

A NOTE ON PRINCIPAL PARTS ON PROJECTIVE SPACE AND LINEAR REPRESENTATIONS

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ABSTRACT. Let H be a closed subgroup of a linear algebraic group G defined over a field of characteristic zero. There is an equivalence of categories between the category of linear finite-dimensional representations of H , and the category of finite rank G -homogeneous vector bundles on G/H . In this paper we will study this correspondence for the sheaves of principal parts on projective space, and we describe the representation corresponding to the principal parts of a line bundle on projective space.

1. INTRODUCTION

In this note we will study the vector bundles of principal parts $\mathcal{P}^k(\mathcal{O}(n))$ of a line bundle on projective space over a field F of characteristic zero from a representation theoretic point of view. We consider projective N -space as a quotient $SL(V)/P$, where V is an $(N + 1)$ -dimensional vector space over F , and P is the subgroup of $SL(V)$ stabilizing a line L in V . There is an equivalence of categories between the category of finite rank $SL(V)$ -homogeneous vector bundles on $SL(V)/P = \mathbf{P}(V^*)$ and the category of linear finite-dimensional representations of P . The principal parts $\mathcal{P}^k(\mathcal{O}(n))$ are $SL(V)$ -homogeneous vector bundles on $\mathbf{P}(V^*)$, and the novelty of this note is that we describe the P -representation corresponding to the principal parts. The main result is Theorem 2.4, which says the following: Let L^* be the dual of the P -module L . Then for all $1 \leq k < n$, the P -representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$. As a corollary, we obtain the splitting type of $\mathcal{P}^k(\mathcal{O}(n))$ on $\mathbf{P}(V^*)$ for all $1 \leq k < n$, and recover results obtained in [6], [7], [8] and [9].

2. PRINCIPAL PARTS ON PROJECTIVE SPACE

In this section we give the representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ on $\mathbf{P}(V^*)$ for all $1 \leq k < n$, where V is an F -vector space of dimension $N + 1$ and F is a field of characteristic 0. A variety is an integral scheme of finite type over F .

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We will consider closed points when we talk about points of a scheme. Let V be a finite-dimensional vector space over F . We let $GL(V)$ denote the group of all invertible linear transformations of V . It is an algebraic group in the sense of [2], Chapt. 1. A *linear algebraic group* is a closed subgroup of $GL(V)$. Let $SL(V)$ be the linear algebraic group of linear transformations of V with determinant 1. Let L in V be a line, and P the closed subgroup of $SL(V)$ stabilizing L . Then the quotient $SL(V)/P$ (which exists by [2], Theorem 6.8) is isomorphic to $\mathbf{P}(V^*)$, the projective space of lines in V (see [1], Section 4.2). This works over any field, not only the complex numbers. There exists a natural left $SL(V)$ -action on $\mathbf{P}(V^*)$, making it into a *homogeneous space* for $SL(V)$. Also by [1], Chapt. 4, there exists an equivalence of categories between the category of finite rank homogeneous vector bundles on $SL(V)/P$ and the category of linear finite-dimensional representations of P , and under this correspondence the dimension of the representation gives the rank of the corresponding vector bundle. Hence any character of P gives a homogeneous line bundle on $SL(V)/P$. The line L corresponds to a character of P , and the bundle corresponding to the dual line L^* is the line bundle $\mathcal{O}(1)$ on $\mathbf{P}(V^*)$ (see [1], Section 4.2). It is also a standard fact that any linear finite-dimensional representation ρ of P lifting to a representation $\tilde{\rho}$ of $SL(V)$ corresponds to a trivial abstract vector bundle on $\mathbf{P}(V^*)$. There exists on any scheme an equivalence of categories between the category of locally free finite rank sheaves and the category of finite rank vector bundles; hence we will use these two notions interchangeably.

