ISOMETRIES FOR KY FAN NORMS BETWEEN MATRIX SPACES

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Abstract. We characterize linear maps between different rectangular matrix spaces preserving Ky Fan norms.

1. Introduction and statements of results

Let $M_{m,n}$ ($M_n$) be the linear space of $m \times n$ ($n \times n$) complex matrices. The singular values of $A \in M_{m,n}$ are the nonnegative square roots of the eigenvalues of $A^*A$, and they are denoted by $s_1(A) \geq \cdots \geq s_n(A)$. For $1 \leq k \leq \min\{m,n\}$, the Ky Fan $k$-norm on $M_{m,n}$ is defined and denoted by

$$
\|A\|_k = s_1(A) + \cdots + s_k(A).
$$

The Ky Fan 1-norm reduces to the operator norm when $m = n$ the Ky Fan $n$-norm is also known as the trace norm.

Evidently, Ky Fan $k$-norms are unitarily invariant norms, i.e.,

$$
\|UAV\|_k = \|A\|_k
$$

for any $A \in M_{m,n}$, and unitary $U \in M_m$ and $V \in M_n$. Actually, they form an important class of unitarily invariant norms; see [1, Chapters 2 and 3]. For instance, given $A, B \in M_{m,n}$,

$$
\|A\|_k \leq \|B\|_k \quad \text{for all } k = 1, \ldots, \min\{m,n\}
$$

if and only if

$$
\|A\| \leq \|B\| \quad \text{for all unitarily invariant norms } \| \cdot \|.
$$

There has been considerable interest in studying isometries for Ky Fan norms on matrix spaces. For example, by a result of Kadison [5], one easily deduces that isometries for the operator norm on $M_n$ have to have the form

(1) $A \mapsto UAV$ or $A \mapsto UA^tV$ for some unitary matrices $U, V \in M_n$. In [4], the authors showed that the same conclusion holds for Ky Fan $k$-norm isometries for any $k = 1, \ldots, \min\{m,n\}$, where the second form in (1) can occur only when $m = n$. In [8], the authors considered the

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problem on block triangular matrix algebras in $M_n$, and showed that the isometries essentially have the same structure. In [3], the authors studied isometries $\phi : (M_n, \| \cdot \|_1) \to (M_p, \| \cdot \|_1)$ for $n \neq p$, and obtained a complete characterization when $p \leq 2n - 2$; moreover, examples were given to show that $\phi$ may have complicated structure for $p > 2n - 2$. In view of these, one may think that isometries $\phi : (M_n, \| \cdot \|_k) \to (M_p, \| \cdot \|_k)$ also have complicated structure for $k > 1$. It turns out that it is not the case as shown in the corollary of our main theorem, which characterizes isometries $\phi : (M_{m,n}, \| \cdot \|_k) \to (M_{p,q}, \| \cdot \|_k)$ provided $k' > 1$. We need some notation and definitions to describe our main result.

For two matrices $A$ and $B$ with $A = (a_{ij})$ denote by $A \otimes B$ the block matrix $(a_{ij}B)$. An $r \times s$ matrix $X$ is called a partial isometry if $X^*X = I_s$, i.e., $X$ has orthonormal columns.

**Theorem 1.1.** Let $1 < k' \leq \min\{m, n\}$ and $1 \leq k \leq \min\{p, q\}$. Suppose $\phi : M_{m,n} \to M_{p,q}$ satisfies

(2) \[ \|\phi(A)\|_k = \|A\|_{k'} \quad \text{for all} \quad A \in M_{m,n}. \]

Then there exist nonnegative integers $c_1$ and $c_2$ with $c_1 + c_2 > 0$, and partial isometries $U$ and $V$ of sizes $p \times (c_1m + c_2n)$ and $q \times (c_1n + c_2m)$, respectively, such that one of the following holds:

(a) $k' < \min\{m, n\}$, $k = k'(c_1 + c_2)$, and $\phi$ has the form

\[ A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)] V^*. \]

(b) $k' = \min\{m, n\}$, $k'(c_1 + c_2) \leq k$, and there are diagonal matrices $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ with positive diagonal entries satisfying $\text{tr} D_1 + \text{tr} D_2 = 1$, such that $\phi$ has the form

\[ A \mapsto U[(D_1 \otimes A) \oplus (D_2 \otimes A^t)] V^*. \]

If $k' = k$, then either $(c_1, c_2) = (1, 0)$ or $(c_1, c_2) = (0, 1)$. By adding columns to $U$ and $V$ to form unitary matrices, we have the following corollary.

