

## FILLING ANALYTIC SETS BY THE DERIVATIVES OF $C^1$ -SMOOTH BUMPS

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ABSTRACT. If  $X$  is an infinite-dimensional Banach space, with separable dual, and  $M \subset X^*$  is an analytic set such that any point  $x^* \in M$  can be reached from 0 by a continuous path contained (except for the point  $x^*$ ) in the interior of  $M$ , then  $M$  is the range of the derivative of a  $C^1$ -smooth function on  $X$  with bounded nonempty support.

### 1. INTRODUCTION

A real-valued function  $f$  defined on a Banach space  $X$  is called a *bump* provided that its *support*  $\text{supp } f := \{x \in X : f(x) \neq 0\}$  is bounded and nonempty. We say that  $f$  is  $C^1$ -smooth if it is (Fréchet) differentiable at each point of  $X$  and its derivative  $f'$  is continuous on  $X$ . If  $X$  is finite-dimensional, then the range  $f'(X)$  of  $f'$  is automatically closed. If  $X$  is separable (infinite-dimensional), then  $f'(X)$  is, by the very definition, an analytic set. So a natural question arises, *whether, any analytic set in a separable dual space  $X^*$ , satisfying some reasonable additional conditions, can be expressed as the range of the derivative of a  $C^1$ -smooth bump on  $X$ .* The aim of this note is to answer this question affirmatively.

This result has the following (pre)history. D. Azagra and R. Deville, encouraged by S. Bates's pioneering paper [6], filled, by the derivatives of a suitable  $C^1$ -smooth bump, all of the dual  $X^*$  [1]. This was then further elaborated in [2]. There, higher-order derivatives of mappings between  $X$  and another Banach space  $Y$ , and also the Gâteaux derivatives were considered. In [7], the problem of exact filling of various closed sets in finite-dimensional spaces by the derivatives of  $C^1$ -smooth bumps was studied. [8] is a continuation of this in infinite dimensions, with focus on open and closed sets. [3] is a kind of a common roof over [2], [7], and [8]. Finally, when the proof of our theorem was done, we learned that T. Gaspari [10] had succeeded to fill in convex analytic sets by derivatives of  $C^1$ -smooth bumps.

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For more information about the ranges of the derivatives of bumps we refer to a survey paper of D. Azagra and M. Jiménez-Sevilla [4].

The main result of this note is the following theorem:

**Theorem.** *Let  $X$  be an infinite-dimensional Banach space with separable dual  $X^*$ . Let  $M \subset X^*$  be an analytic set such that for every  $\xi \in M$  there is a continuous mapping  $\phi: [0, 1] \rightarrow X^*$ , with  $\phi(0) = 0$ ,  $\phi(1) = \xi$ , and  $\phi([0, 1]) \subset \text{int } M$ . Then there exists a  $C^1$ -smooth bump  $b: X \rightarrow \mathbb{R}$  such that  $b'(X) = M$ .*

It is natural to ask whether a converse to our Theorem holds. However, a negative answer follows from [5], where the authors construct a  $C^1$ -smooth bump on  $\ell_2$  such that the range of its gradient has empty interior. When adding this bump with another suitable  $C^1$ -smooth bump, with support disjoint from the support of the first bump, we get a  $C^1$ -smooth bump  $b: X \rightarrow \mathbb{R}$ , with  $\text{int } b'(X) \neq \emptyset$ , and yet satisfying  $b'(X) \setminus \overline{\text{int } b'(X)} \neq \emptyset$ . In Example 2 below, we show that  $b'(X)$  need not satisfy the assumptions of our Theorem even if  $b'(X) \subset \overline{\text{int } b'(X)}$  and  $\text{int } b'(X)$  is connected. Recently, the third author showed that  $\text{int } b'(X)$  need not be connected. This result will be contained in another paper. Nonetheless, it seems to be unknown whether for every  $C^1$ -smooth bump  $b: X \rightarrow \mathbb{R}$  the set  $M = b'(X)$  satisfies the assumptions of our Theorem provided that  $M$  is closed. That our Theorem does not hold (for closed sets) in finite dimensions was shown by T. Gaspari [10]; see Example 3.

## 2. PROOF OF THEOREM

Before proving our Theorem itself we fix some notation and give three lemmas.

Let  $(X, \|\cdot\|)$  be a Banach space. The symbol  $B_X$  denotes its closed unit ball centered at the origin. Given  $u, v \in X$ , we put  $[u, v] = \{\alpha u + (1 - \alpha)v : \alpha \in [0, 1]\}$ . By a *path* in  $X$  we understand any set  $C = \varphi([0, 1])$  where  $\varphi: [0, 1] \rightarrow X$  is a continuous mapping. We then put  $C(0) = \varphi(0)$  and  $C(1) = \varphi(1)$ .

The first two lemmas are basic bricks for building suitable  $C^1$  smooth bumps.

**Lemma 1** ([8, Lemma 2]). *Let  $X$  be a Banach space, with separable dual  $X^*$ , let  $\varepsilon > 0$ , and let  $\xi \in X^*$  be given. Then there exists a  $C^1$ -smooth bump  $f: X \rightarrow [0, 1]$  such that  $\text{supp } f \subset B_X$ ,  $f'(X) \subset [0, \xi] + \varepsilon B_{X^*}$ , and  $(f')^{-1}(\xi)$  is a neighbourhood of 0.*

For another proof of the above Lemma, see [2] or [3, Lemma 2.1].

