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# JOINS OF PROJECTIVE VARIETIES AND MULTISECANT SPACES

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ABSTRACT. Let $X_1, \dots, X_s \subset \mathbf{P}^N$, $s \geq 1$, be integral varieties. For any integers $k_i > 0$, $1 \leq i \leq s$, and $t \geq 0$ set $\vec{k} := (k_1, \dots, k_s)$ and $\vec{X} := (X_1, \dots, X_s)$. Let $\mathrm{Sec}(\vec{X}; t, \vec{k})$ be the set of all linear $t$-spaces contained in a linear $(k_1 + \cdots + k_s - 1)$-space spanned by $k_1$ points of $X_1$, $k_2$ points of $X_2$, ..., $k_s$ points of $X_s$. Here we study some cases where $\mathrm{Sec}(\vec{X}; t, \vec{k})$ *has the expected dimension*. The case $s = 1$ was recently considered by Chiantini and Coppens and we follow their ideas. The two main results of the paper consider cases where each $X_i$ is a surface, more particularly:

$$s = 3, k_1 = k_2 = k_3 = 1 \text{ and } t = 1$$

or

$$s = 2, k_1 = 2, k_2 = 1 \text{ and } t = 1.$$

## 1. INTRODUCTION

L. Chiantini and M. Coppens revived a piece of classical projective geometry (see [6] and references therein): the study of the set of all linear spaces contained in the secant varieties of an integral variety $X \subset \mathbf{P}^N$. For further papers on this topic, see [5], [8] and [9]. Let $G(t+1, N+1)$ be the Grassmannian of all $t$-dimensional linear subspaces of $\mathbf{P}^N$. The order $k$ secant variety of $X$ is the join of $k$ copies of $X$. In this paper we fix $s$ varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$, and consider the closure in $G(t+1, N+1)$ of the set of all $t$-spaces contained in a $(k_1 + \cdots + k_s - 1)$-space spanned by $k_1$ points of $X_1$, $k_2$ points of $X_2$, ...,$k_s$ points of $X_s$. The case $s = 1$ is the case considered in [6] and we will often use the ideas contained in [6].

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. For any two integral varieties $X$, $Y$ of $\mathbf{P}^N$, let $[X; Y]$ be the join of $X$ and $Y$; thus if $X = Y = \{P\}$ with $P$ a point, then $[X; Y] = \{P\}$, while in all other cases $[X; Y]$ is the closure in $\mathbf{P}^N$ of the union of all lines $\langle\{A, B\}\rangle$ spanned by some $A \in X$ and some $B \in Y$ with $A \neq B$.

Fix integers $N \geq 3$, $s > 0$ and $k_i \geq 0$, $1 \leq i \leq s$. Set $\vec{k} := (k_1, \dots, k_s)$ and $|\vec{k}| := k_1 + \cdots + k_s - 1$. Let $t$ be an integer such that $0 \leq t \leq |\vec{k}|$. Fix $s$ irreducible varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$. Usually, we will be interested in the case $X_i \neq X_j$ for $i \neq j$, since the general case may be reduced to this case by decreasing $s$, but with the same

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value of $|\vec{k}|$. The $(t,\vec{k})$-secant variety $\mathrm{Sec}(\vec{X};t,\vec{k})$ of $\vec{X}$ is the closure in $G(t+1,N+1)$ of all $t$-spaces contained in a $|\vec{k}|$-dimensional linear subspace of $\mathbf{P}^N$ spanned by $k_1$ points of $X_1$, $k_2$ points of $X_2$,…,$k_s$ points of $X_s$. Set $\mathrm{Sec}(\vec{X};\vec{k}) := \mathrm{Sec}(\vec{X};0,\vec{k}) \subseteq \mathbf{P}^N$. The $(0,\vec{k})$-secant variety $\mathrm{Sec}(\vec{X};\vec{k})$ of $\vec{X}$ will be called the $\vec{k}$-secant variety of $\vec{X}$. Set $n_i := \dim(X_i)$. We have $\dim(\mathrm{Sec}(\vec{X};\vec{k})) \le \min\{N, \sum_{i=1}^s k_i(n_i+1)-1\}$ and $\dim(\mathrm{Sec}(\vec{X};t,\vec{k})) \le \min\{(t+1)(N-t), \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1)\}$. We will say that $\vec{X}$ is $\vec{k}$-defective (resp. $(t,\vec{k})$-defective) if

$$\dim(\mathrm{Sec}(\vec{X};\vec{k})) < \min\{N, \sum_{i=1}^s k_i(n_i+1)-1\}$$

(resp. $\dim(\mathrm{Sec}(\vec{X};t,\vec{k})) < \min\{(t+1)(N-t), \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1)\}$). If $\vec{X}$ is $\vec{k}$-defective (resp. $(t,\vec{k})$-defective) the integer

$$\delta(\vec{X};\vec{k}) := \sum_{i=1}^s k_i(n_i+1) - 1 - \dim(\mathrm{Sec}(\vec{X};\vec{k}))$$

(resp. $\delta(\vec{X};t,\vec{k}) := \sum_{i=1}^s k_in_i + (|\vec{k}|-t)(t+1) - \dim(\mathrm{Sec}(\vec{X};t,\vec{k})))$ will be called the total order of $\vec{k}$-defectivity (resp. $(t,\vec{k})$-defectivity) of $\vec{X}$.

It seems very natural to start the study of $(t,\vec{k})$-defectivity from the case $\dim(X_i) = 1$ for every $i$. For the case in which each $X_i$ is a non-degenerate curve, see Corollary 1 and Theorem 4 in section 3. For a more general defectivity result for non-degenerate curves, see [3]; in the quoted paper we considered the set of all flags of linear spaces contained in a $|\vec{k}|$-dimensional linear space instead of the set of all $t$-dimensional linear spaces. Obviously, the interested reader may do other related cases (e.g. some degenerate curves or a surface and $s-1$ non-degenerate curves). We stress that degenerate varieties may not give defective $s$-ples (e.g. when $k_i = 1$ for all $i$ take as $X_i$, $1 \le i \le s$, linearly independent linear subspaces). For degenerate varieties, see also Remark 3. We believe that when $s \ge 2$, the mutual position of the varieties is more important than their structure. For examples of defectivity when one of the varieties is a cone, see Remark 1. For a complete analysis of a toy case, see Example 1. We raise the following question.