Pick a basis e_0, \dots, e_N for V . Let x_0, \dots, x_N be the dual basis, and let L be the line spanned by e_0 . Having chosen a basis for V , it follows that $SL(V)$ may be identified with the group of square rank $N + 1$ matrices with determinant equal to 1. The group P may be identified with the subgroup of $SL(V)$ consisting of matrices g of the form

$$g = \begin{pmatrix} a & * & \cdots & * \\ 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

The one-dimensional representation $\chi : P \rightarrow GL(S^n(L^*))$ corresponding to the line bundle $\mathcal{O}(n)$ is given by $\chi(g) = a^{-n}$.

Let X be a smooth variety of dimension d and consider the diagonal Δ in $X \times X$. Let \mathcal{I} be the sheaf of ideals of $\mathcal{O}_{X \times X}$ defining the diagonal Δ , and define \mathcal{O}_{Δ^k} to be $\mathcal{O}_{X \times X} / \mathcal{I}^{k+1}$.

Definition 2.1. Let p, q be the projection maps from $X \times X$ to X , and let \mathcal{E} be an \mathcal{O}_X -module. Define $\mathcal{P}^k(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{E})$ to be the k th order principal parts of the module \mathcal{E} . We put $\mathcal{P}^k(\mathcal{O}_X) = \mathcal{P}^k$.

Note that by [6], if the rank of \mathcal{E} is e , $\mathcal{P}^k(\mathcal{E})$ is a vector bundle of rank $e \binom{d+k}{d}$ on X . Assume that G is an algebraic group, and that X is a homogeneous space for G . Assume furthermore that \mathcal{E} is a G -homogeneous vector bundle on X ; then it follows that $\mathcal{P}^k(\mathcal{E})$ is again a G -homogeneous vector bundle on X . Consider the line bundle $\mathcal{O}(n)$ on $\mathbf{P}(V^*) = SL(V)/P$; then $\mathcal{O}(n)$ is an $SL(V)$ -homogeneous line bundle on $\mathbf{P}(V^*)$ for all n , and we may consider the $SL(V)$ -homogeneous vector bundle $\mathcal{P}^k(\mathcal{O}(n))$. We want to compute the representation ρ of P corresponding to the homogeneous vector bundle $\mathcal{P}^k(\mathcal{O}(n))$ for all $1 \leq k < n$ on $\mathbf{P}(V^*)$. Let, in the following, $X = \mathbf{P}(V^*)$ and consider the projection maps p, q from $X \times X$ to

X . Let \mathcal{I} in $\mathcal{O}_{X \times X}$ be the ideal of the diagonal. We have an exact sequence of $SL(V)$ -homogeneous vector bundles on $X \times X$:

$$(2.1.1) \quad 0 \rightarrow \mathcal{I}^{k+1} \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{O}_{\Delta^k} \rightarrow 0.$$

Apply the functor $p_*(- \otimes q^*\mathcal{O}(n))$ to the sequence (2.1.1) to get a long exact sequence

$$(2.1.2) \quad 0 \rightarrow p_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) \rightarrow p_*q^*\mathcal{O}(n) \rightarrow \mathcal{P}^k(\mathcal{O}(n)) \\ \rightarrow R^1p_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) \rightarrow R^1p_*q^*\mathcal{O}(n) \rightarrow R^1p_*(\mathcal{O}_{\Delta^k} \otimes q^*\mathcal{O}(n)) \rightarrow \dots$$

of vector bundles. The sequence (2.1.2) is a sequence of vector bundles because all sheaves in the sequence are coherent, and it is a standard fact that a homogeneous coherent sheaf on a homogeneous space is locally free. Since the sequence (2.1.2) is a sequence of vector bundles, we get an exact sequence of P -representations when we pass to the fiber at \bar{e} .