**Lemma 2.** *Let  $X$  be a Banach space, with separable dual, let  $C \subset X^*$  be a path, with  $C(0) = 0$ , and let  $\varepsilon > 0$  be given. Then there exists a  $C^1$ -smooth bump  $f: X \rightarrow [0, 1]$  such that  $\text{supp } f \subset B_X$ ,  $f'(X) \subset C + \varepsilon B_{X^*}$  and  $(f')^{-1}(C(1))$  is a neighbourhood of 0.*

*Proof.* First, we shall show the following statement: *For every  $n \in \mathbb{N}$ , every  $\Delta > 0$ , and every  $\xi_1, \dots, \xi_n \in X^*$  there exists a  $C^1$ -smooth bump  $f: X \rightarrow [0, 1]$  such that  $\text{supp } f \subset B_X$ ,  $f'(X) \subset ([0, \xi_1] \cup [\xi_1, \xi_2] \cup \dots \cup [\xi_{n-1}, \xi_n]) + \Delta B_{X^*}$  and  $(f')^{-1}(\xi_n)$  is a neighbourhood of 0.* For  $n = 1$  the statement is reduced to Lemma 1. Assume that we verified the statement for some  $n \in \mathbb{N}$ . Let  $\Delta > 0$  and  $\xi_1, \dots, \xi_n, \xi_{n+1} \in X^*$  be given. Find a  $C^1$ -smooth bump  $g: X \rightarrow [0, +\infty)$  such that  $\text{supp } g \subset B_X$ ,  $g'(X) \subset ([0, \xi_1] \cup [\xi_1, \xi_2] \cup \dots \cup [\xi_{n-1}, \xi_n]) + \Delta B_{X^*}$ , and  $(g')^{-1}(\xi_n)$  is a neighbourhood of 0. By Lemma 1, find a  $C^1$ -smooth bump  $h: X \rightarrow [0, +\infty)$  such that  $\text{supp } h \subset (g')^{-1}(\xi_n) \cap$

$B_X, h'(X) \subset [0, \xi_{n+1} - \xi_n] + \Delta B_{X^*}$ , and  $(h')^{-1}(\xi_{n+1} - \xi_n)$  is a neighbourhood of 0. Then put  $f = g + h$ ; this is a  $C^1$ -smooth bump with  $\text{supp } f \subset B_X$ . Take any  $x \in X$ . If  $x \notin \text{supp } h$ , then  $f'(x) = g'(x) \in ([0, \xi_1] \cup \dots \cup [\xi_{n-1}, \xi_n] \cup [\xi_n, \xi_{n+1}]) + \Delta B_{X^*}$ . If  $x \in \text{supp } h$ , then  $f'(x) = \xi_n + h'(x) \subset \xi_n + [0, \xi_{n+1} - \xi_n] + \Delta B_{X^*} = [\xi_n, \xi_{n+1}] + \Delta B_{X^*} \subset ([0, \xi_1] \cup \dots \cup [\xi_n, \xi_{n+1}]) + \Delta B_{X^*}$ . Also, if  $\|x\|$  is small enough, then  $f'(x) = g'(x) + h'(x) = \xi_n + \xi_{n+1} - \xi_n = \xi_{n+1}$ . Finally, replacing  $f$  by  $\frac{1}{\lambda}f(\lambda \cdot)$ , with a suitable  $\lambda > 0$ , we can force that  $f(X) \subset [0, 1]$ .

Having the above statement verified, we can complete the proof. Write  $C = \varphi([0, 1])$ , with  $\varphi: [0, 1] \rightarrow X$  continuous. Then  $\varphi$  is uniformly continuous, and hence there is  $n \in \mathbb{N}$  so large that  $\|\varphi(t) - \varphi(t')\| < \frac{\varepsilon}{2}$  whenever  $t, t' \in [0, 1]$  and  $|t - t'| < \frac{1}{n}$ . By the statement, find the bump  $f$  for  $\Delta = \frac{\varepsilon}{2}$  and for  $\xi_1 = \varphi(\frac{1}{n}), \dots, \xi_{n-1} = \varphi(\frac{n-1}{n-1}), \xi_n = \varphi(1)$ . Then

$$f'(X) \subset ([\varphi(0), \varphi(\frac{1}{n})] \cup \dots \cup [\varphi(\frac{n-1}{n-1}), \varphi(1)]) + \frac{\varepsilon}{2}B_{X^*} \subset C + \varepsilon B_{X^*}. \quad \square$$

The lemma below is a key statement. It enables us to use the analytic structure of  $M$ . To formulate it we need some more notation.

The set of positive integers is denoted by  $\mathbb{N}$ . We denote by  $\mathbb{N}^{<\mathbb{N}}$  the set of all finite sequences of elements of  $\mathbb{N}$ . We assume that  $\mathbb{N} \subset \mathbb{N}^{<\mathbb{N}}$ . The empty sequence is denoted by  $\emptyset$ ; thus  $\emptyset \in \mathbb{N}^{<\mathbb{N}}$ . The symbol  $|s|$  means the “length” of a sequence  $s \in \mathbb{N}^{<\mathbb{N}}$ . If  $s \in \mathbb{N}^{<\mathbb{N}}$  and  $n < |s|$ , or  $s \in \mathbb{N}^{\mathbb{N}}$ , and  $n = 0, 1, 2, \dots$ , then  $s|_n$  denotes the finite sequence formed by the first  $n$  elements of the sequence  $s$ . For  $s \in \mathbb{N}^{<\mathbb{N}}$  and  $t \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$  the relation  $s \prec t$  means that  $s$  is a “beginning” of  $t$ .