**Corollary 1.2.** Let $1 < k \leq \min\{m, n\}$. Suppose $\phi : M_{m,n} \to M_{p,q}$ satisfies

(3) \[ \|\phi(A)\|_k = \|A\|_k \quad \text{for all} \quad A \in M_{m,n}. \]

Then there are unitary matrices $U \in M_p$ and $V \in M_q$ such that $\phi$ has the form

\[ A \mapsto U[A \oplus 0_{p-m,n-a}] V \quad \text{or} \quad A \mapsto U[A^t \oplus 0_{p-n,q-m}] V. \]

2. Auxiliary results and proofs

Replacing $\phi$ by the mapping(s) $A \mapsto \phi(A^t)$ and/or $A \mapsto [\phi(A)]^t$, we may assume that $m \leq n$ and $p \leq q$. Two nonzero matrices $A, B \in M_{m,n}$ are said to be orthogonal if $AB^* = 0$ and $A^*B = 0$. Equivalently, there are unitary matrices $U$ and $V$ such that $UAV = \sum_{j=1}^r a_{ij} E_{jj}$ and $UBV = \sum_{j=r+1}^{r+s} b_{ij} E_{jj}$ with $a_1 \geq \cdots \geq a_r > 0$ and $b_1 \geq \cdots \geq b_s > 0$ for some $r, s$ with $r + s \leq \min\{m, n\}$. The nonzero matrices $A_1, \ldots, A_d \in M_{m,n}$ are said to be pairwise orthogonal $m \times n$ matrices if $A_iA_j^* = 0$ and $A_jA_i^* = 0$ for any distinct pair $(i, j)$. In such a case, there are unitary $U \in M_m$ and $V \in M_n$, $0 = r_0 < r_1 < \cdots < r_d \leq \min\{m, n\}$ and positive numbers $a_1, \ldots, a_{rd}$ such that $UA_iV = \sum_{r_{i-1} < j \leq r_i} a_j E_{jj}$. 

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We begin with the following lemma from [3] Lemma 5.

**Lemma 2.1.** Let $A, B \in M_{m,n}$ be nonzero. Then $\|aA + bB\|_k = |a|\|A\|_k + |b|\|B\|_k$ for every $a, b \in C$ if and only if $A$ and $B$ are orthogonal and $\text{rank } A + \text{rank } B \leq k$.

By Lemma 2.1 and a simple inductive argument, we have the following.

**Lemma 2.2.** Let $\phi : M_{m,n} \to M_{p,q}$ be a map satisfying (2). Suppose the rank one matrices $A_1, \ldots , A_d \in M_{m,n}$, $d \leq \min\{m,n\}$, are pairwise orthogonal. Then $\phi(A_1), \ldots , \phi(A_d) \in M_{p,q}$ are nonzero and pairwise orthogonal. Furthermore, for any $1 \leq s_1 < \cdots < s_k' \leq d$, $\sum_{j=1}^{k'} s_j$ rank $\phi(A_{s_j}) \leq k$.

**Proof of Theorem 1.1.** For the sufficiency part of Theorem 1.1, one readily sees that singular values of $\phi(A)$ have $c = (c_1 + c_2)$ copies of $s_1(A)/c, \ldots , s_m(A)/c$ if $\phi$ has the form (a). On the other hand, if $k' = m$ and $\phi$ has the form (b), then $k \geq ck'$ and so the Ky Fan $k$-norm of $\phi(A)$ is just the sum of its singular values. Let $D_1 \oplus D_2 = \text{diag}(d_1, \ldots , d_c)$. Then,

$$\|\phi(A)\|_k = d_1\|A\|_{k'} + \cdots + d_c\|A\|_{k'} = \text{tr}(D_1 \oplus D_2)\|A\|_{k'} = \|A\|_{k'}.$$

To prove the necessity part, let $(p', q') = (p - c_1m - c_2n, q - c_1n - c_2m)$. It suffices to prove that there are unitary matrices $U \in M_p$ and $V \in M_q$ such that $\phi$ has the form

(a) $A \mapsto \frac{1}{c_1 + c_2} U[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p', q'}]V^*$ if $k' < m$,

(b) $A \mapsto U[(D_1 \otimes A) \oplus (D_2 \otimes A^t) \oplus 0_{p', q'}]V^*$ if $k' = m$.

