

ON THE UPPER SEMICONTINUITY OF THE WU METRIC

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ABSTRACT. We discuss continuity and upper semicontinuity of the Wu pseudometric.

The Wu pseudometric was introduced by H. Wu in [Wu 1993] (and [Wu]). Various properties of the Wu metric may be found for instance in [Che-Kim 1996], [Che-Kim 1997], [Kim 1998], [Che-Kim 2003], [Juc 2002]. Nevertheless, it seems that even quite elementary properties of this metric are not completely understood, e.g. its upper semicontinuity.

First, let us formulate the definition of the Wu pseudometric in an abstract setting. Let $h : \mathbb{C}^n \rightarrow \mathbb{R}_+$ be a \mathbb{C} -seminorm. Put:

$$I = I(h) := \{X \in \mathbb{C}^n : h(X) < 1\} \text{ (} I \text{ is convex),}$$

$$V = V(h) := \{X \in \mathbb{C}^n : h(X) = 0\} \subset I \text{ (} V \text{ is a vector subspace of } \mathbb{C}^n \text{),}$$

$U = U(h) :=$ the orthogonal complement of V with respect to the standard Hermitian scalar product $\langle z, w \rangle := \sum_{j=1}^n z_j \bar{w}_j$ in \mathbb{C}^n ,

$$I_0 := I \cap U, \quad h_0 := h|_U \text{ (} h_0 \text{ is a norm, } I = I_0 + V \text{).}$$

For any pseudo-Hermitian scalar product $s : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$, let

$$q_s(X) := \sqrt{s(X, X)}, \quad X \in \mathbb{C}^n, \quad \mathbb{E}(s) := \{X \in \mathbb{C}^n : q_s(X) < 1\}.$$

Consider the family \mathcal{F} of all pseudo-Hermitian scalar products $s : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ such that $I \subset \mathbb{E}(s)$, equivalently, $q_s \leq h$. In particular,

$$V \subset I = I_0 + V \subset \mathbb{E}(s) = \mathbb{E}(s_0) + V,$$

where $s_0 := s|_{U \times U}$ (note that $\mathbb{E}(s_0) = \mathbb{E}(s) \cap U$). Let $\text{Vol}(s_0)$ denote the volume of $\mathbb{E}(s_0)$ with respect to the Lebesgue measure of U . Since I_0 is bounded, there exists an $s \in \mathcal{F}$ with $\text{Vol}(s_0) < +\infty$. Observe that for any basis $e = (e_1, \dots, e_m)$ of U ($m := \dim_{\mathbb{C}} U$) we have

$$\text{Vol}(s_0) = \frac{C(e)}{\det S},$$

where $C(e) > 0$ is a constant (independent of s) and $S = S(s_0)$ denotes the matrix representation of s_0 in the basis e , i.e. $S_{j,k} := s(e_j, e_k)$, $j, k = 1, \dots, m$. In particular, if $U = \mathbb{C}^m \times \{0\}^{n-m}$ and $e = (e_1, \dots, e_m)$ is the canonical basis, then $C(e)$ is the volume of the open unit Euclidean ball $\mathbb{B}_m \subset \mathbb{C}^m$. We are interested in

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finding an $s \in \mathcal{F}$ for which $\text{Vol}(s_0)$ is minimal, equivalently, $\det S(s_0)$ is maximal. Observe that if s has this property (with respect to h), then for any \mathbb{C} -linear isomorphism $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$, the scalar product

$$\mathbb{C}^n \times \mathbb{C}^n \ni (X, Y) \xrightarrow{L(s)} s(L(X), L(Y)) \in \mathbb{C}$$

has the extremal property with respect to $h \circ L$. In particular, this permits us to reduce the situation to the case where $U = \mathbb{C}^m \times \{0\}^{n-m}$ and next (by restricting all the above objects to $\mathbb{C}^m \simeq \mathbb{C}^m \times \{0\}^{n-m}$) to assume that $m = n$.