**Lemma 3.** *Let  $X$  and  $M$  be as in our Theorem. Then there are points  $y_s \in \text{int } M$  and paths  $C_s \subset \text{int } M, s \in \mathbb{N}^{<\mathbb{N}}$ , such that the following conditions are fulfilled:*

- (1)  $y_\emptyset = 0$  and  $C_\emptyset = \{0\}$ ;
- (2)  $C_s(0) = y_{s|(|s|-1)}, C_s(1) = y_s$  for every  $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ ;
- (3) for every  $\xi \in M$  there is  $\sigma \in \mathbb{N}^{\mathbb{N}}$  such that  $\|y_{\sigma|n} - \xi\| \rightarrow 0$ ;
- (4) for every  $\sigma \in \mathbb{N}^{\mathbb{N}}$  there is  $\xi \in M$  such that  $\|y_{\sigma|n} - \xi\| \rightarrow 0$  (this is a crucial property);
- (5) the diameter of  $C_s$  is less than  $2^{-|s|+1}$  for every  $s \in \mathbb{N}^{<\mathbb{N}} \setminus \mathbb{N}$ .

*Proof.* From the analyticity of  $M$ , we find a (countable) system  $F_s, s \in \mathbb{N}^{<\mathbb{N}}$ , of nonempty closed subsets of  $\overline{M}$  such that  $M = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} F_{\sigma|n}$ , and such that  $F_s \subset F_t$  whenever  $s, t \in \mathbb{N}^{<\mathbb{N}}$  and  $s \succ t$ ; see, e.g., [9, Chapter III, §35.II].

By induction, we shall construct index sets  $\mathcal{S}_k \subset \mathbb{N}^k \times \mathbb{N}^k, k = 0, 1, 2, \dots$ , and open sets  $G_{s,t} \subset X^*, (s, t) \in \mathcal{S} := \bigcup_{k=0}^{\infty} \mathcal{S}_k$  in such a way that they satisfy:

- (i)  $\mathcal{S}_0 = \{(\emptyset, \emptyset)\}, G_{\emptyset, \emptyset} = X^*$ ;
- (ii)  $\mathcal{S}_k = \{(s^\wedge m, t^\wedge n) : (s, t) \in \mathcal{S}_{k-1}, F_{s^\wedge m} \cap G_{s,t} \neq \emptyset, m, n \in \mathbb{N}\}$  for every  $k \in \mathbb{N}$ ;
- (iii) the diameter of  $G_{s,t}$  is less than  $2^{-|s|}$  for every  $(s, t) \in \mathcal{S} \setminus \{(\emptyset, \emptyset)\}$ ;
- (iv)  $G_{s,t} \cap F_s \neq \emptyset$  for every  $(s, t) \in \mathcal{S}$ ;
- (v)  $\overline{G_{s^\wedge m, t^\wedge n}} \subset G_{s,t}$  whenever  $(s, t) \in \mathcal{S}, m, n \in \mathbb{N}$ , and  $(s^\wedge m, t^\wedge n) \in \mathcal{S}$ ;
- (vi) if  $(s, t) \in \mathcal{S}, m \in \mathbb{N}$ , and  $F_{s^\wedge m} \cap G_{s,t} \neq \emptyset$ , then

$$F_{s^\wedge m} \cap G_{s,t} \subset \bigcup_{n=1}^{\infty} G_{s^\wedge m, t^\wedge n}.$$

The construction starts by fulfilling (i) and then continues by induction. Let  $k \in \mathbb{N}$  be arbitrary fixed and assume that we have already constructed  $\mathcal{S}_{k-1}$  and  $G_{s,t}$  for every  $(s, t) \in \mathcal{S}_{k-1}$ . Define  $\mathcal{S}_k$  as in (ii). Now fix any  $(s, t) \in \mathcal{S}_{k-1}$  and any  $m \in \mathbb{N}$  such that  $F_{s^\wedge m} \cap G_{s,t} \neq \emptyset$ . As  $X^*$  is a separable metric space (and hence hereditarily Lindelöf), we can find a sequence of open sets  $G_{s^\wedge m, t^\wedge n} \subset X^*$ ,  $n \in \mathbb{N}$ , so that  $\text{diam } G_{s^\wedge m, t^\wedge n} < 2^{-k}$ ,  $F_{s^\wedge m} \cap G_{s^\wedge m, t^\wedge n} \neq \emptyset$ ,  $\overline{G_{s^\wedge m, t^\wedge n}} \subset G_{s,t}$  for every  $n \in \mathbb{N}$ , and  $F_{s^\wedge m} \cap G_{s,t} \subset \bigcup_{n=1}^\infty G_{s^\wedge m, t^\wedge n}$ . This completes the induction step.