We divide the proof into three cases:

(I) $k' < m = n$,  \hspace{0.5cm} (II) $k' = m = n$,  \hspace{0.5cm} and  \hspace{0.5cm} (III) $m < n$.

First consider case (I): $k' < m = n$. For any $A \in M_{m,n}$ with singular values $1, 0, \ldots , 0$, there are unitary $X$ and $Y$ such that $A = XE_{1j}Y$. Let $A_j = XE_{1j}Y$ for $j = 1, \ldots , m$. Then $A_1, \ldots , A_m$ are pairwise orthogonal. By Lemma 2.2, $\phi(A_1), \ldots , \phi(A_m)$ are pairwise orthogonal. Thus, there exist unitary $U$ and $V$, $0 = r_0 < r_1 < \cdots < r_d \leq m$ and positive numbers $a_1, \ldots , a_r$ such that

$$B_i = U\phi(A_i)V = \sum_{r_{i-1} < j \leq r_i} a_j E_{jj} \text{ for any } i = 1, \ldots , m.$$

By Lemma 2.2 again, the sum of any $k'$ matrices chosen from $B_1, \ldots , B_m$ has rank at most $k$. Let $1 \leq t_1 < \cdots < t_{k'} \leq m$. Then

$$s_k \left( \sum_{j=1}^{k'} B_{t_j} \right) = 0, \text{ for all } \ell > k.$$

Moreover, if $t \in \{1, \ldots , m\} \setminus \{t_1, \ldots , t_{k'}\}$, we claim that

$$s_1(B_t) \leq s_k \left( \sum_{j=1}^{k'} B_{t_j} \right).$$

If (I) does not hold, then $s_1(B_t) > s_k \left( \sum_{j=1}^{k'} B_{t_j} \right)$, which gives the following contradiction:

$$k' = \left\| A_t + \sum_{j=1}^{k'} A_{t_j} \right\|_{k'} > \left\| B_t + \sum_{j=1}^{k'} B_{t_j} \right\|_k > \sum_{j=1}^{k'} B_{t_j} \left\| \sum_{j=1}^{k'} A_{t_j} \right\|_{k'} = k'.$$

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Let $c = k/k'$. It follows from [2, 3] and [4] that for each $1 \leq j \leq m$, we have $s_j(B_j) = 1/c$ for $1 \leq i \leq c$ and $s_j(B_j) = 0$ for $c < i \leq p$. Thus, we see that

(i) every rank one matrix is mapped to a rank $c$ matrix, and
(ii) every unitary matrix is mapped to a matrix with the singular values $1/c, \ldots, 1/c, 0, \ldots, 0$.

Since (i) holds, by Theorem 2.5 in [7], $\phi$ has the form

$$A \rightarrow R((I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p', q'}) S^*$$

for some invertible $R \in M_p$ and $S \in M_q$. Let $R_1$ (respectively, $S_1$) be obtained from $R$ (respectively, $S$) by removing its last $p'$ (respectively, $q'$) columns. Then

$$R((I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t) \oplus 0_{p', q'}) S^* = R_1[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)] S_1^*.$$

By polar decomposition, there are unitary matrices $U \in M_{c_1 m + c_2 n}$ and $Q \in M_{c_1 n + c_2 m}$ such that

$$R_1 = U \begin{pmatrix} P & \cr 0_{p', c_1 m + c_2 n} & \cr \end{pmatrix} \quad \text{and} \quad S_2 = V \begin{pmatrix} & Q \cr 0_{q', c_1 n + c_2 m} & \cr \end{pmatrix}.$$

Thus,

$$\phi(A) = U \begin{pmatrix} & P[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)] Q^* \oplus 0_{p', q'} \end{pmatrix} V^*.$$

Define $\psi : M_m \rightarrow M_{cm}$ such that $\psi(X) = cP[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)] Q^*$. By (ii), we see that $\psi$ maps unitary matrices to unitary matrices. By the result in [2], we see that $\psi(A) = W_1[(I_{c_1} \otimes A) \oplus (I_{c_2} \otimes A^t)] W_2$ for some unitary $W_1, W_2 \in M_{cn}$. Thus, condition (a) holds.