**Question 1.** Fix integers $s \ge 2$, $k_i > 0$, $1 \le i \le s$, $t > 0$, and integral varieties $X_i \subset \mathbf{P}^N$, $1 \le i \le s$. Set $\vec{X} := (X_1,\dots,X_s)$, $\vec{k} := (k_1,\dots,k_s)$, $\vec{Y} := (X_1,\dots,X_{s-1})$ and $\vec{m} := (k_1,\dots,k_{s-1})$. Assume that $\mathrm{Sec}(\vec{Y};t,\vec{m})$ has the expected dimension and that $X_s$ is a curve. Are there reasonable conditions on $X_s$ assuring that $\mathrm{Sec}(\vec{X};t,\vec{k})$ has the expected dimension? More generally, for any $\vec{Y}$ has $\mathrm{Sec}(\vec{X};t,\vec{k})$ the maximal possible dimension compatible with the dimension of $\mathrm{Sec}(\vec{Y};t,\vec{m})$?

It is well known and easy to show that in the case $t = 0$ a sufficient condition for an affirmative answer to Question 1 is that $X_s$ is a non-degenerate curve (see Corollary 1 and Remark 3 for more precise results). See Theorem 4 for the case $t = 1$ and [3] when $\dim(X_i) = 1$ and each $X_i$ is non-degenerate.

Our main results are non-existence results for the $(1,\vec{k})$-defectivity of joins of surfaces. In section 2 we will prove the following results.

**Theorem 1.** *Let $X_1$, $X_2$ and $X_3$ be integral non-degenerate surfaces of $\mathbf{P}^N$, $N \ge 5$, such that $X_i \ne X_j$ for $i \ne j$. Assume $\dim([X_1;X_2]) = 5$ and that $X_3$ is not* *a cone. Set* $\vec{k} = (1,1,1)$*. Then* $\vec{X} := (X_1, X_2, X_3)$ *is not* $(1,\vec{k})$*-defective, i.e.* $\dim(\mathrm{Sec}(\vec{X};1,\vec{k})) = 8$.

**Theorem 2.** *Let* $X_1$ *and* $X_2$ *be integral non-degenerate surfaces of* $\mathbf{P}^N$*,* $N \geq 5$*, such that* $X_1 \neq X_2$*. Assume that neither* $X_1$ *nor* $X_2$ *is a cone. Set* $\vec{k} := (2,1)$*. Then* $\vec{X} := (X_1, X_2)$ *is not* $(1,\vec{k})$*-defective.*

The condition $\dim([X_1; X_2]) = 5$ in the statement of Theorem 1 is very mild (see [1]). It implies that neither $X_1$ nor $X_2$ is a cone. We do not know if the condition that no $X_i$ is a cone is always necessary (see Remark 1), but certainly $X_1$, $X_2$ and $X_3$ cannot be cones with the same vertex (see Example 1). We do not have any construction (except cones) to obtain defective $s$-ples.

In section 3 we will give two general results on the $(t,\vec{k})$-defectivity of varieties of arbitrary dimension: an easy extension of [8] to the case of joins of different varieties (Theorem 3) and a non-defectivity result with respect to lines (Theorem 4).

## 2. Proofs of Theorems 1 and 2

In this section we will prove Theorems 1 and 2 and give the toy example and the remark on cones promised in the introduction.

For any subset $S$ of $\mathbf{P}^N$, let $\langle S\rangle$ be its linear span. We start with a baby example.

**Example 1.** Fix integers $s \geq 2$, $n_i \geq 2$, $1 \leq i \leq s$, $k_i > 0$, $1 \leq i \leq s$, $N > \min_{1\leq i\leq s}\{n_i\}$ and $t$ such that $0 \leq t \leq k_1 + \cdots + k_s - 1$. Fix $P \in \mathbf{P}^N$ and non-degenerate varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$, such that $\dim(X_i) = n_i$. Set $\vec{X} := (X_1,\dots,X_s)$ and $\vec{k} := (k_1,\dots,k_s)$. Assume that each $X_i$ is a cone with vertex containing $P$. For all hyperplanes $H, M$ of $\mathbf{P}^N$ such that $P \notin H \cup M$ the $s$-ples $(X_1 \cap H, \dots, X_s \cap H)$ and $(X_1 \cap M, \dots, X_s \cap M)$ (respectively seen as $s$-ples in $H$ and in $M$) are projectively equivalent (use the linear projection from $P$). Fix $H$ and set $Y_i := X_i \cap H$ and $\vec{Y} := (Y_1,\dots,Y_s)$. Fix $S_i \subset Y_i$, $1 \leq i \leq s$, such that $\mathrm{card}(S_i) := k_i$ and $\dim(\langle S_1 \cup \cdots \cup S_s\rangle) = k_1 + \cdots + k_s - 1$. Thus $\dim(\langle S_1 \cup \cdots \cup S_s \cup \{P\}\rangle) = k_1 + \cdots + k_s$, and for every $t$-dimensional linear space $D \subset \langle S_1 \cup \cdots \cup S_s\rangle$ the $(t+1)$-dimensional linear space $[D;\{P\}]$ contains a $(t+1)$-dimensional family of $t$-dimensional linear spaces not containing $P$ and mapped isomorphically onto $D$ by the linear projection from $P$. Fix any such $t$-dimensional linear space $D'$. There is a hyperplane $M$ of $\mathbf{P}^N$ such that $D' \subset M$ and $P \notin M$. Set

$$S_i^M := \bigcup_{Q\in S_i} \langle\{Q,P\}\rangle \cap M \subset X_i.$$

Thus $\dim(\langle S_1^M \cup \cdots \cup S_s^M\rangle) = k_1 + \cdots + k_s - 1$, $D' \subset \langle S_1^M \cup \cdots \cup S_s^M\rangle$ and hence $D' \in \mathrm{Sec}(\vec{X};t,\vec{k})$. Thus if $\mathrm{Sec}(\vec{Y};t,\vec{k}) = G(t+1,N)$, then

$$\mathrm{Sec}(\vec{X};t,\vec{k}) = G(t+1,N+1),$$

while if $\mathrm{Sec}(\vec{Y};t,\vec{k}) \neq G(t+1,N)$, then $\delta(\vec{X};t,\vec{k}) = \delta(\vec{Y};t,\vec{k}) + k_1 + \cdots + k_s - 1 - t$. Thus in the former case $(\vec{X},\vec{k})$ is not $(t,\vec{k})$-defective. In the latter case if $t < k_1 + \cdots + k_s - 1$, then $\vec{X}$ is $(t,\vec{k})$-defective. The fact that in the definition of defectivity we have to take the cut-off function min implies that very natural constructions do not always (but only almost always) give degenerate $s$-ples.