Let  $0 \in D \subset \text{int } M$  be a countable set, dense in  $\text{int } M$ . We shall construct index sets  $\mathcal{T}_k \subset \mathbb{N}^k \times \mathbb{N}^k \times \mathbb{N}^k$ ,  $k = 0, 1, \dots$ , and for every  $(s, t, u) \in \mathcal{T} := \bigcup_{k=0}^\infty \mathcal{T}_k$  points  $\tilde{y}_{s,t,u} \in D$  and paths  $\tilde{C}_{s,t,u}$  in  $\text{int } M$ . Put  $\mathcal{T}_0 = \{(\emptyset, \emptyset, \emptyset)\}$ ,  $\tilde{y}_{\emptyset, \emptyset, \emptyset} = 0 \in X^*$ , and  $\tilde{C}_{\emptyset, \emptyset, \emptyset} = \{0\} \subset X^*$ . Let  $k \in \mathbb{N}$  be arbitrary fixed, and suppose we have defined  $\mathcal{T}_{k-1}$  and  $\tilde{y}_{s,t,u}, \tilde{C}_{s,t,u}$  for every  $(s, t, u) \in \mathcal{T}_{k-1}$ . Moreover, suppose that  $(s, t) \in \mathcal{S}_{k-1}$  whenever  $(s, t, u) \in \mathcal{T}_{k-1}$ . For  $(s, t, u) \in \mathcal{T}_{k-1}$  and for  $m, n \in \mathbb{N}$  such that  $(s^\wedge m, t^\wedge n) \in \mathcal{S}_k$ , put

$$D_{s,t,u}^{m,n} = \{ \xi \in D \cap G_{s^\wedge m, t^\wedge n} : \tilde{y}_{s,t,u} = C(0), \xi = C(1) \text{ for a path } C \subset \text{int } M \cap G_{s,t} \}.$$

Clearly, this set is either empty or countable. In the second case, we enumerate it as  $D_{s,t,u}^{m,n} = \{ \tilde{y}_{s^\wedge m, t^\wedge n, u^\wedge l} : l \in \mathbb{N} \}$ . Let  $\tilde{C}_{s^\wedge m, t^\wedge n, u^\wedge l}$  denote a path  $C$  corresponding to the point  $\tilde{y}_{s^\wedge m, t^\wedge n, u^\wedge l}$ . Further, put

$$\mathcal{T}_k = \{ (s^\wedge m, t^\wedge n, u^\wedge l) : (s, t, u) \in \mathcal{T}_{k-1}, (s^\wedge m, t^\wedge n) \in \mathcal{S}_k, D_{s,t,u}^{m,n} \neq \emptyset, m, n, l \in \mathbb{N} \},$$

$$[\mathcal{T}] = \{ (\sigma, \tau, v) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : \forall k \in \mathbb{N} (\sigma|k, \tau|k, v|k) \in \mathcal{T}_k \},$$

and

$$\mathcal{U} = \{ (\sigma|k, \tau|k, v|k) : (\sigma, \tau, v) \in [\mathcal{T}], k \in \mathbb{N} \} \cup \{ (\emptyset, \emptyset, \emptyset) \}.$$

We claim that *the set  $[\mathcal{T}]$  is large enough, i.e., for every  $\xi \in M$  there is  $(\sigma, \tau, v) \in [\mathcal{T}]$  such that  $\tilde{y}_{\sigma|k, \tau|k, v|k}$  converge to  $\xi$  as  $k \rightarrow \infty$* . Indeed, let  $\xi \in M$  be arbitrary. Then there is some  $\sigma \in \mathbb{N}^{\mathbb{N}}$  such that  $\xi \in \bigcap_{k=1}^\infty F_{\sigma|k}$ . Then  $(\emptyset, \emptyset) \in \mathcal{S}_0$ ,  $\xi \in F_{\sigma|1} \cap G_{\emptyset, \emptyset}$ , and (vi) yields  $t_1 \in \mathbb{N}$  such that  $(\sigma|1, (t_1)) \in \mathcal{S}_1$  and  $\xi \in G_{\sigma|1, (t_1)}$ . Thus  $\xi \in F_{\sigma|2} \cap G_{\sigma|1, (t_1)}$  and (vi) yields  $t_2 \in \mathbb{N}$  such that  $(\sigma|2, (t_1, t_2)) \in \mathcal{S}_2$ . For a general  $k \in \mathbb{N}$ , if we already found  $t_1, \dots, t_k \in \mathbb{N}$  such that  $(\sigma|k, (t_1, \dots, t_k)) \in \mathcal{S}_k$  and  $\xi \in F_{\sigma|(k+1)} \cap G_{\sigma|k, (t_1, \dots, t_k)}$ , (vi) yields  $t_{k+1} \in \mathbb{N}$  such that  $(\sigma|(k+1), (t_1, \dots, t_k, t_{k+1})) \in \mathcal{S}_{k+1}$  and  $\xi \in G_{\sigma|(k+1), (t_1, \dots, t_k, t_{k+1})}$ . Performing this for every  $k \in \mathbb{N}$ , it is clear that there is  $\tau \in \mathbb{N}^{\mathbb{N}}$  such that for every  $k \in \mathbb{N}$  we have  $(\sigma|k, \tau|k) \in \mathcal{S}_k$  and  $\xi \in G_{\sigma|k, \tau|k}$ . Let  $\varphi: [0, 1] \rightarrow M$  be a continuous mapping such that  $\varphi(0) = 0$ ,  $\varphi(1) = \xi$ , and  $\varphi([0, 1]) \subset \text{int } M$ . For every  $k \in \mathbb{N}$  find  $r_k \in (0, 1)$  so that  $\varphi([r_k, 1]) \subset G_{\sigma|k, \tau|k}$ . By a microscopic change of  $\varphi$ , we can reach that  $\varphi(r_k) \in D$  for every  $k \in \mathbb{N}$ . Thus, by the definition of  $\mathcal{T}$ , there is  $v \in \mathbb{N}^{\mathbb{N}}$  such that for every  $k \in \mathbb{N}$  we have  $(\sigma|k, \tau|k, v|k) \in \mathcal{T}_k$  and  $\varphi(r_k) = \tilde{y}_{\sigma|k, \tau|k, v|k}$ . This proves the Claim.