Theorem 1 ([Wu], [Wu 1993]). *There exists exactly one element $s^h \in \mathcal{F}$ such that*

$$\text{Vol}(s_0^h) = \min\{\text{Vol}(s_0) : s \in \mathcal{F}\} < +\infty.$$

Put $\widehat{s}^h := m \cdot s^h$ ($m := \dim U(h)$), $\mathbb{W}h := q_{\widehat{s}^h}$. Obviously, $\mathbb{W}h \leq \sqrt{m}h$ and $\mathbb{W}h \equiv \sqrt{m}h$ iff $h = q_s$ for some pseudo-Hermitian scalar product s . For instance, $\mathbb{W}\| \cdot \| = \sqrt{n}\| \cdot \|$, where $\| \cdot \|$ is the Euclidean norm in \mathbb{C}^n . Moreover, $\mathbb{W}(\mathbb{W}h) \equiv \sqrt{m}\mathbb{W}h$.

Remark 2. Assume that $U(h) = \mathbb{C}^n$. Let $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a \mathbb{C} -linear isomorphism such that $|\det L| = 1$ and $h \circ L = h$. Then $\text{Vol}(s^h) = \text{Vol}(L(s^h))$ and hence $s^h = L(s^h)$, i.e. $s^h(X, Y) = s^h(L(X), L(Y))$, $X, Y \in \mathbb{C}^n$.

Theorem 3 ([Wu], [Wu 1993]¹). (a) $h \leq \mathbb{W}h \leq \sqrt{m}h$.

(b) *If $h(X) := \max\{h_1(X_1), h_2(X_2)\}$, $X = (X_1, X_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$, then*

$$\widehat{s}^h(X, Y) = \widehat{s}^{h_1}(X_1, Y_1) + \widehat{s}^{h_2}(X_2, Y_2), \quad X = (X_1, X_2), Y = (Y_1, Y_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2},$$

In particular,

$$\mathbb{W}h(X) = \left((\mathbb{W}h_1(X_1))^2 + (\mathbb{W}h_2(X_2))^2 \right)^{1/2}, \quad X = (X_1, X_2).$$

For a domain $G \subset \mathbb{C}^n$, let $\mathcal{M}(G)$ denote the space of all *pseudometrics*

$$\eta : G \times \mathbb{C}^n \rightarrow \mathbb{R}_+, \quad \eta(a; tX) = |t|\eta(a; X), \quad a \in G, X \in \mathbb{C}^n, t \in \mathbb{C},$$

such that

$$\forall a \in G \quad \exists M, r > 0 : \eta(z; X) \leq M\|X\|, \quad z \in \mathbb{B}(a, r) \subset G, X \in \mathbb{C}^n,$$

where $\mathbb{B}(a, r)$ is the open Euclidean ball centered at a with radius r .

For $\eta \in \mathcal{M}(G)$ we define the *Wu pseudometric*

$$(\mathbb{W}\eta)(a; X) := (\mathbb{W}\widehat{\eta}(a; \cdot))(X), \quad a \in G, X \in \mathbb{C}^n,$$

where

$$\widehat{\eta}(a; X) := \sup\{h(X) : h \text{ is a } \mathbb{C}\text{-seminorm, } h \leq \eta(a; \cdot)\}, \quad a \in G, X \in \mathbb{C}^n,$$

denotes the *Buseman pseudometric* associated to η (cf. [J-P 1993], §4.3). Observe that $\mathbb{W}\eta \in \mathcal{M}(G)$.

Recall that an upper semicontinuous metric $\eta \in \mathcal{M}(G)$ is said to be *complete* if any $\int\eta$ -Cauchy sequence is convergent to a point from G , where $\int\eta$ denotes the *integrated form of η* (cf. [J-P 1993], §§4.3, 7.3).

¹See also M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis — revisited*, to appear.

Proposition 4. (a) If $\eta \in \mathcal{M}(G)$ is a continuous metric, then so is $\mathbb{W}\eta$ (cf. Example 6).

(b) If $\eta \in \mathcal{M}(G)$ is a continuous complete metric, then so is $\mathbb{W}\eta$.