*Remark* 1. Fix $P \in \mathbf{P}^N$ and also fix a locally closed and irreducible subset $T$ of $G(t+1, N+1)$ such that $P \notin A$ for every $A \in T$. Let $T * P$ be the closure in $G(t+1, N+1)$ of the set of all $B \in G(t+1, N+1)$ contained in some $(t+1)$-dimensional linear space $\langle A \cup \{P\}\rangle$ for some $A \in T$. Fix integers $s \geq 1$, $k_i > 0$, $1 \leq i \leq s$, and integral varieties $X_i \subset \mathbf{P}^N$, $1 \leq i \leq s$. Assume that $X_s$ is a positive-dimensional cone with vertex containing $P \in \mathbf{P}^N$, say $X_s = [D; \{P\}]$ with $\dim(D) = \dim(X_s) - 1$ and $P \notin D$. Set $\vec{X} := (X_1, \dots X_{s-1}, X_s)$ and $\vec{Y} := (X_1, \dots, X_{s-1}, D)$. Obviously, $\mathrm{Sec}(\vec{X}; \vec{k})$ is the cone with $P$ as vertex and $\mathrm{Sec}(\vec{Y}; \vec{k})$ as a basis. Hence $\vec{X}$ is $\vec{k}$-defective if $k_s \geq 2$ and $\dim(\mathrm{Sec}(\vec{Y}; \vec{k})) \leq N - 2$. Now assume $t > 0$. Let $\mathrm{Sec}(\vec{Y}; t, \vec{k})'$ be the open subset of $\mathrm{Sec}(\vec{Y}; t, \vec{k})$ formed by the $t$-planes not containing $P$. It is easy to check that $\mathrm{Sec}(\vec{X}; t, \vec{k}) = \mathrm{Sec}(\vec{Y}; t, \vec{k})' * P$. Since $\dim(G(t+1, t+2)) = t+1$, we obtain $\dim(\mathrm{Sec}(\vec{X}; t, \vec{k})) \leq \dim(\mathrm{Sec}(\vec{Y}; t, \vec{k})) + t+1$. Hence $\mathrm{Sec}(\vec{X}; t, \vec{k})$ is $(t, \vec{k})$-defective if $k_s \geq t+2$ and $\dim(\mathrm{Sec}(\vec{Y}; t, \vec{k})) + t + 1 < (t+1)(N-t)$.

*Remark* 2. Let $X_i \subset \mathbf{P}^m$, $1 \leq i \leq 3$, $m \geq 3$, be integral surfaces such that $\langle X_1 \cup X_2 \cup X_3\rangle = \mathbf{P}^m$ and either $X_1 \neq X_2$ or $X_1$ is not a plane. It is easy to check that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ and a general $(B_1, B_2, B_3, B_4) \in X_1 \times X_2 \times X_3 \times X_4$ we have $\dim(\langle\{A_1, A_2, A_3\}\rangle) = 2$ and $\dim(\langle\{B_1, B_2, B_3, B_4\}\rangle) = 3$.

**Lemma 1.** *Let $X_i \subset \mathbf{P}^4$, $1 \leq i \leq 2$, be integral non-degenerate surfaces. Then for a general $(A_1, A_2) \in X_1 \times X_2$ we have $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2) = \{A_1, A_2\}$.*

*Proof.* If $X_1 = X_2$, then the lemma is [6], Cor. 1.3, for the invariants $r = 4$, $n = 2$ and $k = 1$. Assume $X_1 \neq X_2$. First assume that $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \in X_1 \backslash \{A_1\}$. Then for a general $A_2 \in X_2$ the restriction to $X_1$ of the linear projection $\mathbf{P}^4 \backslash \{A_2\} \to \mathbf{P}^3$ from $A_2$ is not birational. Since $X_2 \neq X_1$, this implies that $X_2$ is contained in the so-called Segre locus $\Sigma(X_1)$ of $X_1$, contradicting the inequality $\dim(\Sigma(X_1)) \leq 1$ proved in [4], Th. 1. Similarly, if $\langle\{A_1, A_2\}\rangle \cap (X_1 \cup X_2)$ contains $B \in X_2 \backslash \{A_2\}$, we see that $X_1$ is contained in the Segre locus of $X_2$, contradicting [4], Th. 1. □

**Lemma 2.** *Let $X_i \subset \mathbf{P}^5$, $1 \leq i \leq 3$, be integral surfaces such that $X_i \neq X_j$ and $\langle X_i \cup X_j\rangle = \mathbf{P}^5$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. Then for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) = \{A_1, A_2, A_3\}$.*

*Proof.* Assume that for a general $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$ we have $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3) \neq \{A_1, A_2, A_3\}$. Just to fix the notation assume that $\langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$ contains $B \in X_1 \backslash \{A_3\}$. Let $f : \mathbf{P}^5 \backslash \{A_3\} \to \mathbf{P}^4$ be the linear projection from the point $A_3$ and $g : \mathbf{P}^4 \backslash \{f(A_2)\} \to \mathbf{P}^3$ the linear projection from the point $f(A_2)$. Using that $X_3 \neq X_2$, $X_3 \neq X_1$, $X_3$ is not contained in $\langle X_i\rangle$ $(i = 1, 2)$ if $X_i$ is degenerate and $A_3$ is general in $X_3$, we obtain $A_3 \notin X_1 \cup X_2$, $\langle f(X_1) \cup f(X_2)\rangle = \mathbf{P}^4$ and $\dim(\langle f(X_i)\rangle = \min\{4, \dim(\langle X_i\rangle)\}$ for $i = 1, 2$. Hence either $\langle f(X_1)\rangle = \mathbf{P}^4$ or $f(X_2) \nsubseteq \langle f(X_1)\rangle$. If $\langle f(X_1)\rangle \neq \mathbf{P}^4$, by the generality of $f(A_2)$ in $f(X_2)$ we obtain that $g|f(X_1)$ is injective (here we just take any $f(A_2) \notin \langle f(X_1)\rangle$), contradicting the existence of $A_1$ and $B$ (even if $A_1$ is not assumed to be general in $X_1$) such that $A_1 \neq B$ and $B \in \langle\{A_1, A_2, A_3\}\rangle \cap (X_1 \cup X_2 \cup X_3)$. Hence we may assume $\langle f(X_1)\rangle = \mathbf{P}^4$. To obtain a contradiction it is sufficient to show that $g|f(X_1)$ is birational. Assume that $g|f(X_1)$ is not birational. Since $f(A_2)$ is a general point of $f(X_2)$ and $f(X_1) \neq f(X_2)$, we obtain that a general point of $f(X_2) \backslash f(X_1)$ is in the Segre locus $\Sigma(f(X_1))$ of $f(X_1)$, contradicting [4], Th. 1. □