Further, fix any  $(\sigma, \tau, v) \in [\mathcal{T}]$ . For every  $k \in \mathbb{N}$ ,  $k \geq 2$ , the set

$$\bigcup \{ \tilde{C}_{s,t,u} : (s, t, u) \in \mathcal{U}, \sigma|k \prec s, \tau|k \prec t, v|k \prec u \}$$

is contained in  $G_{\sigma|(k-1), \tau|(k-1)}$ , and hence has diameter less than  $2^{-k+1}$ . In particular, using this and the completeness of  $X^*$ , we have that  $\tilde{y}_{\sigma|k, \tau|k, v|k}$  converges to some  $\xi \in X^*$  whenever  $(\sigma, \tau, v) \in [\mathcal{T}]$ . Further,  $\tilde{y}_{\sigma|k, \tau|k, v|k} \in G_{\sigma|k, \tau|k}$ ; hence, by (iii) and (iv), the distance of  $\tilde{y}_{\sigma|k, \tau|k, v|k}$  from  $F_{\sigma|l}$  is at most  $2^{-k}$  for every  $k, l \in \mathbb{N}$ ,  $k \geq l$ .

It follows that  $\xi \in F_{\sigma|l}$  for every  $l \in \mathbb{N}$ , and therefore  $\xi \in \bigcap_{l=1}^{\infty} F_{\sigma|l} \subset M$ . Thus we proved that for every  $(\sigma, \tau, \nu) \in [T]$  the sequence  $(\tilde{y}_{\sigma|k, \tau|k, \nu|k})_{k \in \mathbb{N}}$  converges to an element of the set  $M$ .

Now, we are going to relabel the points  $\tilde{y}_{s,t,u}$  and the paths  $\tilde{C}_{s,t,u}$ , and thus we shall obtain the points  $y_s \in D$  and paths  $C_s \subset \text{int } M$ ,  $s \in \mathbb{N}^{<\mathbb{N}}$ . We define a bijection  $\psi: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathcal{U}$  as follows. Put  $\psi(\emptyset) = (\emptyset, \emptyset, \emptyset)$ . Take any  $r \in \mathbb{N}^{<\mathbb{N}}$  and assume that  $\psi(r) \in \mathcal{U}$  was already defined. Write  $\psi(r) = (s, t, u)$ . From the above construction, it follows that the set  $\{(s^\wedge m, t^\wedge n, u^\wedge l) : m, n, l \in \mathbb{N}\} \cap \mathcal{U}$  is infinite. We enumerate it as  $\{\psi(r^\wedge i) : i \in \mathbb{N}\}$ . Having  $\psi$  thus defined, for every  $r \in \mathbb{N}^{<\mathbb{N}}$  we put  $y_r = \tilde{y}_{\psi(r)}$  and  $C_r = \tilde{C}_{\psi(r)}$  for  $r \in \mathbb{N}^{<\mathbb{N}}$ . From the above paragraphs we can easily deduce that these families have all the required properties (1) – (5).  $\square$

*Remark.* If for every  $(s, t) \in \mathcal{S}$  we can choose the set  $G_{s,t}$  in such a way that  $G_{s,t} \cap \text{int } M$  is connected, then the argument can be simplified. In particular, considering the “third” index  $u$  is not necessary, and joining the points  $y_{\sigma|k}$ ,  $k \in \mathbb{N}$ , by paths causes no problems. This happens, for instance, if  $\text{int } M$  is convex.

*Proof of the Theorem.* Let  $(y_s : s \in \mathbb{N}^{<\mathbb{N}})$  and  $(C_s : s \in \mathbb{N}^{<\mathbb{N}})$  be the families found in Lemma 3. For every  $s \in \mathbb{N}^{<\mathbb{N}}$  and every  $n \in \mathbb{N}$  choose  $0 < \varepsilon_{s^\wedge n} < 2^{-|s|+1}$  such that  $C_{s^\wedge n} + \varepsilon_{s^\wedge n} B_{X^*} \subset \text{int } M$ . By Lemma 2, there is a  $C^1$ -smooth bump  $h_{s^\wedge n}: X \rightarrow [0, 1]$  satisfying the following conditions:

- (a)  $h'_{s^\wedge n}(X) \subset C_{s^\wedge n} - y_s + \varepsilon_{s^\wedge n} B_{X^*}$ ;
- (b)  $(h'_{s^\wedge n})^{-1}(y_{s^\wedge n} - y_s)$  is a neighborhood of 0;
- (c)  $\text{supp } h_{s^\wedge n} \subset B_X$ .

