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NASH EQUIDIMENSIONALITY THEOREM

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ABSTRACT. Consider a Nash mapping of Nash subsets. After a finite number of Nash blowings-up, the Nash mapping induced from it has equidimensional fibers. The purpose of this short note is to show this Nash equidimensionality theorem.

1. Introduction

Parusinski proved the following local equidimensionality theorem.

Theorem 1.1 ([2]). *Let $f : X \to M$ be a morphism of real analytic spaces, and assume that M is nonsingular. Let L and K be compact subsets of X and M, respectively. Then there exist a finite number of analytic morphisms $s_\alpha : W_\alpha \to M$ such that*

(1) *each s_α is the composition of a finite sequence of real local blowings-up with smooth nowhere dense centers;*
(2) *for each α there exists a compact subset K_α of W_α such that $\bigcup_\alpha s_\alpha(K_\alpha) = K$;*
(3) *the strict transforms $\tilde{f}_\alpha : \tilde{X}_\alpha \to \tilde{W}_\alpha$ of a complexification of f by complexifications of s_α satisfy, at every point $x \in \tilde{X}_\alpha$ corresponding to L, the equidimensionality property:*

$$\dim_{\mathbb{C}}(\tilde{f}_\alpha^{-1}(\tilde{f}_\alpha(x)) \cap \tilde{X}'_\alpha, x) = \dim_{\mathbb{C}}(\tilde{X}_\alpha, x) - \dim_{\mathbb{R}} M,$$

for every irreducible component $\tilde{X}'_\alpha$ of $\tilde{X}_\alpha$ at x.

We prove the global equidimensionality theorem for any real closed field in the present paper only when f is a Nash mapping of Nash sets. Roughly speaking, our global equidimensionality theorem claims that there exists a composition of a finite sequence of blowings-up whose centers are Nash sets such that the strict transforms of f satisfy the equidimensionality property. We need some definitions to state our result truly. We define Nash blowings-up in Section 2 and describe our result with accuracy in Section 3.

Throughout the present paper, R denotes a real closed field.

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2. Nash blowings-up

A Nash submanifold M of R^n is a semialgebraic subset of R^n that is simultaneously a C^∞ submanifold of R^n, and a Nash function on M is a semialgebraic C^∞ function on M. Let $\mathcal{N}(M)$ denote the ring of Nash functions on M. Remember that a *Nash subset* of M is the zero set of a Nash function on M. Let X and Y be Nash subsets of Nash manifolds. A C^∞ semialgebraic mapping $f : X \to Y$ is called a *Nash mapping*. For a Nash subset X of a Nash submanifold M of some Euclidean space, we also call the zero set of a Nash function on X a *Nash subset of* X. Remark that a Nash subset of X is also a Nash subset of M by [1, Corollary 13, Corollary 15].

We define $\mathcal{N}_M^{\mathrm{sa}}$ as the sheaf of Nash functions on M for the semialgebraic topology. Set $\mathcal{N}_X^{\mathrm{sa}} := \mathcal{N}_M^{\mathrm{sa}}/\mathcal{I}_X$, where $\mathcal{I}_X$ denotes the sheaf of ideals of $\mathcal{N}_M^{\mathrm{sa}}$ vanishing on X.

We will define the blowing-up for a general real closed field.

Definition 2.1. Let X be as above, and let $\mathcal{I}$ be a finite sheaf of ideals of $\mathcal{N}_X^{\mathrm{sa}}$ with support $\neq X$. A Nash mapping $\sigma : X' \to X$ is called the Nash blowing-up of X with center $\mathcal{I}$ if the following conditions are satisfied:

(1) $\mathcal{I}\mathcal{N}_{X'}^{\mathrm{sa}}$ is invertible;
(2) for any Nash mapping $f : T \to X$ such that $\mathcal{I}\mathcal{N}_T^{\mathrm{sa}}$ is invertible, there exists only one Nash mapping $f' : T \to X'$ with $\sigma \circ f' = f$.

We can define the *strict transform* of a Nash set by a Nash blowing-up in the same way as Hironaka's definition.