**Lemma 3.** *Let $X_i \subset \mathbf{P}^5$, $1 \le i \le 2$, be integral surfaces such that $X_1 \ne X_2$, $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$ and $\dim(\langle X_1\rangle) \ge 4$. Then for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2) = \{A_1, A_2, A_3\}$.*

*Proof.* Assume that for a general $(A_1, A_2, B) \in X_1 \times X_1 \times X_2$ we have $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2) \ne \{A_1, A_2, B\}$. Let $f : \mathbf{P}^5 \backslash \{B\} \to \mathbf{P}^4$ be the linear projection from $B$. First assume that $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2)$ contains $D \in X_1 \backslash \{A_1, A_2\}$. Since $B$ is general in $X_2$, $f(X_1)$ spans $\mathbf{P}^4$. Hence a general secant line of $f(X_1)$ is not a trisecant line ([6], Cor. 1.3). Since $f(D) \in \langle f(A_1), f(A_2)\rangle$, we obtain that either $f(D) = f(A_1)$ or $f(D) = f(A_2)$. Just to fix the notation we assume $f(D) = f(A_1)$. Since $A_1$ is general in $X_1$, we obtain that $f|X_1$ is not birational. Hence a general $B \in X_2$ is in the Segre locus $\Sigma(X_1)$ of $X_1$, contradicting [4], Th. 1. Now assume that $\langle\{A_1, A_2, B\}\rangle \cap (X_1 \cup X_2)$ contains $C \in X_2 \backslash \{B\}$. We obtain that any secant line to $f(X_1)$ intersects $f(X_2 \backslash \{B\})$. Hence $f(X_2) \subseteq \Sigma(f(X_1))$, a contradiction. □

**Lemma 4.** *Let $C, D \subset \mathbf{P}^m$, $m \ge 3$, be integral non-degenerate curves. Assume $C \ne D$. Then a general secant line to $C$ is not secant to $D$.*

*Proof.* Assume that this is not true and fix a general $P \in C$. By assumption for a general $Q \in C$ the line $\langle\{P, Q\}\rangle$ is secant to $D$. Hence the linear projection from $P$ is not birational. Thus a general $P \in C$ is contained in the Segre locus $\Sigma(D)$ of $D$, contradicting [4], Th. 1. □

**Lemma 5.** *Let $X_i \subset \mathbf{P}^5$, $1 \le i \le 2$, be integral surfaces such that $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$ and neither $X_1$ nor $X_2$ is a plane. Then for a general $(A_1, A_2, B_1, B_2) \in X_1 \times X_1 \times X_2 \times X_2$ the set $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap (X_1 \cup X_2)$ is finite.*

*Proof.* Assume that the lemma is false and that for instance $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap (X_1 \cup X_2)$ contains an integral curve $C \subset X_2$. Let $f : \mathbf{P}^5 \backslash \langle\{A_1, A_2\}\rangle \to \mathbf{P}^3$ be the linear projection from the line $\langle\{A_1, A_2\}\rangle$. First assume that $C$ is not contained in a plane containing $\langle\{A_1, A_2\}\rangle$. Thus $f(C \backslash C \cap \langle\{A_1, A_2\}\rangle)$ is a curve. Since $f(C \backslash C \cap \langle\{A_1, A_2\}\rangle)$ is contained in the line $\langle\{f(B_1), f(B_2)\}\rangle$, we obtain $\langle\{f(B_1), f(B_2)\}\rangle \subseteq f(X_2 \backslash X_2 \cap \langle\{A_1, A_2\}\rangle)$. Since $(f(B_1), f(B_2))$ is general in $f(X_2) \times f(X_2)$, $f(X_2)$ is a plane. Thus $\dim(\langle X_2 \cup \{A_1, A_2\}\rangle) = 4$. By the generality of $A_1$ and $A_2$ and the assumption $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$, we obtain that $X_2$ is a plane, a contradiction. Now assume that $C$ is contained in a plane $M$ containing $\langle\{A_1, A_2\}\rangle$. Varying $A_1$ and $A_2$ we obtain that $X_2$ contains at least a two-dimensional family of plane curves. If $\dim(\langle X_2\rangle) \ge 4$, we obtain that $C$ is a plane conic and $X_2$ is either the Veronese surface or a projection of the Veronese surface ([6], Segre's lemma at p. 623). We have $M = \langle C\rangle$, $\{A_1, A_2\} \subset M$ and the scheme-theoretic intersection of $\langle\{A_1, A_2\}\rangle$ with $C$ has length two. Hence any secant line to $X_1$ is secant to $X_2$. Take a general hyperplane $H$ of $\mathbf{P}^5$ and apply Lemma 4 to a general projection of the curves $X_1 \cap H$ and $X_2 \cap H$ in $\mathbf{P}^3$ to obtain a contradiction. Now assume $\dim(\langle X_2\rangle) \le 3$. Hence $\dim(\langle X_2\rangle) = 3$. Since $\langle X_1 \cup X_2\rangle = \mathbf{P}^5$, for general $(A_1, A_2) \in X_1 \times X_1$ we have $\langle\{A_1, A_2\}\rangle \cap \langle X_2\rangle = \emptyset$, and hence $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap X_2$ is contained in $\langle\{B_1, B_2\}\rangle \cap X_2$ and hence it is finite, a contradiction. □

Look at the set-up of Lemma 5. If $X_2$ is a plane, then $\langle\{A_1, A_2, B_1, B_2\}\rangle \cap X_2$ is a line.