The desired bump  $b$  will be a sum of suitably translated and scaled functions  $h_{s^\wedge n}$ . We shall define the translation and the scaling by induction.

We start by putting  $r_\emptyset = 1$ ,  $z_\emptyset = 0 \in X$  and finding an infinite  $\frac{1}{4}$ -discrete set  $\{z_n : n \in \mathbb{N}\}$  in  $\frac{1}{2} B_X$ . Then put  $g_n(x) = \frac{1}{16} h_n(16(x - z_n))$ ,  $x \in X$ , for every  $n \in \mathbb{N}$ .

Next, suppose that for some  $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$  we have already defined a  $C^1$ -smooth bump  $g_s: X \rightarrow [0, 1]$  such that  $(g'_s)^{-1}(y_s - y_{s|(|s|-1)})$  has nonempty interior. Then the latter set contains an open ball  $B(u_s, r_s)$  with centre  $u_s \in X$  and radius  $0 < r_s < 2^{-|s|}$ . Choose an infinite  $\frac{1}{4}r_s$ -discrete set  $\{z_{s^\wedge n} : n \in \mathbb{N}\}$  in  $B(u_s, \frac{1}{2}r_s)$ , and put

$$g_{s^\wedge n}(x) = \frac{r_s}{16} h_{s^\wedge n}\left(\frac{16}{r_s}(x - z_{s^\wedge n})\right), \quad x \in X,$$

for every  $n \in \mathbb{N}$ . By the construction,  $g_{s^\wedge n}$  is a  $C^1$ -smooth bump,  $|g_{s^\wedge n}(x)| \leq \frac{1}{16 \cdot 2^{|s|}}$  for every  $x \in X$ , and  $g'_{s^\wedge n}(X) = h'_{s^\wedge n}(X)$ . Also, by (5) and (a),  $\|g'_{s^\wedge n}(x)\| < 2^{-|s|+2}$  whenever  $|s| \geq 1$  and  $x \in X$ . Moreover,

$$\text{supp } g_{s^\wedge n} \subset B\left(z_{s^\wedge n}, \frac{r_s}{16}\right) \subset B\left(u_s, \frac{9}{16}r_s\right) \subset (g'_s)^{-1}(y_s - y_{s|(|s|-1)}).$$

Hence, for every  $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ , the family of sets  $\{\text{supp } g_{s^\wedge n} : n \in \mathbb{N}\}$  is discrete, i.e., every  $x \in X$  has a neighbourhood that intersects  $\text{supp } g_{s^\wedge n}$  for at most one  $n \in \mathbb{N}$ .

Having this information and the fact that  $\text{supp } g_{s^\wedge n} \subset \text{supp } g_s$  at hand, it follows easily by induction that  $\{\text{supp } g_s : s \in \mathbb{N}^{<\mathbb{N}}, |s| = n\}$  is a discrete family for every

$n \in \mathbb{N}$ . Therefore, if for  $n \in \mathbb{N}$  we define

$$f_n(x) = \sum_{|s|=n} g_s(x), \quad x \in X,$$

we obtain a  $C^1$ -smooth bump satisfying  $f'_n(x) = \sum_{|s|=n} g'_s(x)$  for every  $x \in X$  and  $f'_n(x) = g'_s(x) = y_s - y_{s|(s-1)}$  whenever  $s \in \mathbb{N}^n$  and  $x \in B(u_s, r_s)$ .

Finally, put

$$b(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in X.$$

As  $|f_n(x)| \leq 2^{-n-4}$  and  $\|f'_n(x)\| < 2^{-n+2}$  for every  $x \in X$  and every  $n \geq 2$ , the function  $b$  is a well-defined  $C^1$ -smooth bump and  $b'(x) = \sum_{n=1}^{\infty} f'_n(x)$  for every  $x \in X$ .

We observe that  $b'(X) \subset M$ . Indeed, by the construction, given any  $x \in X$ , then: either there is  $K \in \mathbb{N}$  so that  $b'(x) = \sum_{n=1}^K f'_n(x)$  and there is  $s \in \mathbb{N}^{<\mathbb{N}}$  such that  $|s| = K$ ,  $x \in \text{supp } g_s$ , and  $b'(x) = \sum_{n=1}^{K-1} (y_{s|n} - y_{s|(n-1)}) + g'_s(x) \in \text{int } M$ , by the choice of  $\varepsilon_s$ ; or there is  $\sigma \in \mathbb{N}^{\mathbb{N}}$  such that  $x \in \bigcap_{n \in \mathbb{N}} \text{supp } g_{\sigma|n} \subset \bigcap_{n \in \mathbb{N}} B(u_{\sigma|n}, r_{\sigma|n})$  and hence  $b'(x) = \sum_{n=1}^{\infty} (y_{\sigma|n} - y_{\sigma|(n-1)}) = \lim_{n \rightarrow \infty} y_{\sigma|n} \in M$ , by (4).