Proposition 2.2. *There exists a Nash blowing-up $\sigma : X' \to X$ for any finite sheaf of ideals of $\mathcal{N}_X^{sa}$ with support $\neq X$.*

Proof. The sheaf $\mathcal{I}$ is generated by $g_1, \dots, g_m \in \mathcal{N}(M)$ by [1, Corollary 13, Corollary 15]. Set

$$Z = \bigcap_{i=1}^{m} g_i^{-1}(0) \cap X \text{ and}$$
$$X'' = \{(x,s) \in X \times \mathbb{P}^{m-1}(R); x \notin Z, g_i(x)s_j = g_j(x)s_i \text{ for all } i \neq j\},$$

where $\mathbb{P}^{m-1}(R)$ denotes the $(m-1)$-dimensional projective space. Define X' as the closure of X'' in $X \times P^{m-1}(R)$ and $\sigma : X' \to X$ as the natural projection. Then σ obviously satisfies the first condition of the above definition.

We show that σ satisfies the second condition. Let $T \to X$ be a Nash mapping such that $(g_1 \circ f, \dots, g_m \circ f)\mathcal{N}_T^{\mathrm{sa}}$ is invertible. We have an open semialgebraic covering $\{T_i\}_{i=1,\dots,m}$ of T such that $g_i \circ f$ generate $(g_1 \circ f, \dots, g_m \circ f)\mathcal{N}_T^{\mathrm{sa}}|_{T_i}$. We have only to show the case when $g_1 \circ f$ generates $(g_1 \circ f, \dots, g_m \circ f)\mathcal{N}_T^{\mathrm{sa}}$ by the definition of Nash blowings-up. Define f_j' as the Nash function on T with $g_j \circ f = f_j' g_1 \circ f$ for all $j \neq 1$. Then the Nash mapping $f' : T \to X'$ defined by $f'(t) = (f(t), (1 : f_2'(t) : \ldots : f_n'(t)))$ satisfies the equation $\sigma \circ f' = f$.

We show the uniqueness of f'. Let $f'' : T \to X'$ be another Nash mapping satisfying $\sigma \circ f'' = f$. Since $g_i \circ f$ generates $(g_1 \circ f, \dots, g_m \circ f)\mathcal{N}_T^{\mathrm{sa}}|_{T_i}$, $f''(T_i)$ is contained in $X_i' = \{(x,s) \in X'; s_i \neq 0\}$. It is easy to see that $f''(t)$ is determined only by $\sigma \circ f''(t) = f(t)$ on each X_i' by the definition of X'. We have finished the proof of uniqueness. □

Proposition 2.3. *Let $f : X \to Y$ be Nash mappings between Nash subsets, and let $\mathcal{J}$ be a finite sheaf of ideals of $\mathcal{N}_Y^{sa}$. Then there exist a finite sheaf $\mathcal{I}$ of ideals of $\mathcal{N}_X^{sa}$ and a Nash mapping $f' : X' \to Y'$ with $f \circ \sigma_X = \sigma_Y \circ f'$. Here $\sigma_X : X' \to X$ and $\sigma_Y : Y' \to Y$ denote Nash blowings-up with centers $\mathcal{I}$ and $\mathcal{J}$, respectively.*

Proof. By the definition of a Nash subset, a Nash subset Y is the zero set of a Nash function on some Nash manifold N. Let $h_1, \dots, h_m \in \mathcal{N}(N)$ be generators of $\mathcal{J}$. Define $\mathcal{I}$ as the sheaf of ideals of $\mathcal{N}_X^{\mathrm{sa}}$ generated by $h_1 \circ f, \dots, h_m \circ f$. As shown in the proof of Proposition 2.2, X' and Y' are subsets of $X \times \mathbb{P}^{m-1}(R)$ and $Y \times \mathbb{P}^{m-1}(R)$, respectively. Consider the restriction of $(f, \mathrm{id}) : X \times \mathbb{P}^{m-1}(R) \to Y \times \mathbb{P}^{m-1}(R)$ to X'. It is easy to see that the image $(f, \mathrm{id})(X')$ is contained in Y'. Hence this restriction $f' = (f, \mathrm{id})|_{X'} : X' \to Y'$ satisfies the requirement. $\square$

3. Nash equidimensionality theorem

The following lemma was proved in [3]. We will give the proof here for completeness.

