2r-1} is $\mathcal{A}_r$.

Let E_i be a component of π^*L different from L and E_n. It has self-intersection strictly smaller than -1: its initial self-intersection is -1, it decreases by 1 when we blow up at $x_{i+1} \in E_i$. As E_n is the exceptional divisor of the last blowing up it has self-intersection -1. The strict transform of the line at infinity L should have self-intersection -1, otherwise in the contraction process $\phi\circ\pi$ the only possible divisor to start with is E_n, and it is dicritical. Before we start blowing up, L has self-intersection 1; as x_1 meets L the self-intersection of L becomes 0 after π_1; as we have to decrease it to -1 another blowing up center should meet L, the only possible one is x_2 (use that x_i should meet E_{i-1}). After π_2 the self-intersection of L is already -1 and hence no more blowing up centers meet L. The center x_3 can be either $E_1 \cap E_2$ or a point in $\dot{E}_2$. In the last case the claim is true for $r = 2$. Hence we assume that $x_3 = E_1 \cap E_2$. Let r be the maximal number such that $x_i = E_1 \cap E_{i-1}$ for any $3 \leq i \leq r$. The divisor E_r is nondicritical; otherwise it should be possible to successively contract all the components except E_r starting with L. The self-intersection of E_1 is $-r$ and the divisors L, E_2, ..., E_{r-1} are separated from E_1 by E_r, hence in the contraction process E_1 would never increase its self-intersection, and hence it could never be contracted. We conclude that there is a further blowing up π_{r+1} in the blowing up process π. The center of π_{r+1} should be either $E_{r-1} \cap E_r$ or a point of $\dot{E}_r$.

If $x_{r+1} = E_{r-1} \cap E_r$, then, after π, the self-intersection of E_{r-1} is upper bounded by -3. Remembering that only the nondicriticals with self-intersection -1 can be contracted we easily see that we have to contract successively L, E_2,...,E_{r-2}. After this E_{r-1} gets self-intersection upper bounded by -2, as we have contracted only one component that meets it. The self-intersection of the rest of the remaining components is not affected by the contractions. Then the only component with self-intersection -1 is the dicritical component and hence we cannot finish the contraction procedure. We conclude that $x_{r+1} \in \dot{E}_r$.

Let s be the maximal integer such that x_{r+i} belongs to $\dot{E}_{r+i-1}$ for $1 \leq i \leq s$. We prove that $s \geq r-1$; as this is trivial for $r = 2$, we deal with $r \geq 3$. We assume that $s < r-1$. Because of the definition of s we have that either E_{r+s} is dicritical or x_{r+s+1} equals $E_{r+s-1} \cap E_{r+s}$. In both cases the divisor L has self-intersection -1, the divisor E_1 has $-r$, the divisors E_i have -2, for $2 \leq i \leq s+r-2$. If E_{r+s} is dicritical, then the self-intersection of E_{r+s-1} equals -2. If we contract successively the nondicritical components with self-intersection -1 we will reach a point in which the only remaining components will be E_{r+s}, that is dicritical, and E_1, with self-intersection $-r+s < -1$. Hence the contraction process cannot be completed. If $x_{r+s+1} = E_{r+s-1} \cap E_{r+s}$, then the self-intersection of E_{r+s-1} is strictly smaller than -2. We can contract successively L, E_2,...,E_r,...,E_{r+s-2}. After this E_{r+s-1} has self-intersection strictly smaller than -1, because the only component meeting it before being contracted was E_{r+s-2}. The rest of the remaining nondicritical components have self-intersection upper bounded by -2 because they are separated from the contracted components by E_{r+s-1}. Hence the contraction process again cannot be completed. This proves that $s \geq r-1$, and this, in turn, implies that the graph of σ_{2r-1} is $\mathcal{A}_r$, as we claimed.

Let $\sigma_{2r,n}$ be the composition $\pi_{2r}\circ ... \circ\pi_n$. Define σ'_{2r-1} as the composition of the first $2r-1$ contractions of π' and $\sigma'_{2r,n}$ as the composition of the rest of the contractions. We have that $\phi = H\circ\pi'\circ\pi^{-1}$. If $\sigma_{2r,n}$ is not trivial (when $\pi \neq \sigma_{2r-1}$), then, by the argument of the previous paragraph, the point x_{2r} is in $\dot{E}_{2r-1}$. Consequently the centers of the blowing ups of $\sigma_{2r,n}$ are not located in any of the divisors contracted by σ'_{2r-1}, and therefore performing the blowing up process $\sigma_{2r,n}^{-1}$ and then the contraction σ'_{2r-1} is the same as making first the contraction after the blowing up process. This implies the commutativity

$$\sigma'_{2r,n}\circ\sigma'_{2r-1}\circ\sigma_{2r,n}^{-1}\circ\sigma_{2r-1}^{-1} = \sigma'_{2r,n}\circ\sigma_{2r,n}^{-1}\circ\sigma'_{2r-1}\circ\sigma_{2r-1}^{-1}.$$

As the graph of σ_{2r-1} is $\mathcal{A}_r$, there exists a unique isomorphism F from the target of σ'_{2r-1} to $\mathbb{P}^2$ that makes $\phi' := F\circ\sigma'_{2r-1}\sigma_{2r-1}^{-1}$ an automorphism of $\mathbb{T}_r$ (recall that $x_1 = (0:1:0)$), and hence triangular. If we define $\phi'' := H\circ\sigma'_{2r,n}\circ\sigma_{2r,n}^{-1}\circ F^{-1}$, then we have the factorisation $\phi = \phi''\circ\phi'$, where ϕ' is triangular and ϕ'' needs less blowing ups than ϕ to resolve its indetermination. Hence the theorem is proved by induction on the number of blowing ups needed to resolve the indetermination. $\square$

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