Consider the diagram

$$\begin{array}{ccc} \text{Spec}(\kappa(\bar{e})) \times X & \xrightarrow{j} & X \times X \\ \downarrow \pi & & \downarrow p \\ \text{Spec}(\kappa(\bar{e})) & \xrightarrow{i} & X \end{array}$$

Then by [5], Chapt. III, Sect. 12, we get maps

$$\phi^i : R^i p_*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n))(\bar{e}) \rightarrow R^i \pi_*(j^*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)))$$

of \mathcal{O}_X -modules. Put, for any $\mathcal{O}_{X \times X}$ -module \mathcal{E} ,

$$h^i(y, E) = \dim_{\kappa(y)} H^i(X_y, \mathcal{E}_y),$$

where X_y is the fiber $p^{-1}(y)$ and \mathcal{E}_y is the restriction of \mathcal{E} to X_y . We see that

$$h^i(y, \mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) = \dim_{\kappa(y)} H^i(X, \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n))$$

is a constant function of y for $i = 0, 1, \dots$ for the following reason: Consider the commutative diagram

$$\begin{array}{ccc} \text{Spec}(\kappa(y)) \times X & \xrightarrow{\tilde{g}} & \text{Spec}(\kappa(gv)) \times X \\ \downarrow j & & \downarrow k \\ X \times X & \xrightarrow{g} & X \times X \end{array}$$

where the action of $SL(V)$ on $X \times X$ is given by $g(x, y) = (gx, gy)$. In general, if $G \times Y \rightarrow^\sigma Y$ is an algebraic group acting on a scheme Y , and \mathcal{E} is a G -linearized sheaf on Y , then there exists an isomorphism $I : \sigma^*\mathcal{E} \rightarrow p^*\mathcal{E}$, where $p : G \times Y \rightarrow Y$ is the projection map. It follows that for all $g \in G$ we get an isomorphism $g^*\mathcal{E} \cong \mathcal{E}$ of sheaves. Then, since $\mathcal{O}(n)$ and $\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)$ are $SL(V)$ -homogeneous sheaves, we have an isomorphism

$$\tilde{g}^*(\mathfrak{m}_{gy}^{k+1} \otimes \mathcal{O}(n)) = j^*g^*(\mathcal{I}^{k+1} \otimes q^*\mathcal{O}(n)) = \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n);$$

hence since \tilde{g} is an isomorphism, we see that we have an isomorphism

$$\mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n) \cong \mathfrak{m}_{gy}^{k+1} \otimes \mathcal{O}(n)$$

of sheaves for all g in $SL(V)$. It follows that

$$\dim_{\kappa(y)} H^i(X, \mathfrak{m}_y^{k+1} \otimes \mathcal{O}(n)) = \dim_{\kappa(gy)} H^i(X, \mathfrak{m}_{gy}^{k+1}, \mathcal{O}(n))$$

for all g in $SL(V)$; hence by [5], Chapt. III, Corr. 12.9, it follows that the maps ϕ^i are isomorphisms for $i = 0, 1, \dots$. Here \mathfrak{m}_y is the sheaf of ideals corresponding to the point y in X . We get an exact sequence

$$(2.1.3) \quad \begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) &\rightarrow H^0(X, \mathcal{O}(n)) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\bar{e}) \\ &\rightarrow H^1(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) \rightarrow H^1(X, \mathcal{O}(n)) \rightarrow \dots \end{aligned}$$

of P -representations.

Lemma 2.2. *For all $1 \leq k < n$ we have that $H^1(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) = 0$.*