Next, we turn to case (II) : $k' = m = n$. From the first part of the proof in case (I), we can see that for any unitary $X, Y \in M_m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$, $\sum_{i=1}^m \lambda_i \phi(X E_{ii} Y)$ has rank at most $k$. Hence, $\phi(A)$ has rank at most $k$ for all $A \in M_m$. We may assume that $p = q$ by appending $q - p$ zero rows to $\phi(A)$ for each $A \in M_m$. So, we assume that $\phi : M_m \rightarrow M_p$ and suppose $\phi(I_m) = D$ is a nonnegative diagonal matrix with diagonal entries arranged in descending order. For any Hermitian $X \in M_m$ with trace zero and spectrum in $[-1, 1]$ and $t \in [-1, 1]$,

$$\|\phi(I_m + tX)\|_k = \|I_m + tX\|_{k'} = k' = \|I_m\|_{k'} = \|\phi(I_m)\|_k = \text{tr } D.$$

Let $Y = \phi(X)$. Then $\text{tr } Y = 0$ because

$$\|\psi(I_m + tX)\|_p = \|\psi(I_m + tX)\|_k = \text{tr } D$$

for $t = \pm 1$. Moreover,

$$k' = \text{tr } (D \pm Y) \leq \|\phi(I_m + tX)\|_p = \|\phi(I_m + tX)\|_k = k'.$$

By [3] Corollary 3.2, we conclude that $D \pm Y$ is positive semidefinite. As a result, if $\phi(I_m) = D = \text{diag}(d_1, \ldots, d_r, 0, \ldots, 0)$ with $d_1 \geq \cdots \geq d_r > 0$, then $\phi(X)$ has the form $Y \oplus 0_{p-r}$. We may now consider $\psi : M_m \rightarrow M_r$ such that $\phi(A) = \psi(A) \oplus 0_{p-r}$. It follows from the above argument that $\psi$ maps Hermitian matrices to Hermitian matrices and $\|\psi(A)\|_p = \|\phi(A)\|_k = \|A\|_{k'}$. We claim that

(i) $\psi$ maps positive semidefinite matrices to positive semidefinite matrices, and
(ii) $\psi$ maps invertible Hermitian matrices to invertible Hermitian matrices.
To see (i), suppose that \( A \in M_m \) is positive semidefinite. Let \( D_1 = \psi(I_m) = \text{diag}(d_1, \ldots, d_r) \). Choose \( t > 0 \) such that \( D_1 + t\psi(A) \) is positive semidefinite. Then we have

\[
\text{tr}(D_1 + t\psi(A)) = \|D_1 + t\psi(A)\|_r = \|I_m + tA\|_{kr} = \text{tr}(I_m) + t\text{tr}(A)
\]

\[
= \|I_m\|_{kr} + t\|A\|_{kr} = \|\psi(I_m)\|_r + t\|\psi(A)\|_r = \text{tr}D_1 + t\|\psi(A)\|_r.
\]

Thus, \( \text{tr}\psi(A) = \|\psi(A)\|_r \), and it follows from [6 Corollary 3.2] again that \( \psi(A) \) is positive semidefinite.

To prove (ii), let

\[
A = U^* \left( \sum_{j=1}^{m} \lambda_j E_{jj} \right) U
\]

for some unitary \( U \) and \( \lambda_j \in \mathbb{R} \setminus \{0\} \) for \( j = 1, \ldots, m \). Since \( \phi(U^*E_{11}U), \ldots, \phi(U^*E_{mm}U) \) are pairwise orthogonal and \( \phi(I_m) = D_1, \phi(U^*E_{jj}U) = V^*F_jV \oplus 0_{p-r} \) for \( j = 1, \ldots, m \) such that \( F_j = \sum_{r_{j-1} < s \leq r_j} a_{ss} E_{ss} \) for \( 0 = r_0 < \cdots < r_m = r \) and positive numbers \( a_1, \ldots, a_r \). Therefore,

\[
\psi(A) = V^* \left( \sum_{j=1}^{m} \lambda_j \left( \sum_{r_{j-1} < s \leq r_j} a_{ss} E_{ss} \right) \right) V
\]

is also invertible. Thus, condition (ii) holds.