(c) If $(\delta_G)_G$ is a holomorphically contractible family of pseudometrics (cf. [J-P 1993]), then for any biholomorphic mapping $F : G \rightarrow D$ ($G, D \subset \mathbb{C}^n$) we have

$$(\mathbb{W}\delta_D)(F(z); F'(z)X) = (\mathbb{W}\delta_G)(z; X), \quad z \in G, X \in \mathbb{C}^n.$$

(d) If $(\delta_G)_G$ is a holomorphically contractible family of pseudometrics, then for any holomorphic mapping $F : G \rightarrow D$ ($G \subset \mathbb{C}^{n_1}, D \subset \mathbb{C}^{n_2}$) we have

$$(\mathbb{W}\delta_D)(F(z); F'(z)X) \leq \sqrt{n_2}(\mathbb{W}\delta_G)(z; X), \quad z \in G, X \in \mathbb{C}^n,$$

but, for example, the family $(\mathbb{W}\varkappa_G)_G$ is not holomorphically contractible, where \varkappa_G is the Kobayashi–Royden pseudometric of G (cf. Example 5).

In the case $\eta = \varkappa_G$, the above properties (a)–(d) were formulated (without proof) in [Wu], [Wu 1993].

Proof. (a) Fix a point $z_0 \in G \subset \mathbb{C}^n$. Let $s^z := s^{\widehat{\eta}(z, \cdot)}$, $z \in G$. We are going to show that $s^z \xrightarrow{z \rightarrow z_0} s^{z_0}$.

By our assumptions, there exist $r > 0, c > 0$ such that

$$\eta(z; X) \geq c\|X\|, \quad z \in \mathbb{B}(z_0, r) \subset G, X \in \mathbb{C}^n.$$

In particular, the sets

$$I(z) := \{X \in \mathbb{C}^n : \widehat{\eta}(z; X) < 1\}, \quad z \in \mathbb{B}(z_0, r),$$

are contained in the ball $\mathbb{B}(0, C)$ with $C := 1/c$. Moreover,

$$|\widehat{\eta}(z; X) - \widehat{\eta}(z_0; X)| \leq \varphi(z)\|X\|, \quad X \in \mathbb{C}^n,$$

where $\varphi(z) \xrightarrow{z \rightarrow z_0} 0$. Hence

$$(1 + C\varphi(z))^{-1}I(z) \subset I(z_0) \subset (1 + C\varphi(z))I(z), \quad z \in \mathbb{B}(z_0, r),$$

and consequently,

$$\begin{aligned} (*) \quad I(z_0) &\subset (1 + C\varphi(z))\mathbb{E}(s^z) = \mathbb{E}((1 + C\varphi(z))^{-2}s^z), \\ I(z) &\subset (1 + C\varphi(z))\mathbb{E}(s^{z_0}) = \mathbb{E}((1 + C\varphi(z))^{-2}s^{z_0}), \quad z \in \mathbb{B}(z_0, r). \end{aligned}$$

Hence

$$\begin{aligned} \text{Vol}(s^{z_0}) &\leq \text{Vol}((1 + C\varphi(z))^{-2}s^z) = (1 + C\varphi(z))^{2n} \text{Vol}(s^z), \\ \text{Vol}(s^z) &\leq \text{Vol}((1 + C\varphi(z))^{-2}s^{z_0}) = (1 + C\varphi(z))^{2n} \text{Vol}(s^{z_0}), \quad z \in \mathbb{B}(z_0, r). \end{aligned}$$

Thus $\text{Vol}(s^z) \xrightarrow{z \rightarrow z_0} \text{Vol}(s^{z_0})$.

Take a sequence $z_\nu \rightarrow z_0$. Since

$$|s^{z_\nu}(e_j, e_k)| \leq \eta(z_\nu; e_j)\eta(z_\nu; e_k), \quad j, k = 1, \dots, n, \nu \in \mathbb{N},$$

we may assume that $s^{z_\nu} \rightarrow s^*$, where s^* is a pseudo–Hermitian scalar product. We already know that $\text{Vol}(s^*) = \text{Vol}(s^{z_0})$. Moreover, by (*), $I(z_0) \subset \mathbb{E}(s^*)$. Consequently, the uniqueness of s^{z_0} implies that $s^* = s^{z_0}$.

(b) Recall that $\int \eta = \int \widehat{\eta}$ — cf. [J-P 1993], Proposition 4.3.5(b). By (a), $\mathbb{W}\eta$ is a continuous metric. In particular, the distance $\int(\mathbb{W}\eta)$ is well defined. By Theorem 3(a) we get

$$\int \widehat{\eta} \leq \int(\mathbb{W}\eta),$$

which directly implies the required result.