*Proof of Theorem 1.* We divide the proof into 9 steps. Steps 1 to 7 are just the translation in our set-up of the corresponding steps in the proof of the Theorem in section 2 of [6]. The degree 3 curve arising in Step 10 of [6] does not appear in our proof of Theorem 1 because the integer 3 is now distributed between $X_1$, $X_2$ and $X_3$. Instead, in our proof of Theorem 1 we obtain a one-dimensional family $\Phi$ of lines contained in $X_3$. Furthermore, in Step 9 we will use again that $X_2 \neq X_3$. Therefore the proof of Theorem 1 is shorter and easier than the proof of the Theorem in [6], §2.

*Step 1.* Taking a general linear projection into $\mathbf{P}^5$ we reduce to the case $N = 5$. By assumption we have $\dim([X_1; X_2]) = 5$, i.e. $[X_1; X_2] = \mathbf{P}^5$. Let $J := \{(\Pi, Q) : Q \in \Pi\} \subset \mathrm{Sec}(\vec{X}; \vec{k}) \times \mathbf{P}^5$ be the incidence variety and $q : J \to \mathrm{Sec}(\vec{X}; \vec{k})$, $p : J \to \mathbf{P}^5$ the projections. We have $\dim(J) = 8$ (see e.g. the proof of [6], Prop. 1.1). Since $[X_1; X_2] = \mathbf{P}^5$, $p$ is surjective. Thus for a general $P \in \mathbf{P}^5$ every irreducible component of $p^{-1}(P)$ has dimension 3. Fix a general $P \in \mathbf{P}^5$ and choose one irreducible component $L_P$ of $p^{-1}(P)$.

*Step 2.* Let $W_P := p(q^{-1}(q(L_P)))$ be the union of all planes belonging to $L_P$. In this step we will check the existence of a choice of the component $L_P$ of $p^{-1}(P)$ such that $W_P$ is an irreducible variety containing $X_3$. $W_P$ is irreducible because $L_P$ is irreducible and $q$ is equidimensional and with irreducible fibers. Since $[X_1; X_2] = \mathbf{P}^5$ and $P$ is general, there are $A \in X_1$ and $B \in X_2$ such that $P \in \langle\{A, B\}\rangle$. Hence for a general $Q \in X_3$ the plane $\langle\{A, B, Q\}\rangle$ belongs to $\mathrm{Sec}(\vec{X}; \vec{k})$ and contains $P$, i.e. $\langle\{A, B, Q\}\rangle \in p^{-1}(P)$. Thus $X_3$ is contained in $p(q^{-1}(q(p^{-1}(P))))$. Since $X_3$ is irreducible, there is at least one irreducible component $L_P$ of $p^{-1}(P)$ such that $X_3 \subseteq p(q^{-1}(q(L_P)))$.

*Step 3.* In order to obtain a contradiction, from now on we assume that $\vec{X}$ is $(1, \vec{k})$-defective. Here we will check that $\dim(W_P) = 4$. Assume $\dim(W_P) = 5$. Then for a general $Q \in \mathbf{P}^5$ there is $\Pi \in L_P$ such that $Q \in \Pi$. Thus the line $\langle\{P, Q\}\rangle$ is contained in $W_P$. By the generality of $P$ and $Q$ we obtain $G(2, 6) = \mathrm{Sec}(\vec{X}; 1, \vec{k})$, a contradiction. Now assume $\dim(W_P) \leq 3$. Since $W_P$ is irreducible and contains $X_3$ (Step 2) and $P \notin X_3$, $\dim(W_P) = 3$. For a general $Q \in X_3$ there is $\Pi \in W_P$ such that $Q \in \Pi$. Thus $\langle\{P, Q\}\rangle \subset \Pi$. Hence $W_P$ is the cone $[X_3; \{P\}]$. Since $W_P$ contains a 3-dimensional family of planes, the projection of $X_3$ from $P$ is a surface $Y$ containing a 3-dimensional family of lines. No such surface $Y$ exists because any two general points of it would be contained in a line contained in $Y$; hence $Y$ would be a plane, while a plane does not contain a 3-dimensional family of lines.

*Step 4.* Choose $A \in X_1$ and $B \in X_2$ such that $P \in \langle\{A, B\}\rangle$. Since $P$ is general, the pair $(A, B)$ is general in $X_1 \times X_2$. From now on we fix a general $(A, B) \in X_1 \times X_2$ and a general $P \in \langle\{A, B\}\rangle$. Let $\Psi$ be the rational map from $X_3$ into $G(3, 6)$ that sends a general $C \in X_3$ into the plane $\langle\{A, B, C\}\rangle \in G(3, 6)$. Call $L'_P$ the closure of $\mathrm{Im}(\Psi)$. Clearly, $L'_P$ is irreducible and by construction it lies in $q(p^{-1}(P))$. We choose as $L_P$ a component of $q(p^{-1}(P))$ containing $L'_P$.

*First Claim:* We have $\dim(L'_P) = 2$, $p(q^{-1}(L'_P)) = W_P = [\{A\}; [\{B\}; X_3]]$. With this choice of $L_P$ we have $X_3 \subseteq W_P$, i.e., the statement of Step 2 holds for this component of $q(p^{-1}(P))$.

*Proof of the First Claim* It is easy to check (see Lemma 2 or Lemma 5 for stronger statements) that $\Psi$ has finite fibers. Hence $\dim(L'_P) = 2$. Since $L'_P \subseteq L_P$, we have $p(q^{-1}(L'_P)) \subseteq W_P$. By the very definition of the rational map $\Psi$ we have $p(q^{-1}(L'_P)) = [\{A\}; [\{B\}; X_3]]$. Hence to prove the First Claim it is sufficient to prove that the cone $[\{A\}; [\{B\}; X_3]]$ has dimension 4, i.e. that $X_3$ is not a cone with vertex containing $B$ and that the vertex of the cone $[\{B\}; X_3]$ does not contain

$A$. Since $X_1$ and $X_2$ are non-degenerate and the pair $(A, B)$ is general in $X_1 \times X_2$, both assertions are obvious.

*Step 5.* For any $\Pi \in L_P \backslash L'_P$ write $\Lambda_\Pi$ for the linear span of $\Pi$ and the line $\langle\{A, B\}\rangle$.