Now, \( \psi(I_m) \) is positive definite and \( \psi \) maps invertible Hermitian matrices to invertible Hermitian matrices. By (the proof of) [7 Proposition 3.4], we see that

\[
\psi(X) = T^*[ (I_{c_1} \otimes X) \oplus (I_{c_2} \otimes X^t) ] T
\]

for some invertible \( T \in M_{c} \). In particular, we see that

(iii) \( \psi \) maps rank \( s \) matrices to rank \( cs \) matrices for \( s = 1, \ldots, m \).

Next, we show that \( \psi \) has the form \( X \mapsto U^*[(D_1 \otimes X) \oplus (D_2 \otimes X^t)]U \) for some unitary matrix \( U \) and diagonal matrices \( D_1 \) and \( D_2 \) with positive diagonal entries such that \( trD_1 + trD_2 = 1 \). Equivalently, we show that \( \psi \) has the form

\[
A = (a_{uv}) \mapsto V^*BV,
\]

where \( B = (B_{uv})_{1 \leq u, v \leq m} \) with \( B_{uv} = a_{uv}I_{c_1} \oplus a_{uv}I_{c_2} \) for some unitary \( V \). First, by a suitable permutation, we can rewrite \( \psi \) in (5) as

\[
A = (a_{uv}) \mapsto S^*BS
\]

with \( B = (B_{uv})_{1 \leq u, v \leq m} \) with \( B_{uv} = a_{uv}I_{c_1} \oplus a_{vu}I_{c_2} \) for some nonsingular \( S \in M_r \). By Lemma 2.2, we see that \( \phi(E_{11}), \ldots, \phi(E_{mm}) \) are pairwise orthogonal. Then for any distinct pair \( i \) and \( j \),

\[
[S^*(E_{ii} \otimes I_c)S]^*[S^*(E_{jj} \otimes I_c)S] = \psi(E_{ii})^*\psi(E_{jj}) = 0.
\]

Thus, \( (E_{ii} \otimes I_c)SS^*(E_{jj} \otimes I_c) = 0 \) whenever \( i \neq j \). It follows that \( SS^* = S_1 \oplus \cdots \oplus S_n \) where \( S_i \in M_{c_i} \).

Let \( i > 1 \), \( X = E_{i1} + E_{1i} \) and \( Y = E_{i1} - E_{1i} \). From (6), \( \psi(X) = S^*(B_{x})S \) and \( \psi(Y) = S^*(C_{x})S \) so that

\[
\tilde{B} = \begin{pmatrix} B_{11} & B_{1i} \\ B_{i1} & B_{ii} \end{pmatrix}, \quad \begin{pmatrix} I_c \oplus I_{c_2} \\ 0_c \end{pmatrix}, \quad \begin{pmatrix} 0_c \oplus I_{c_2} \\ I_{c_1} \oplus 0_c \end{pmatrix},
\]

\[
\tilde{C} = \begin{pmatrix} C_{11} & C_{1i} \\ C_{i1} & C_{ii} \end{pmatrix}, \quad \begin{pmatrix} 0_c \oplus I_{c_2} \\ I_{c_1} \oplus 0_c \end{pmatrix}, \quad \begin{pmatrix} I_{c_1} \oplus 0_c \\ 0_c \oplus I_{c_2} \end{pmatrix}
\]
and all other $B_{uv}$ and $C_{uv}$ are 0. Let $J_1 = I_c \oplus 0_{c_2}$ and $J_2 = 0_{c_1} \oplus I_{c_2}$. Since $X$ and $Y$ are orthogonal, so are $\psi(X)$ and $\psi(Y)$. Hence $B^*(SS^*)C = 0$ and $B(SS^*)C^* = 0$. Thus,

\[
\begin{pmatrix}
J_2S_1J_1 & S_1J_2 - J_2S_1 \\
0 & J_1S_1J_2
\end{pmatrix}
= \tilde{B}^*(S_1 \oplus S_i)\tilde{C} = 0
= \tilde{B}(S_1 \oplus S_i)\tilde{C}^* =
\begin{pmatrix}
J_1S_1J_2 & S_1J_1 - J_1S_1 \\
0 & J_2S_1J_2
\end{pmatrix}.
\]