In order to show that  $M \subset \overline{b'(X)}$ , let  $\xi \in M$  be arbitrary. Find  $\sigma \in \mathbb{N}^{\mathbb{N}}$  by (3). Then, since  $B(u_{\sigma|n}, r_{\sigma|n}) \supset \overline{B(u_{\sigma|(n+1)}, r_{\sigma|(n+1)})}$  and  $r_{\sigma|n} < 2^{-n}$  for every  $n \in \mathbb{N}$ , these balls have a one-point intersection,  $x \in X$ , say. Thus

$$b'(x) = \sum_{n=1}^{\infty} (y_{\sigma|n} - y_{\sigma|(n-1)}) = \lim_{n \rightarrow \infty} y_{\sigma|n} = \xi.$$

This completes the proof of our Theorem. □

### 3. EXAMPLES AND REMARKS

In the first example we shall show that our Theorem is false if the property of the set  $M$  is weakened a little. In fact, the construction works in any at least two-dimensional Banach space.

**Example 1 (Comb).** *If  $X$  is any Banach space, of dimension at least two, then there exists a closed pathwise connected subset  $M \subset X^*$  such that  $0 \in \text{int } M$  and  $\text{int } M$  is connected and dense in  $M$ , but  $M$  is the range of the derivative of no  $C^1$ -smooth bump on  $X$ .*

*Proof.* Put

$$N = [-1, 0] \times [-1, 1] \cup [0, 1] \times \left( \left[-1, \frac{1}{2}\right] \cup \left[\frac{2}{3}, \frac{3}{4}\right] \cup \left[\frac{4}{5}, \frac{5}{6}\right] \cup \dots \cup \{1\} \right).$$

This set is equal to the closure of its interior and  $(0, 0)$  belongs to its interior. Moreover, it is easy to check that  $N$  is pathwise connected and  $\text{int } N$  is connected. Let  $X$  be any Banach space, with dimension at least 2. Then we can write  $X = \mathbb{R}^2 \times Y$ , with a suitable (possibly trivial) subspace  $Y \subset X$ . Then put

$$M = N \times B_{Y^*};$$

this set is equal to the closure of its interior and the origin belongs to its interior. Clearly,  $\text{int } M$  is connected and every point of  $M$  can be reached by a continuous path in  $M$  beginning at the origin (i.e.,  $M$  is pathwise connected). However, the paths corresponding to the points of the form  $((t, 1), \eta)$ ,  $t \in (0, 1]$ ,  $\eta \in B_{Y^*}$ , do not always go through  $\text{int } M$ , as is required in our Theorem. We shall show that

$M = f'(X)$  for no  $C^1$ -smooth bump  $f: X \rightarrow \mathbb{R}$ . Assume that there exists such an  $f$ . Consider  $\xi = ((1, 1), 0) \in M$  and take the direction  $h = ((0, 1), 0) \in X$ . By the proof of [7, Theorem 2], we find  $x \in X$  and a sequence  $(z_n)_{n \in \mathbb{N}}$  in  $X$  so that  $f'(x) = \xi$ ,  $z_n \rightarrow x$ , and  $f'(z_n)(h) \neq \xi(h) = 1$  for every  $n \in \mathbb{N}$ . Consider the neighbourhood  $U = (\frac{2}{3}, \frac{4}{3}) \times (\frac{2}{3}, \frac{4}{3}) \times B_{Y^*}$  of  $\xi$ . From the continuity of  $f'$ , we find a convex neighbourhood  $V$  of  $x$  so that  $f'(V) \subset U$ . From the above, we find  $z \in V$  such that  $f'(z)(h) \neq 1$ . Then, as  $f'(z) \in U \cap M$ , there is  $m \in \mathbb{N}$  such that  $f'(z) \in [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}$ . Define

$$\varphi(t) = f'(tx + (1 - t)z), \quad t \in [0, 1];$$

this is a continuous function, and  $\varphi(t) \in f'(V) \subset U \cap M$  for every  $t \in [0, 1]$ . Put

$$t_0 = \sup \{t \in [0, 1] : \varphi(t) \in [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}\}.$$

Clearly,  $t_0 < 1$ . Now, for every  $t \in (t_0, 1]$  we have  $\varphi(t) \notin [0, 1] \times [\frac{m}{m+1}, \frac{m+1}{m+2}] \times B_{Y^*}$ . Hence,  $\varphi(t)$ , with such  $t$ , belongs either to  $(\frac{2}{3}, 1] \times \{1\} \times B_{Y^*}$  or to  $(\frac{2}{3}, 1] \times [\frac{k}{k+1}, \frac{k+1}{k+2}] \times B_{Y^*}$  for some  $k \neq m$ . Thus  $\|\varphi(t_0) - \varphi(t)\| \geq \frac{m+2}{m+3} - \frac{m+1}{m+2} > 0$  for every  $t \in (t_0, 1]$ , which contradicts the continuity of  $\varphi$  at  $t = t_0$ .  $\square$

The next example shows that even a weak converse to our Theorem is false in  $\ell_2$ .