(c) The result is obvious because for any $z \in G$, the mapping $F'(z)$ is a \mathbb{C} -linear isomorphism and $\delta_D(F(z); F'(z)X) = \delta_G(z; X)$, $X \in \mathbb{C}^n$.

(d) It is known that the family $(\widehat{\delta}_G)_G$ is holomorphically contractible ([J-P 1993], Theorem 4.3.10(c)). Hence, using Theorem 3(a), we get

$$\begin{aligned} (\mathbb{W}\delta_D)(F(z); F'(z)X) &\leq \sqrt{n_2}\widehat{\delta}_D(F(z); F'(z)X) \\ &\leq \sqrt{n_2}\widehat{\delta}_G(z; X) \leq \sqrt{n_2}(\mathbb{W}\delta_G)(z; X), \quad z \in G, X \in \mathbb{C}^n. \end{aligned}$$

□

Example 5. Let $G := \{(z_1, z_2) \in \mathbb{B}_2 : |z_1| < \varepsilon\}$, $0 < \varepsilon < 1/\sqrt{2}$. Recall that $\varkappa_{\mathbb{B}_2}(0; X) = \|X\|$ and $\varkappa_G(0; X) = \max\{\|X\|, |X_1|/\varepsilon\}$, $X = (X_1, X_2)$. Then $(\mathbb{W}\varkappa_{\mathbb{B}_2})(0; (0, 1)) = \sqrt{2} > (\mathbb{W}\varkappa_G)(0; (0, 1))$. In particular, the family $(\mathbb{W}\varkappa_D)_D$ is not contractible with respect to inclusions.

We point out that Proposition 4(a) gives us the continuity of $\mathbb{W}\eta$ only in the case where η is a continuous metric. It is natural to conjecture that in the general case, where η is only an upper semicontinuous (pseudo)metric, $\mathbb{W}\eta$ remains to be upper semicontinuous. The following Example 6 shows that in general this is not true. In the case where η is a continuous pseudometric, we do not know whether $\mathbb{W}\eta$ is upper semicontinuous. Observe that the upper semicontinuity (or at least Borel measurability) of $\mathbb{W}\eta$ appears in a natural way when one defines $\int(\mathbb{W}\eta)$. In the case where $\eta = \varkappa_G$, the upper semicontinuity of $\mathbb{W}\varkappa_G$ is claimed for instance in [Wu 1993] (Theorem 1), [Che-Kim 1996] (Proposition 2), [Juc 2002] (Theorem 0), but so far there is no proof.

Example 6. There is an upper semicontinuous metric η such that $\mathbb{W}\eta$ is not upper semicontinuous.

Indeed, let $\eta : \mathbb{B}_2 \times \mathbb{C}^2 \rightarrow \mathbb{R}_+$, $\eta(z; X) := \|X\|$ for $z \neq 0$, and $\eta(0, X) := \max\{\|X\|, |X_1|/\varepsilon\}$, $X = (X_1, X_2) \in \mathbb{C}^2$ ($\varepsilon > 0$ small). Then $(\mathbb{W}\eta)(z; X) = \sqrt{2}\|X\|$ for $z \neq 0$, and $\{X \in \mathbb{C}^2 : (\mathbb{W}\eta)(0; X) < 1\} \setminus \mathbb{B}(0, 1/\sqrt{2}) \neq \emptyset$, so $\mathbb{W}\eta$ is not upper semicontinuous (cf. Example 5).

Example 7. There exists a bounded domain $G \subset \mathbb{C}^2$ such that $\mathbb{W}\varkappa_G$ is not continuous (see Proposition 2 in [Che-Kim 1996], where such a continuity is claimed).

Indeed, let $D \subset \mathbb{C}^2$ be a domain such that (cf. [J-P 1993], Example 3.5.10):

- there exists a dense subset $M \subset \mathbb{C}$ such that $(M \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}) \subset D$,
- $\varkappa_D(z; (0, 1)) = 0$, $z \in A := M \times \mathbb{C}$,
- there exists a point $z^0 \in D \setminus A$ such that $\varkappa_D(z^0; X) \geq c\|X\|$, $X \in \mathbb{C}^2$, where $c > 0$ is a constant.