Since $J_2S_1J_1 = J_1S_1J_2 = J_2S_1J_1 = J_1S_1J_2 = 0$, each of the matrices $S_1$ and $S_i$ is a direct sum of a matrix in $M_{c_1}$ and a matrix in $M_{c_2}$. Furthermore, we can conclude that $S_1 = S_i = P_1 \oplus P_2$, where $P_1 \in M_{c_1}$ and $P_2 \in M_{c_2}$, from $S_1J_1 - J_1S_1 = 0 = S_1J_2 - J_2S_1$. Since $i$ is arbitrary, $SS^* = I_m \otimes (P_1 \oplus P_2)$ where $P_1$ and $P_2$ are both positive definite. Thus there exist unitary $U_1 \in M_{c_1}$ and $U_2 \in M_{c_2}$ such that $U_1P_1U_1^* = D_1$ and $U_2P_2U_2^* = D_2$, where $D_1$ and $D_2$ are diagonal matrices with positive diagonal entries.

Let $U = I_m \otimes (U_1 \oplus U_2)$ and $\tilde{S} = US$. Then $\tilde{S}S^* = I_m \otimes (D_1 \oplus D_2)$. Since the row vectors of $\tilde{S}$ form an orthogonal basis, we may write $\tilde{S} = DV$, where $D = I_m \otimes (D_1 \oplus D_2)^{1/2}$ and $V$ is unitary.

On the other hand, we have $U^*BU = B$ for the block matrix $B$ in [4], since

\[a_{uv}I_{c_1} \oplus a_{uv}I_{c_2} = (U_1 \oplus U_2)^*(a_{uv}I_{c_1} \oplus a_{uv}I_{c_2})(U_1 \oplus U_2).\]

Then $S^*BS = S^*U^*BUS = \tilde{S}^*B\tilde{S} = V^*D^*BDV$. In fact, the $(i,j)$-th block of $D^*BD$ is equal to

\[\frac{1}{2}(a_{uv}I_{c_1} \oplus a_{uv}I_{c_2})(D_1 \oplus D_2)^{1/2} = a_{uv}D_1 \oplus a_{uv}D_2.\]

Thus, $\phi$ has the asserted form. Since $\|I_m \otimes (D_1 \oplus D_2)\|_{k'} = \|\psi(I_m)\|_r = \|I_m\|_{k'} = m$, it follows that $\text{tr} (D_1 \oplus D_2) = \text{tr} D_1 + \text{tr} D_2 = 1$.

Finally, we consider case (III) : $m < n$. We prove the desired conclusion by induction on $n-m$ starting from $n-m = 0$, which follows from cases (I) and (II). Suppose that $n-m = r > 0$ and that the result holds for the cases when $n-m < r$. Applying the assumption on the restriction of $\phi$ on $M^0_{m,n}$, the subspace of $M_{m,n}$ that consists of matrices with zero $n$-th column, we conclude that for any $A \in M^0_{m,n}$,

\[\phi(A) = U[(D_1 \otimes \tilde{A}) \oplus (D_2 \otimes \tilde{A}^t) \otimes 0_{p',q'}]V\]

where $\tilde{A}$ denotes the $m \times (n-1)$ matrix obtained from $A$ by deleting the $n$-th column, $(p',q') = (p-c_1m - c_2(n-1), q-c_1(n-1) - c_2m)$, $U \in M_p$ and $V \in M_q$ are unitary, and the following holds:

(a) if $k' < m$ and $c = c_1 + c_2 = k/k'$, then $D_1 = \frac{1}{k'}I_{c_1}$ and $D_2 = \frac{1}{k'}I_{c_2}$;
(b) if $k' = m$ and $c = c_1 + c_2 \leq k/k'$, then $D_1 \in M_{c_1}$ and $D_2 \in M_{c_2}$ are diagonal matrices with positive diagonal entries such that $\text{tr} D_1 + \text{tr} D_2 = 1$.