Lemma 3.1. *Let Z be an algebraic subset of R^n, and let $p : R^n \to R^m$ (resp. $q : R^n \to R^{n-m-1}$) be the projection forgetting the last $n - m$ factors (resp. the first $m + 1$ factors). Assume $n > m$, and set $k = n - m - 1$. Assume furthermore that Z does not have an irreducible component of the form*

$$R^{m+1} \times X.$$

Then there exist $\tau_1, \dots, \tau_k \in R[y]$ such that the polynomial map defined by

$$\tau(x, y, z) = (x, y, z_1 + \tau_1(y), \dots, z_k + \tau_k(y))$$

satisfies that $(p, q) \circ \tau|_Z$ is a finite-to-one mapping. Here $x = (x_1, \dots, x_m)$, y and $z = (z_1, \dots, z_k)$ denote the coordinate functions of R^m, R and R^k, respectively.

Proof. Let $g \in R[x, y, z]$ be a polynomial function with $g^{-1}(0) = Z$. We choose $\tau'_j \in R^k$ ($j \in \mathbb{N} \cup \{0\}$) satisfying the following condition. For any $j \in \mathbb{N}$, there exists $i > j$ with

$$(*) \quad \bigcap_{l=0}^{i} \{(x, y, z) \in R^n; g(x, l, z - \tau'_l) = 0\} \neq \bigcap_{l=0}^{j} \{(x, y, z) \in R^n; g(x, l, z - \tau'_l) = 0\}$$

if the latter set is not empty.

Set $\tau'_0 = 0$ first. Choose $i > j$ such that the zero set of $g(x^i, i, z) \in R[z]$ is neither empty nor the entire space R^k for some $x^i \in R^m$. We can in fact choose such a positive integer i and variable x^i, for otherwise Z must have an irreducible component of the form $R^{m+1} \times X$, which contradicts the assumption. Choose $v^i, w^i \in R^k$ with $g(x^i, i, v^i) \neq 0$ and $g(x^i, l, w^i) = 0$ for all $0 \leq l \leq j$. We have only to set $\tau'_i = w_i - v_i$ and choose arbitrary $\tau'_l \in R^k$ for all $j < l < i$. Then the constructed numbers $\tau'_i \in R^k$ satisfy the expected inequality $(*)$.

Let $J \in \mathbb{N}$ satisfy

$$\bigcap_{l=0}^{J} \{(x, y, z) \in R^n; g(x, l, z - \tau'_l) = 0\} = \emptyset.$$

There exists $\tilde{\tau} = (\tau_1, \dots, \tau_k) \in R[y]^k$ with $\tilde{\tau}(l) = \tau'_l$ for any $0 \leq l \leq J$.

Set $W = \{(x,z) \in R^{n-1}; \dim(p,q)^{-1}(x,y) \cap \tau(Z) = 1\}$, where τ is the mapping defined by

$$\tau(x,y,z) = (x, y, z_1 + \tau_1(y), \ldots, z_k + \tau_k(y)).$$

The set W is contained in the intersection

$$\bigcap_{l=0}^{\infty} \{(x,y,z) \in R^n; g(x,l,z-\tau_l') = 0\} = \emptyset.$$

Hence $(p,q) \circ \tau$ is finite-to-one. □

Lemma 3.2. *Let Z be an algebraic subset of R^n, and let $p : R^n \to R^m$ be the projection forgetting the last $n-m$ factors. Set*

$$s = m + \max\{\dim(p^{-1}(x) \cap Z); x \in R^m\}.$$

Then there exists a Nash mapping $\pi : R^n \to R^s$ such that $\pi|_Z$ is finite-to-one and $p' \circ \pi = p$, where p' denotes the projection of R^s forgetting the last $s-m$ factors.

Proof. We prove this lemma by the induction on $n-s$.

When $s = n$, this lemma is clear.