For $R > 0$ let $D_R := \{z = (z_1, z_2) \in D : |z_j - z_j^0| < R, j = 1, 2\}$. It is known that $\varkappa_{D_R} \searrow \varkappa_D$ when $R \nearrow +\infty$. Observe that $z^0 \in D_R$ and

$$\varkappa_{D_R}(z^0; X) \geq \varkappa_D(z^0; X) \geq c\|X\|, \quad X \in \mathbb{C}^2.$$

Hence, by Theorem 3(a), $(\mathbb{W}\varkappa_{D_R})(z^0; X) \geq c\|X\|$, $X \in \mathbb{C}^2$. In particular,

$$(\mathbb{W}\varkappa_{D_R})(z^0; (0, 1)) \geq c.$$

Fix a sequence $M \ni z_k \rightarrow z_1^0$. Note that $\{z_k\} \times (z_2^0 + RE) \subset D_R$, which implies that $\varkappa_{D_R}((z_k, z_2^0); (0, 1)) \leq 1/R$, $k = 1, 2, \dots$. In particular,

$$(\mathbb{W}\varkappa_{D_R})((z_k, z_2^0); (0, 1)) \leq \sqrt{2}\varkappa_{D_R}((z_k, z_2^0); (0, 1)) \leq \sqrt{2}/R, \quad k = 1, 2, \dots$$

Now it clear that if $R > \frac{\sqrt{2}}{c}$, then

$$\limsup_{k \rightarrow +\infty} (\mathbb{W}\varkappa_{D_R})((z_k, z_2^0); (0, 1)) \leq \sqrt{2}/R < c \leq (\mathbb{W}\varkappa_{D_R})(z^0; (0, 1)),$$

which shows that for $G := D_R$ the pseudometric $\mathbb{W}\varkappa_G$ is not continuous.

Remark 8. We point out the influence of the factor \sqrt{m} in the definition of \mathbb{W} to its upper semicontinuity.

Suppose we defined $\widetilde{\mathbb{W}}h := q_{s^h}$, $\widetilde{\mathbb{W}}\eta(a; X) = (\widetilde{\mathbb{W}}\eta(a; \cdot))(X)$. Then, using the product formula (Theorem 3(b)), we would get a domain $G \subset \mathbb{C}^3$ such that $\widetilde{\mathbb{W}}\varkappa_G$ is not upper semicontinuous.

Indeed (the example is due to W. Jarnicki), let $D \subset \mathbb{C}^2$ and $D \ni z_k \rightarrow z_0 \in D$ be such that:

- $\varkappa_D(z_k; \cdot)$ is not a metric (in particular, $m(k) := \dim U(\widehat{\varkappa}_D(z_k; \cdot)) \leq 1$, $k \in \mathbb{N}$),
- $\varkappa_D(z_0; \cdot)$ is a metric.

Take, for instance, the domain D as in Example 7.

Put $G := D \times E \subset \mathbb{C}^3$. Then $\widetilde{\mathbb{W}}\varkappa_G$ is not upper semicontinuous at $((z_0, 0), (0, 1))$ because

$$\begin{aligned} (\widetilde{\mathbb{W}}\varkappa_G)^2((z_k, 0); (0, 1)) &= s^{\varkappa_G((z_k, 0); \cdot)}((0, 1), (0, 1)) = \frac{1}{m(z_k) + 1} \geq \frac{1}{2}, \quad k \in \mathbb{N}, \\ (\widetilde{\mathbb{W}}\varkappa_G)^2((z_0, 0); (0, 1)) &= s^{\varkappa_G((z_0, 0); \cdot)}((0, 1), (0, 1)) = \frac{1}{m(z_0) + 1} = \frac{1}{3}. \end{aligned}$$

We conclude the paper by repeating the main open question.

Problem. Let $\eta \in \{\gamma_G^{(k)}, A_G, \varkappa_G\}$ (cf. [J-P 1993]). Is $\mathbb{W}\eta$ upper semicontinuous?

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