Proof. Consider the exact sequence (2.1.3). We prove that

$$\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\bar{e}),$$

and then the result follows by counting dimensions. We have that $H^0(X, \mathcal{O}(n))$ equals $S^n(V^*)$, where V^* is the F -vector space on the basis x_0, \dots, x_N . We also see that $H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1})$ equals $m^{k+1}S^{n-(k+1)}(V^*)$ considered as a subspace of $S^n(V^*)$. Here m is the F -vector space on the basis x_1, \dots, x_N and $m^{k+1}S^{n-(k+1)}(V^*)$ is the image of the natural map

$$S^{k+1}(m) \otimes S^{n-(k+1)}(V^*) \rightarrow S^n(V^*).$$

Write V^* as the direct sum $Fx_0 \oplus m$. Then it follows that

$$m^{k+1}S^{n-(k+1)}(V^*) = x_0^{n-(k+1)}m^{k+1} \oplus \dots \oplus x_0m^{n-1} \oplus m^n;$$

hence we see that the dimension of $m^{k+1}S^{n-(k+1)}(V^*)$ equals $\sum_{i=k+1}^n \binom{i+N-1}{N-1}$. We also see that the dimension of $S^n(V^*)$ equals $\sum_{i=0}^n \binom{i+N-1}{N-1}$, and it follows that

$$\dim_F S^n(V^*) - \dim_F m^{k+1}S^{n-(k+1)}(V^*) = \sum_{i=0}^k \binom{i+N-1}{N-1}.$$

It follows that

$$\sum_{i=0}^k \binom{i+N-1}{N-1} = \binom{k+N}{N} = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\bar{e}),$$

and we have proved that

$$\dim_F H^0(X, \mathcal{O}(n)) - \dim_F H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) = \dim_F \mathcal{P}^k(\mathcal{O}(n))(\bar{e}).$$

The result follows from the fact that the sequence (2.1.3) is exact and that $H^1(X, \mathcal{O}(n)) = 0$ for $n \geq 1$. □

Note that by Lemma 2.2 and the sequence (2.1.3), there exists for all $1 \leq k < n$ an exact sequence of P -representations

$$0 \rightarrow H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) \rightarrow H^0(X, \mathcal{O}(n)) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\bar{e}) \rightarrow 0.$$

Since the representation $H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1})$ equals $m^{k+1}S^{n-(k+1)}(V^*)$ as a subrepresentation of $H^0(X, \mathcal{O}(n)) = S^n(V^*)$, it follows that we have an exact sequence of P -representations

$$(2.2.1) \quad 0 \rightarrow m^{k+1}S^{n-(k+1)}(V^*) \rightarrow S^n(V^*) \rightarrow \mathcal{P}^k(\mathcal{O}(n))(\bar{e}) \rightarrow 0.$$

From the exact sequence

$$0 \rightarrow m \rightarrow V^* \rightarrow V^*/m \rightarrow 0,$$

where m is the F -vector space on x_1, \dots, x_N , we see that the representation V^*/m is the representation corresponding to the module L^* of P , giving the line bundle $\mathcal{O}(1)$ on $X = SL(V)/P$.

Lemma 2.3. *For all $1 \leq k < n$ there exists a surjective map of P -representations*

$$\phi : S^n(V^*) \rightarrow S^{n-k}(L^*) \otimes S^k(V^*).$$

Proof. Recall that we have chosen a basis e_0, \dots, e_N for V , with the property that \bar{x}_0 is a basis for L^* . The P -representation m with basis x_1, \dots, x_N gives an exact sequence

$$0 \rightarrow m \rightarrow V^* \rightarrow L^* \rightarrow 0$$

of P -representations. Define a map

$$\phi : S^n(V^*) \rightarrow S^{n-k}(L^*) \otimes S^k(V^*)$$

as follows: $\phi(f) = \bar{x}_0^{n-k} \otimes \partial_0^{n-k}(f)$, where ∂_0^{n-k} is the $n - k$ times partial derivative with respect to the x_0 -variable. Let g be an element of P . Then by induction on the degree of the differential operator ∂_0^{n-k} and applying the chain rule for derivation, it follows that

$$\begin{aligned} \phi(gf) &= \bar{x}_0^{n-k} \otimes \partial_0^{n-k}(gf) = \bar{x}_0^{n-k} \otimes a^{-(n-k)}g(\partial_0^{n-k}f) \\ &= a^{-(n-k)}\bar{x}_0^{n-k} \otimes g(\partial_0^{n-k}f) = g(\bar{x}_0^{-(n-k)} \otimes \partial_0^{n-k}f) = g\phi(f), \end{aligned}$$

and we see that ϕ is P -linear. It is clearly surjective, and the lemma follows. \square