Now replacing $\phi$ by $X \mapsto U^*\phi(X)V^*$, we may assume that $U = I_p$ and $V = I_q$.

For any $x \in M_{m,1}$, let $A$ be the $m \times n$ matrix with $x$ as the $n$-th column and zero in the other columns, and $X = (X_{uv})_{1 \leq u,v \leq m+1} = \phi(A)$, where $X_{uv} \in M_{m,n-1}$ for $1 \leq u \leq c_1$, $X_{ac} \in M_{m-1,n-1}$ for $c_1 < u \leq c$ and $X_{c+1,c+1} \in M_{p',q'}$.

Take any nonzero $y \in M_{m,1}$ such that $x^*y = 0$. (Note that $1 < k \leq m$ and hence $y$ exists.) For any $l < n$, let $B$ be the $m \times n$ matrix with $y$ as the $l$-th column and zero in all other columns. Then $Y = \phi(B) = (D_1 \otimes \tilde{B}) \oplus (D_2 \otimes \tilde{B}^t) \otimes 0_{p',q'}$. 

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Since $A$ and $B$ are orthogonal, $X^*Y = 0_q$ and $XY^* = 0_p$. It follows from the structure of $Y$ that

\[
\begin{align*}
X_{uv}^* \tilde{B} &= 0 \text{ when } 1 \leq u \leq c_1 \text{ and } 1 \leq v \leq c + 1, \\
X_{uv}^* \tilde{B}^t &= 0 \text{ when } c_1 < u \leq c \text{ and } 1 \leq v \leq c + 1, \\
X_{uv} \tilde{B}^* &= 0 \text{ when } 1 \leq u \leq c + 1 \text{ and } 1 \leq v \leq c_1, \\
X_{uv} (\tilde{B}^t)^* &= 0 \text{ when } 1 \leq u \leq c + 1 \text{ and } c_1 < v \leq c.
\end{align*}
\]

Since the $l$-th column of the $m \times (n-1)$ matrix $\tilde{B}$ is the nonzero vector $y$, if $X_{uv}^* \tilde{B}^* = 0$, then the $l$-th row of $X_{uv}$ must be the zero vector. Furthermore, since $l$ can be any integer in $\{1, \ldots, n-1\}$, we conclude that $X_{uv} = 0$. Similarly, $X_{uv}^*$ must be the zero matrix if $X_{uv}^* \tilde{B} = 0$.

On the other hand, if $X_{uv}^* \tilde{B} = 0$, then all the columns of $X_{uv}$ must be orthogonal to $y$. Since $y$ can be any vector orthogonal to $x$, all columns of $X_{uv}$ must be multiples of $x$. Hence, $X_{uv} = xw^t$ for some vector $w$ of suitable size. Similarly, since $X_{uv} (\tilde{B}^t)^* = 0$, we have $X_{uv} = zx^t$ for some $z$.

By the arguments in the last two paragraphs, if $1 \leq u \leq c_1$ and $c_1 < v \leq c$, then $xw^t = X_{uv} = zx^t$ for some $w$ and $z$ of suitable sizes. Thus, $w = \lambda x$ for some constant $\lambda$ in $\mathbb{C}$, that is, $X_{uv} = \lambda xx^t$.

Combining the above analysis, we know that

\[
\phi[0_{m,n-1} \mid x] = \begin{pmatrix}
0_{c_1 m, c_1 n} & E(x) & F(x) \\
0_{c_2 n, c_1 n} & 0_{c_2 n, c_2 m} & 0_{c_2 n, q'} \\
0_{p', c_1 n} & G(x) & H(x)
\end{pmatrix}
\]

where

\[
E(x) = (\lambda_{uv} xx^t)_{1 \leq u \leq c_1, 1 \leq v \leq c_2}, \\
F(x) = \begin{pmatrix}
xx_1^t \\
\vdots \\
xx_n^t
\end{pmatrix}, \\
G(x) = (z_1 w^t \cdots z_n x^t),
\]

$H(x)$, $\lambda_{uv}$, $w_u$ and $z_v$ all depend on $x$. By linearity of $\phi$, $\lambda_{uv}$, $w_u$ and $z_v$ must be the same for all $x$, and $\lambda_{uv}$ must be zero, i.e., $E(x) = 0_{c_1 m, c_2 m}$.