*Second Claim:* $\Lambda_\Pi$ is a 3-dimensional linear space contained in $W_P$, and $W_P$ is the closure of the union of the spaces $\Lambda_\Pi$ as $\Pi$ varies in $L_P \backslash L'_P$. For a general $\Pi \in L_P$ the scheme $\Lambda_\Pi \cap X_3$ contains a curve.

*Proof of the Second Claim* Since $P \in \langle\{A, B\}\rangle \cap \Pi$ and $\Pi \notin L'_P$, we have $\dim(\Lambda_\Pi) = 3$. By the First Claim for a general $\Pi \in L_P$ and a general $Q \in \Pi$ there is $C \in X_3$ such that $Q \in \langle\{A, B, C\}\rangle$. Thus $\langle\{A, B, Q\}\rangle \subseteq W_P$ and hence $\Lambda_\Pi \subseteq W_P$. Since $\Lambda_\Pi \neq \Lambda_{\Pi'}$ for a general pair $(\Pi, \Pi') \in L_p \times L_P$ and $\dim(W_P) = 4$, $W_P$ is the closure of the union of the spaces $\Lambda_\Pi$ as $\Pi$ varies in $L_P \backslash L'_P$. Since $X_3 \subset W_P$, for a general $\Pi$ the set $\Lambda_\Pi \cap X_3$ contains a curve, proving the Second Claim.

*Step 6.* Here we will check that $\Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$. Assume on the contrary that $\Lambda_\Pi \cap \Lambda_{\Pi'}$ is a plane, $V$. By the Linear Lemma in [6], §1, this implies that either all 3-spaces $\Lambda_\Pi$ are contained in a 4-dimensional linear space $M$ or for every $R \in L_P \backslash L'_P$ the 3-space $\Lambda_R$ contains $V$. The first possibility cannot occur because $X_3$ is non-degenerate and contained in $W_P$ (Step 2) and $W_P$ is the closure of the union all $\Lambda_R$ (First Claim). Assume that for every $R \in L_P \backslash L'_P$ the 3-space $\Lambda_R$ contains $V$. The linear projection $\alpha : X_3 \backslash X_3 \cap V \to \mathbf{P}^2$ is dominant because the last assertion of the Second Claim implies that $\alpha$ does not contract infinitely many lines. Hence the linear projection of $X_3$ from the line $\langle\{A, B\}\rangle$ into $\mathbf{P}^3$ is a cone. By the Lemma proved in [6], Step 6 at p. 625, $X_3$ is a cone, a contradiction.

*Step 7.* Here we will check that $\langle\{A, B\}\rangle$ is the only line containing $P$ and intersecting $X_1 \backslash X_1 \cap X_2$ and $X_2 \backslash X_1 \cap X_2$. Since the tangent developable of $X_3$ has dimension 4 and $P$ is general, $P$ is not contained in any line tangent to $X_3$ at one of its smooth points. Since $[X_1; X_2] = \mathbf{P}^5$, the set $\mathcal{D}$ of all lines containing $P$ and intersecting both $X_1 \backslash X_1 \cap X_2$ and $X_2 \backslash X_1 \cap X_2$ is finite. Now we will check that $\mathcal{D} = \{\langle\{A, B\}\rangle\}$. Take any $D \in \mathcal{D}$. By the finiteness of $\mathcal{D}$, $D$ must be fixed as $\Pi$ varies. Hence $D \subseteq \Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$ (Step 6).

*Step 8.* Call $\Gamma_\Pi$ the union of the one-dimensional components of $\Lambda_\Pi \cap X_3$. Here we will check that for general $\Pi$ the curve $\Gamma_\Pi$ is a line. Recall that $W_P$ is the closure of the union of all spaces $\Lambda_\Pi$ with $\Pi \in L_P$. Let $Y \subseteq \mathbf{P}^r$ be an irreducible $m$-dimensional variety, $m \geq 2$, containing a two-dimensional family of $(m-1)$-dimensional linear spaces. By [6], Lemma in Step 9 of §2, $Y$ is a linear space. Thus $W_P$ contains only a one-dimensional family of distinct 3-spaces $\Lambda_\Pi$. Since $\dim(L_P) = 3$ and each plane of $L_P$ belongs to some 3-space $\Lambda_\Pi$ contained in $W_P$, it follows that the general plane $U$ of $\Lambda_\Pi$ containing $P$ intersects $X_1$, $X_2$ and $X_3$ and that, for general $P$, $\Pi$ and $U$, it intersects each $X_i$ exactly at one point (see [6], Cor. 1.3). Hence $\Gamma_\Pi$ is a line. Hence the variety $X_3$ contains an irreducible family $\Phi$ of lines $\Gamma_\Pi$, $\Pi$ general in $L_P$, with $\Gamma_\Pi \subset \Lambda_\Pi$. Since $\Lambda_\Pi \cap \Lambda_{\Pi'} = \langle\{A, B\}\rangle$ for a general pair $(\Pi, \Pi')$ (Step 6), we have $\dim(\Phi) > 0$. Since $X_3$ is not a plane, we have $\dim(\Phi) = 1$. If all lines $\Gamma_\Pi$ pass through a common point $Q$, then $X_3$ is a cone with vertex $Q$, contradicting our assumptions. Since not all lines $\Gamma_\Pi$ pass through a common point and $X_3$ is not a plane, we have $\Gamma_\Pi \cap \Gamma_{\Pi'} = \emptyset$ for a general pair $(\Pi, \Pi')$ (Linear Lemma in [6], §1). We now give a side remark. Since $X_3$ is neither a plane nor a smooth quadric surface, $\Phi$ is the only positive-dimensional irreducible family of lines contained in $X_3$. Hence $\Phi$ does not depend on the choice of $P$, $A$ and $B$. For a general $B_3 \in X_3$ there is a unique line $D(B_3)$ such that $B_3 \in D(B_3) \subset X_3$. Since $\Phi$ covers $X_3$, we have $D(B_3) \in \Phi$. Notice that $D(B_3)$ depends only on $B_3$ and $X_3$, not on $X_1$, $X_2$ and the choices of $A$, $B$ and $P$ that we made to construct $\Phi$.