Theorem 2.4. *For all $1 \leq k < n$, the representation corresponding to $\mathcal{P}^k(\mathcal{O}(n))$ is $S^{n-k}(L^*) \otimes S^k(V^*)$.*

Proof. By Lemma 2.3 there exists a surjective map of P -representations

$$\phi : S^n(V^*) \rightarrow S^{n-k}(L^*) \otimes S^k(V^*).$$

We claim that $m^{k+1}S^{n-(k+1)}(V^*)$ equals $\ker \phi$: We first prove the inclusion

$$m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi.$$

Pick a monomial $x_0^{p_0}x_1^{p_1} \dots x_N^{p_N}$ in $m^{k+1}S^{n-(k+1)}(V^*)$; then $p_0 + \dots + p_N = n$ and $p_0 < n - k$. These monomials form a basis for $m^{k+1}S^{n-(k+1)}(V^*)$. We see that $\partial_0^{n-k}(x_0^{p_0} \dots x_N^{p_N})$ is zero; hence, since ϕ is a linear map, it follows that we have an inclusion

$$m^{k+1}S^{n-(k+1)}(V^*) \subseteq \ker \phi$$

of vector spaces. The reverse inclusion follows from counting dimensions and the fact that ϕ is surjective: We have that

$$\begin{aligned} \dim_F \ker \phi &= \dim_F S^n(V^*) - \dim_F S^{n-k}(L^*) \otimes S^k(V^*) \\ &= \sum_{i=0}^n \binom{i+N-1}{N-1} - \sum_{i=0}^k \binom{i+N-1}{N-1} = \sum_{i=k+1}^n \binom{i+N-1}{N-1}, \end{aligned}$$

and we see that $\dim_F \ker \phi = \dim_F m^{k+1} S^{n-(k+1)}(V^*)$. It follows that

$$m^{k+1} S^{n-(k+1)}(V^*) = \ker \phi;$$

hence we have an exact sequence of P -representations

$$0 \rightarrow m^{k+1} S^{n-(k+1)}(V^*) \rightarrow S^n(V^*) \rightarrow S^{n-k}(L^*) \otimes S^k(V^*) \rightarrow 0.$$

Using sequence (2.2.1) we get isomorphisms

$$\begin{aligned} \mathcal{P}^k(\mathcal{O}(n))(\bar{e}) &\cong H^0(X, \mathcal{O}(n)) / H^0(X, \mathcal{O}(n) \otimes \mathfrak{m}_{\bar{e}}^{k+1}) \\ &\cong S^n(V^*) / m^{k+1} S^{n-(k+1)}(V^*) \cong S^{n-k}(V^*) \otimes S^k(V^*), \end{aligned}$$

and it follows that $\mathcal{P}^k(\mathcal{O}(n))(\bar{e})$ is isomorphic to $S^{n-k}(L^*) \otimes S^k(V^*)$ as a representation. \square

Note that the result in Theorem 2.4 is true if $\text{char}(F) > n$.

Corollary 2.5. *For all $1 \leq k < n$, $\mathcal{P}^k(\mathcal{O}(n))$ splits as an abstract vector bundle as $\bigoplus \binom{N+k}{N} \mathcal{O}(n-k)$.*

Proof. Since $S^k(V^*)$ corresponds to the trivial rank $\binom{N+k}{N}$ abstract vector bundle on $\mathbf{P}(V^*)$, and $S^{n-k}(L^*)$ corresponds to the line bundle $\mathcal{O}(n-k)$, the assertion is proved. \square

We see that we recover results on the splitting type of the principal parts obtained in [6], [7], [8] and [9].

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