**Example 2.** *There is a  $C^1$ -smooth bump  $b: \ell_2 \rightarrow \mathbb{R}$  such that the set  $M = b'(\ell_2)$  does not satisfy the assumptions of our Theorem and yet  $0 \in \text{int } M$ ,  $M \subset \overline{\text{int } M}$  and  $\text{int } M$  is connected.*

*Proof.* In [5], a  $C^1$ -smooth bump  $b_1: \ell_2 \rightarrow \mathbb{R}$  is constructed such that

$$b'_1(\ell_2) \subset Z := \{0\} \cup \{(z_n) \in \ell_2 : z_n \neq 0 \text{ for infinitely many } n \in \mathbb{N}\}$$

(see the last two paragraphs of the proof of Theorem 2.2 and Corollary 2.3 in [5]).

Fix any nonzero  $\xi \in b'_1(\ell_2)$ , put  $r = \|\xi\|/2$ , and choose  $n_0 \in \mathbb{N}$ , with  $n_0 > \frac{1}{r}$ . Put

$$\begin{aligned} F_0 &= (\{0\} \cup \{\pm \frac{1}{n} : n > n_0\}) \times [-r, r], \\ N_0 &= \mathbb{R}^2 \setminus F_0, \\ N' &= N_0 \times \ell_2 \subset \ell_2. \end{aligned}$$

Then  $N'$  is an open connected subset of  $\ell_2$ , dense in  $\ell_2$  such that  $0 \notin N'$  and there is no continuous  $\varphi: [0, 1] \rightarrow \ell_2$  with  $\varphi(1) = 0$  and  $\varphi([0, 1)) \subset N'$ . Put  $N = N' + \xi$ . Then  $N$  is an open connected set in  $\ell_2$  containing 0, and hence by our Theorem (in fact already by [8] or [10]) there is a  $C^1$ -smooth bump  $b_2: \ell_2 \rightarrow \mathbb{R}$  with  $b'_2(\ell_2) = N$ . Up to a translation, we may assume that  $\text{supp } b_1 \cap \text{supp } b_2 = \emptyset$ . Then put  $b = b_1 + b_2$ . Then  $b$  is a  $C^1$ -smooth bump such that  $b'(\ell_2) = N \cup b'_1(\ell_2)$ . We claim that  $\text{int } b'(\ell_2) = N$ . The inclusion  $\supset$  is clear. Let us show the reverse one. Suppose that there is some  $x = (x_n) \in \text{int } b'(\ell_2) \setminus N$ . Then necessarily  $x$  is a relative interior point in  $H$  of  $b'_1(\ell_2) \cap H$ , where  $H = \{(z_n) \in \ell_2 : z_1 = x_1, |z_2| \leq r\}$ . However,  $Z \cap H$  has empty interior in  $H$ , a contradiction.

So,  $\text{int } b'(\ell_2) = N$  is connected and dense in  $\ell_2$  and, moreover, there is no path  $\varphi: [0, 1] \rightarrow \ell_2$  with  $\varphi(1) = \xi$  and  $\varphi([0, 1)) \subset \text{int } b'(\ell_2)$ . This completes the proof.  $\square$

The last example is from [10] and shows that our Theorem does not hold in finite dimensions, even for closed sets. (Note that the range of the gradient of any

$C^1$ -smooth bump on a finite-dimensional space is closed, due to the compactness of closed balls.)

**Example 3.** *If  $n \in \mathbb{N}$ ,  $n \geq 2$ , then there exists a closed subset  $M \subset \mathbb{R}^n$  that is the range of the gradient of no  $C^1$ -smooth bump on  $\mathbb{R}^n$ , and yet for every  $\xi \in M$  there is a continuous path from 0 to  $\xi$  through the points of  $\text{int } M$ . Moreover, these paths can have uniformly bounded lengths.*

*Proof.* Put

$$N_1 = N \cup [-1, 1] \times [1, 2], \quad \text{and} \quad M = N_1 \times [-1, 1]^{n-2},$$

where  $N$  is from Example 1. (If  $n = 2$ , put  $M = N_1$ .) This set has all the required properties. Yet there is no  $C^1$ -smooth bump  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  with  $f'(X) = M$ . Indeed, if there were such an  $f$ , then  $M$  would be a continuous image of a compact locally connected space (a sufficiently large closed ball in  $\mathbb{R}^n$ ), and hence  $M$  itself would be locally connected. However, it is obvious that  $M$  is not locally connected.  $\square$

Next we shall formulate an open problem which seems to be essential for understanding what kind of property a set needs to have “to be the range of the derivative of a  $C^1$ -smooth bump”.

**Problem.** *Let  $X$  be a Banach space with separable dual and  $M \subset X^*$  the range of the derivative of a  $C^1$ -smooth bump on  $X$ . Suppose that  $h: X^* \rightarrow X^*$  is a homeomorphism (or even a linear isomorphism) of  $X^*$  onto  $X^*$  and  $0 \in \text{int } h(M)$ . Is then  $h(M)$  also the range of the derivative of a  $C^1$ -smooth bump?*

Let us remark that the answer is positive if  $h = L^*$  when  $L$  is any linear isomorphism of  $X$  onto  $X$ . It can also be easily checked that  $M - \xi$  is the range of the derivative of a  $C^1$ -smooth bump whenever  $M$  has this property,  $0, \xi \in \text{int } M$  and  $\xi$  can be reached from 0 by a path contained in  $\text{int } M$ .

*Remark.* We are convinced that our method (Lemma 3) can be used for several “analytic” statements which may hold. We refer the reader to [3] for more fantasy on how to formulate and prove such statements.

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