*Step 9.* Take a general triple $(A_1, A_2, A_3) \in X_1 \times X_2 \times X_3$. Hence $\langle\{A_1, A_2, A_3\}\rangle$ is a plane and a general element of $\mathrm{Sec}(\vec{X}; 2, \vec{k})$, and hence $\langle\{A_1, A_2, A_3\}\rangle \cap X_i = \{A_i\}$ (Lemma 2). A general $P \in \langle\{A_1, A_2, A_3\}\rangle$ may be considered as a general element of $\mathbf{P}^5$ because $[X_1; [X_2; X_3]] = \mathbf{P}^5$. Since $[X_1; X_2] = \mathbf{P}^5$, there is $(B_1, B_2) \in X_1 \times X_2$ such that $B_1 \neq B_2$ and $P \in \langle\{B_1, B_2\}\rangle$; furthermore, there are only finitely many such pairs $(B_1, B_2)$. Conversely, given a general quadruple $(A_1', A_2', B_1', B_2') \in X_1 \times X_1 \times X_2 \times X_2$, the 3-dimensional linear space $\langle\{A_1, A_2, B_1, B_2\}\rangle$ intersects $X_3$ in a non-empty set and one point of this set, for fixed $A_1'$, $A_2'$ but for a general pair $(B_1', B_2')$, may be considered as a general point $A_3'$ of $X_3$; furthermore, for general $A_1'$ and $A_2'$ we may find $A_3'$ not collinear with $A_1'$ and $A_2'$. Hence $\langle\{A_1', A_2', A_3'\}\rangle$ is a plane of $\langle\{A_1, A_2, B_1, B_2\}\rangle$, and thus it intersects the line $\langle\{B_1', B_2'\}\rangle$. Since $\dim(\langle\{A_1, A_2, B_1, B_2\}\rangle) = 3$, for general $B_1'$ and $B_2'$ we have $\langle\{A_1', A_2'\}\rangle \cap \langle\{B_1', B_2'\}\rangle = \emptyset$. Thus we may do the construction of Step 4 starting from $B_1'$, $B_2'$ and $P$ instead of $A$, $B$ and $P$. By construction $\langle\{A_1', A_2', A_3'\}\rangle \in \Lambda_P \backslash \Lambda_{P'}$. Thus by Step 8 the 3-dimensional linear space $G$ spanned by $B_1'$, $B_2'$ and $\langle\{A_1', A_2', A_3'\}\rangle$ intersects $X_3$ in a line $D(A_1', B_1', A_2', B_2') \in \Phi$. However, $G = \langle\{A_1', A_2', B_1', B_2'\}\rangle$. Thus for a general quadruple $(A_1', A_2', B_1', B_2') \in X_1 \times X_1 \times X_2 \times X_2$ the set $X_3 \cap \langle\{A_1, A_2, B_1, B_2\}\rangle$ contains a line. Furthermore, $\langle\{A_1, A_2, B_1, B_2\}\rangle$ is spanned by $D(A_1', B_1', A_2', B_2')$, $A_1'$ and $A_2'$. Since $\dim(\Phi) = 1$, we may find a one-dimensional irreducible family, $\Delta$, of pairs $(B_1'', B_2'') \in X_1 \times X_2$ such that $X_3 \cap \langle\{A_1, A_2, B_1, B_2\}\rangle = D(A_1', B_1', A_2', B_2')$ for every $(B_1'', B_2'') \in \Delta$. Hence $\langle\{A_1', A_2', B_1', B_2'\}\rangle$ contains all such pairs $(B_1'', B_2'')$, contradicting Lemma 5 and hence concluding the proof.

*Proof of Theorem 2.* If $X \subset \mathbf{P}^m$, $m \geq 5$, is a non-degenerate surface such that $\dim([X; X]) = 4$, then $X$ is either a cone or a Veronese surface ([7]). Notice that the role of the surface $X_3$ in the proof of Theorem 1 was quite different from the roles of $X_1$ and $X_2$, while the roles of $X_1$ and $X_2$ were exactly the same. The same proof works in the case $X_1 = X_2$ (and proves Theorem 2), except that in Steps 4 and 9 we need to quote Lemma 3 instead of Lemma 2 and that in the proof of Theorem 1 we assumed and heavily used that $\dim([X_1; X_2]) = 5$. Hence to complete the proof of Theorem 2 it is sufficient to check it when $\dim([X_1; X_1]) \leq 4$ and $X_1$ is not a cone, i.e. when $X_1$ is the Veronese surface. Assume $N = 5$ and let $S \subset \mathbf{P}^5$ be the Veronese surface. Let $Y \subset \mathbf{P}^5$ be an integral non-degenerate surface with $Y \neq S$ and $Y$ not a cone. The pair $(S, Y)$ is not a $(1, (2, 1))$-defective pair if and only if a general line $D \subset \mathbf{P}^5$ is contained in a plane spanned by two points of $S$ and one point of $Y$. Let $Z \subset \mathbf{P}^5$ be the secant variety of $S$. Thus $S$ is the hypersurface of $\mathbf{P}^5$ union all planes spanned by the conics contained in $S$. Let $D \subset \mathbf{P}^5$ be a general line. Fix $P \in D \cap Z$ and call $E \subset Z$ the plane such that $P \in E$ and $E \cap S = C$, where $C$ is a smooth conic. Set $M := \langle E \cup D\rangle$. Thus $\dim(M) = 3$. Take $Q \in M \cap Y$ and set $F := E \cap \langle\{Q\} \cup D\rangle$. Hence $F$ is the intersection of two planes contained in $M$. The linear space $M$ moves if we move $D$ among the lines through $P$. For a general line $D$ containing $P$ the set $F$ is a line not tangent to $C$.

Hence the general line $D$ containing $P$ is contained in the plane spanned by $Q$ and the two points of $F \cap C \subset S$, concluding the proof. $\square$

## 3. Further results on $(t, \vec{k})$-defectivity

Let $\sigma : \mathbf{P}^t \times \mathbf{P}^N \to \mathbf{P}^{tN+t+N}$ be the Segre embedding. The proof of [8], Th. 2.1 (i.e. a computation of a certain Jacobian matrix), gives the following result.