Assume that $s < n$. Changing the last $n-m$ coordinate functions linearly, we may suppose that Z satisfies the assumption of Lemma 3.1. We can therefore choose the Nash mapping $r : R^n \to R^{n-1}$ such that $p'' \circ r = p$ and $r|_Z$ is finite-to-one by Lemma 3.1, where p'' denotes the projection forgetting the last $n-m-1$ factors. The image $r(Z)$ is a semialgebraic set by the Tarski-Seidenberg Principle. Let Z' be the Zariski closure of $r(Z)$. Then, by the induction hypothesis, there exists a Nash mapping $\tilde{\pi} : R^{n-1} \to R^s$ such that $\tilde{\pi}|_{Z'}$ is finite-to-one and $p' \circ \tilde{\pi}$ coincides with the projection of R^{n-1} forgetting the last $n-1-m$ factors. The Nash mapping $\pi = \tilde{\pi} \circ r$ satisfies the requirement. □

Theorem 3.3. *Let $f : X \to Y$ be a Nash mapping of Nash subsets. Then there exist finite compositions of Nash blowings-up $\sigma_X : \tilde{X} \to X$ and $\sigma_Y : \tilde{Y} \to Y$ such that the induced Nash mapping $\tilde{f} : \tilde{X} \to \tilde{Y}$ with $\sigma_Y \circ \tilde{f} = f \circ \sigma_X$ has constant dimensional fibers.*

Proof. We first consider that X and Y are contained as semialgebraic sets in Nash submanifolds $M \subset R^{n-m}$ and $N \subset R^m$, respectively. Set $u = \min\{\dim f^{-1}(y); y \in Y\}$ and $v = \max\{\dim f^{-1}(y); y \in Y\}$. We prove the theorem by the induction on $v-u$. We have nothing to prove when $u = v$. Consider the case when $u < v$. Assume that there exists a finite sheaf $\mathcal{J}$ of ideals of $\mathcal{N}_Y^{\mathrm{sa}}$ such that the induced Nash mapping $f' : X' \to Y'$ given in Proposition 2.3 satisfies the inequality $\max\{(f')^{-1}(y); y \in Y\} < v$. Then the theorem holds true by the induction hypothesis. Hence, we have only to show that such a finite sheaf $\mathcal{J}$ exists.

Consider the Zariski closure Z in R^n of the graph of the Nash mapping f. Let $p : R^n \to R^m$ denote the projection forgetting the last $n-m$ factors. Apply Lemma 3.2 to Z and p. There exists a Nash mapping $\pi : R^n \to R^{m+v}$ such that $\pi|_Z$ is finite-to-one and $p' \circ \pi = p$, where p' denotes the projection of R^{m+v} forgetting the last v factors. Let Z' be the Zariski closure of the semialgebraic set $\pi(Z)$. If the claim is true for Z' and $p'|_{Z'}$, then it is also true for X and f. Hence we may assume that

- $X \subset R^n$ is an algebraic set,
- Y is a Euclidean space,

- f is the restriction of the projection $p : R^n \to R^m$ to X and
- $v = n - m$.

Choose a polynomial function $F \in R[x_1, \ldots, x_m, y_1, \ldots, y_v]$ with $F^{-1}(0) = X$. Consider the expansion

$$F = \sum_{(\alpha_1,\ldots,\alpha_v)\in(\mathbb{N}\cup\{0\})^v} a_\alpha(x)y^\alpha,$$

where $a_\alpha \in R[x_1, \ldots, x_m]$ and y^α denotes the monomial $y_1^{\alpha_1} \cdots y_v^{\alpha_v}$. Let $\mathcal{J}$ be the sheaf of ideals of $\mathcal{N}_Y^{\text{sa}}$ generated by a_α. Consider the induced mapping $f' : X' \to Y'$. Let $\{Y'_\alpha\}_\alpha$ be a semialgebraic open covering of Y' such that a_α generates $\mathcal{J}\mathcal{N}_{X'}^{\text{sa}}$. There exists a Nash function $F'_\alpha \in \mathcal{N}_Y(Y'_\alpha)[y_1, \ldots, y_v]$ such that $a_\alpha F'_\alpha = F$. We define p as the projection of R^n forgetting the last v factors. Especially, $p^{-1}(y) \cap (F'_\alpha)^{-1}(0)$ is of dimension $< v$ for all $y \in Y'_\alpha$. The blow-up $X' \cap p^{-1}(Y'_\alpha)$ is contained in the zero set $(F'_\alpha)^{-1}(0)$ by the definition. We have finished the proof of the claim. $\square$

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