Now we consider the orthogonal pair $A = E_{11} + E_{1n}$ and $B = -E_{21} + E_{2n}$. Let $e_i$ be the $i$-th column of $I_m$. Then

\[
\phi(A) = \begin{pmatrix}
D_1 \otimes E_{11} & 0_{c_1 m, c_2 m} & F(e_1) \\
0_{c_2 n, c_1 n} & D_2 \otimes E_{11} & 0_{c_2 n, q'} \\
0_{p', c_1 n} & G(e_1) & H(e_1)
\end{pmatrix}
\]

and

\[
\phi(B) = \begin{pmatrix}
D_1 \otimes -E_{21} & 0_{c_1 m, c_2 m} & F(e_2) \\
0_{c_2 n, c_1 n} & D_2 \otimes -E_{21} & 0_{c_2 n, q'} \\
0_{p', c_1 n} & G(e_2) & H(e_2)
\end{pmatrix}.
\]
Set \( W = \begin{pmatrix} w_1^t \\ \vdots \\ w_c^t \end{pmatrix} \). Since \( \phi(A)\phi(B)^* = 0 \), the \((1, 1)\)-th block equals

\[
0_{c_1m} = (D_1 \otimes \widetilde{E}_{11})(D_1 \otimes -\tilde{E}_{21})^* + F(e_1)F(e_2)^* \\
= -(D_1^2 \otimes E_{12}) + (WW^* \otimes E_{12}) \\
= (WW^* - D_1^2) \otimes E_{12}.
\]

Thus, \( WW^* = D_1^2 \). Let \( D_1 = \text{diag}(d_1, \ldots, d_{c_1}) \). Hence, \( \{w_1/d_1, \ldots, w_{c_1}/d_{c_1}\} \) is a set of orthonormal vectors. Let \( U \in M_{q'} \) be a unitary matrix with \( w_{c_1}/d_{c_1} \) as the first \( c_1 \) rows. Then \( F'(x) = F(x)U^* = [D_1 \otimes x \mid 0_{c_1m,q'-c_1}] \).

Similarly, by considering \( \phi(A)^*\phi(B) = 0 \), we write

\[
G'(x) = V^*G(x) = \begin{pmatrix} D_2 \otimes x^t \\ 0_{p'-c_2,c_2m} \end{pmatrix}
\]

for some unitary \( V \). Now, we write

\[
\phi[0_{m-1} \mid x] = (I_{cn} \oplus V) \begin{pmatrix} 0_{c_1m,c_1n} & 0_{c_1m,c_2m} & F'(x) \\ 0_{c_2n,c_1n} & 0_{c_2n,c_2m} & 0_{c_2n,q'} \\ 0_{p',c_1n} & G'(x) & H'(x) \end{pmatrix} (I_{cn} \oplus U).
\]

On the other hand, by applying the assumption on the restriction of \( \phi \) on the subspace of \( M_{m,n} \) that consists of matrices with zero in the \((n-1)\)-th column, we conclude that

\[
\text{rank } \phi[0_{m-1} \mid x] = \text{rank } \phi[x \mid 0_{m-2} \mid x] = \text{rank } \phi[x \mid 0_{m-1}] = c.
\]

(Note that here we used the fact that \( n > m \geq 2 \) to ensure nontrivial consideration.) Therefore, \( H'(x) = 0 \) for all \( x \). Finally, there exist permutation matrices \( P \) and \( Q \) such that for \( A = [0_{m,n-1} \mid x] \),

\[
\phi(A) = (I_{cn} \oplus V)P[(D_1 \otimes A) \oplus (D_2 \otimes A^t) \oplus 0_{p'-c_2,q'-c_1}]Q(I_{cn} \oplus U).
\]

The result follows.

\[
\square
\]

REFERENCES


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