**Theorem 3.** *$\vec{X}$ is $(t, \vec{k})$-defective if and only if $\sigma(\mathbf{P}^t \times \vec{X}) := (\sigma(\mathbf{P}^t \times X_1), \dots, \sigma(\mathbf{P}^t \times X_s))$ is $\vec{k}$-defective and the total order of $(t, \vec{k})$-defectivity of $\vec{X}$ and the $\vec{k}$-defectivity of $\sigma(\mathbf{P}^t \times \vec{X})$ are the same.*

To obtain results for the $(t, \vec{k})$-defectivity it is essential to have results for the $\vec{k}$-defectivity of other joins; Theorem 3 is just one reason. Since the role of the varieties $X_1, \dots, X_s$ is not the same if $k_i \neq k_j$ for some $i, j$ or the geometric properties (and even the dimensions) of the varieties $X_1, \dots, X_s$ may be quite different (as, for instance, in Question 1), we introduce the following definition.

**Definition 1.** Fix $N, s, k_i, n_i, 1 \leq i \leq s$, and $t$ as above and assume that $\sum_{i=1}^{s} k_i(n_i + 1) \leq N - 1$. Fix an integer $i$ with $1 \leq i \leq s$ such that $k_i > 0$. Set $\vec{k}(i) := (k'_1, \dots, k'_s)$ with $k'_j = k_j$ if $j \neq i$ and $k'_i = k_i - 1$. Set $\delta(\vec{X}; \vec{k}; i) := \delta(\vec{X}; \vec{k}) - \delta(\vec{X}; \vec{k}(i))$. The integer $\delta(\vec{X}; \vec{k}; i)$ will be called the $\vec{k}$-defect of $\vec{X}$ for the factor $i$.

The proof of [2], Th. 1.1, gives the following result.

**Lemma 6.** *Fix $N$, $\vec{k}$, $i$ and $\vec{X}$ as above and assume that each variety $X_j$ is non-degenerate. Assume $\delta(\vec{X}; \vec{k}; i) > 0$. Fix a general $P_{u,v} \in X_u$, $1 \leq u \leq s$ and $1 \leq v \leq k_u$, and let $A$ be the linear span of the tangent spaces $(TX_u)_{P_{u,v}}$, $1 \leq u \leq s$ and $1 \leq v \leq k_u$. Then the general hyperplane $H$ of $\mathbf{P}^N$ containing $A$ is tangent to $X_i$ at least along an irreducible variety of dimension $\delta(\vec{X}; \vec{k}; i)$ containing one of the points $P_{i,j}$ with $1 \leq j \leq k_i$.*

**Corollary 1.** *Fix $N$, $\vec{k}$, $i$ and $\vec{X}$ as above and assume that each variety $X_j$ is non-degenerate. Assume $\delta(\vec{X}; \vec{k}; i) > 0$. Then $\dim(X_i) > \delta(\vec{X}; \vec{k}; i)$ and in particular $X_i$ is not a curve.*

*Remark* 3. In Lemma 6 and Corollary 1 we may substitute the condition that each variety $X_j$ is non-degenerate with the condition that for each proper subspace $M$ of $\mathbf{P}^N$ containing some of the varieties $X_j$, say $X_j$ for $j \in S \subset \{1, \dots, s\}$, we have $\sum_{j \in S} k_j(\dim(X_j) + 1) \leq \dim(M)$.

**Theorem 4.** *Fix an integer $s \geq 2$, positive integers $k_1, \dots, k_s$ such that $k_s = 1$ and integral subvarieties $X_i \subset \mathbf{P}^N$ such that $X_j$ is non-degenerate for every $j < s$ and $\dim(X_s) = 1$. Set $\vec{k} := (k_1, \dots, k_s)$, $\vec{k}(s) := (k_1, \dots, k_{s-1})$, $\vec{X} := (X_1, \dots, X_s)$ and $\vec{X}(s) := (X_1, \dots, X_s)$. Assume that $\vec{X}(s)$ is neither $\vec{k}(s)$-defective nor $(1, \vec{k}(s))$-defective and that $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$ is not a cone with vertex containing $X_s$. Then $\vec{X}$ is not $(1, \vec{k})$-defective.*

*Proof.* Assume that $\vec{X}$ is $(1, \vec{k})$-defective. A general line $L \in \mathrm{Sec}(\vec{X}; 1, \vec{k})$ is obtained by taking general $P_{i,j} \in X_i$, $1 \leq i \leq s$, $1 \leq j \leq k_i$, and then taking a general line $L$ contained in the $|\vec{k}|$-dimensional linear space spanned by the points $P_{i,j}$. Let

$A(s)$ be the join of all points $P_{i,j}$ with $1 \le i \le s-1$ and $1 \le j \le k_i$. Set $Q(L) := A(s) \cap L$. Hence $Q(L) \in \mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By the generality of the points $P_{i,j}$ and of the line $L$ the point $Q(L)$ may be considered as a general point of $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By assumption there is a one-dimensional family of $|\vec{k}|$-planes, say $\{\Pi_t\}_{t \in T}$ with $T$ irreducible curve such that each $\Pi_t$ intersects each $X_i$ at $k_i$ distinct points, say $P_{i,j}(t)$, $L \subset \Pi_t$ for every $t$, and there is $o \in T$ such that $P_{i,j}(o) = P_{i,j}$ for all $i, j$. Let $A(s,t)$ be the join of all points $P_{i,j}(t)$ with $1 \le i \le s-1$ and $1 \le j \le k_i$. Set $Q(L,t) := A(s,t) \cap L$. Hence $Q(L,t) \in A(s,t)$. First assume $Q(L,t) = Q(L)$ for every $t$. Since $Q(L,t) := A(s,t) \cap L$, this implies that $Q(L)$ is contained in infinitely many $(|\vec{k}| - 1)$-dimensional linear spaces belonging to $\mathrm{Sec}(\vec{X}(s); |\vec{k}(s)| - 1, \vec{k}(s))$. Since $Q(L)$ is general in $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$, this implies $\dim(\mathrm{Sec}(\vec{X}(s); \vec{k}(s)))) \le\le \sum_{i=1}^{s-1} k_i(\dim(X_i) + 1) - 2$, a contradiction. If $Q(L,t) \neq Q(L)$ for general $t \in T$, then $L$ is contained in $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$. By the generality of $L$ we obtain $\mathrm{Sec}(\vec{X}; \vec{k}) = \mathrm{Sec}(\vec{X}(s); \vec{k}(s))$, i.e. $\mathrm{Sec}(\vec{X}(s); \vec{k}(s))$ is a cone with vertex containing $X_s$, a contradiction. □

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