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NON-ADDITIVITY FOR TRIPLE POINT NUMBERS ON THE CONNECTED SUM OF SURFACE-KNOTS

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ABSTRACT. Any surface-knot F in 4-space can be projected into 3-space with a finite number of triple points, and its triple point number, $\mathrm{t}(F)$, is defined similarly to the crossing number of a classical knot. By definition, we have $\mathrm{t}(F_1\#F_2) \leq \mathrm{t}(F_1)+\mathrm{t}(F_2)$ for the connected sum. In this paper, we give infinitely many pairs of surface-knots for which this equality does not hold.

A *surface-knot* F is an (orientable or non-orientable) connected, closed surface smoothly embedded in Euclidean 4-space $\mathbb{R}^4$. Two surface-knots F and F' are *equivalent*, denoted by $F \cong F'$, if there is an ambient isotopy of $\mathbb{R}^4$ that maps F to F'. For a fixed projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, we can isotope F slightly so that the projection $\pi|_F$ into $\mathbb{R}^3$ is a generic map (cf. [4]). The set of triple points of such a generic map is discrete. The *triple point number* of F, denoted by $\mathrm{t}(F)$, is the minimal number of triple points for all possible generic projections of F. There are several studies on triple point numbers: [9, 10, 12, 13, 14, 16], for example.

The triple point number has an analogy to the crossing number $\mathrm{c}(K)$ of a classical knot K. The connected sum $K_1\#K_2$ of classical knots K_1 and K_2 satisfies

$$\mathrm{c}(K_1\#K_2) \leq \mathrm{c}(K_1)+\mathrm{c}(K_2), \tag{1}$$

and it is still an open problem whether the equality in (1) holds for any K_1 and K_2. Similarly, for the connected sum $F_1\#F_2$ of surface-knots F_1 and F_2, we have the following by definition:

$$\mathrm{t}(F_1\#F_2) \leq \mathrm{t}(F_1)+\mathrm{t}(F_2). \tag{2}$$

Hence, it is natural to ask whether the equality in (2) holds for any F_1 and F_2. The aim of this paper is to give a negative answer to this question.

Let $\tau^n K$ denote the n-twist-spin of a classical knot K (cf. [17]). Also, let $P_g(e)$ denote the non-orientable trivial surface-knots of genus g specified with the normal Euler number e (cf. [8]).

Theorem 1. *Assume that K is a 2-bridge knot and $n \geq 2$. Then we have*

$$\tau^n K\#P_3(\pm 2) \cong \begin{cases} \tau^0 K\#P_3(\pm 2) & \text{if } n \text{ is even,} \\ P_3(\pm 2) & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

It follows that the equality in (2) does not hold for any pair $\{F_1, F_2\} = \{\tau^n K, P_3(\pm 2)\}$.

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Let $\sigma^n K = \tau^n K + h_1$ denote the surface-knot of a torus obtained from $\tau^n K$ by surgery along a 1-handle h_1 contained in the axis-plane of twisting [2].

Lemma 2. *For any classical knot K, we have*

$$\sigma^n K \# P_1(\pm 2) \cong \sigma^{n+2} K \# P_1(\pm 2). \tag{4}$$

Proof. Recall that a projection of $\tau^n K$ is constructed by (i) making n writhes on the embedded sphere in $\mathbb{R}^3$, (ii) taking a simple closed curve L on the sphere that travels around all the writhes, and (iii) replacing the neighborhood of L by the product of L and a tangle diagram of K. Here, a *writhe* is regarded as a pile of motions representing Reidemeister moves I. Refer to [1] for more details. Hence, $\sigma^n K$ has a projection as shown in Figure 1.

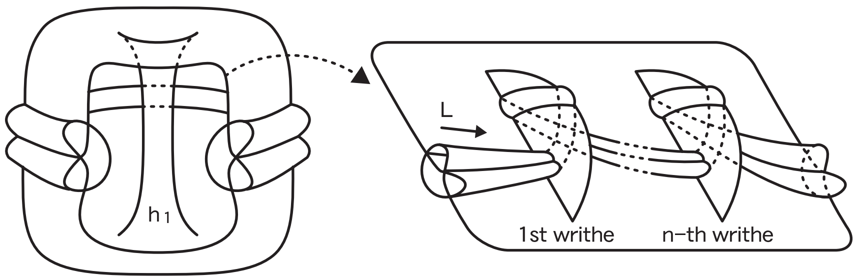

FIGURE 1.

We will eliminate the writhes from the diagram as follows. First, we move the attaching region of the handle h_1 close to the feet of the tangle diagram, and then expand and extend the inside of h_1 along L (the left of Figure 2). Next, we push the tube formed by the knot diagram of K out of the writhes, and then eliminate them by an ambient isotopy of $\mathbb{R}^4$ (the right of Figure 2). Hence, $\sigma^n K$ is equivalent to the surface-knot obtained from a surface-link $S_0 \cup_n T^n K$ by surgery along the 1-handle h_2, where S_0 is the trivial sphere-knot (cf. [6]), $T^n K$ is the n-turned torus-knot of K defined by Boyle [3], and $\cup_n$ means that $T^n K$ links S_0 by n times. By taking the connected sum of $P_1(\pm 2)$ with S_0, we can regard $\sigma^n K \# P_1(\pm 2)$ as $\big(P_1(\pm 2) \cup_n T^n K\big) + h_2$.

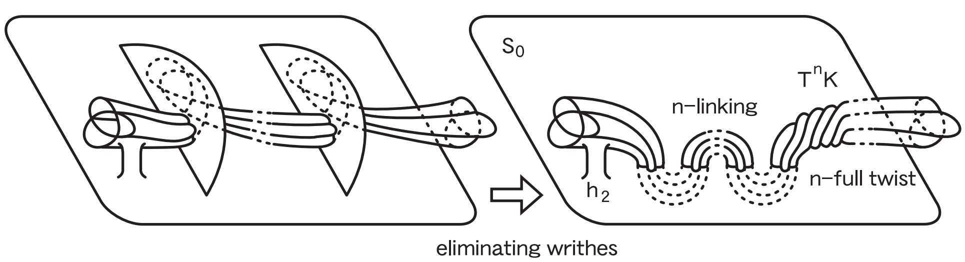

FIGURE 2.

Since $P_1(\pm 2)$ has the fundamental group $\mathbb{Z}_2$ of the complement in $\mathbb{R}^4$, the linking number between $P_1(\pm 2)$ and $T^n K$ is changeable up to the parity and relative to the handle h_2. It follows that

$$\big(P_1(\pm 2) \cup_n T^n K\big) + h_2 \cong \big(P_1(\pm 2) \cup_{n+2} T^n K\big) + h_2.$$

This equivalence is similar to Viro's work in [15]. Since $T^nK \cong T^{n+2}K$ relative to S_0 and h_2 (cf. [3]), we have the equivalence (4). □

Proof of Theorem 1. Let h_0 be the trivial 1-handle on $\tau^n K$. Then it is easy to see that $\tau^n K \# P_3(\pm 2) \cong (\tau^n K + h_0) \# P_1(\pm 2)$. On the other hand, Boyle [2] proved that if K is a 2-bridge knot, then h_0 and h_1 are equivalent relative to $\tau^n K$. It follows that $\tau^n K + h_0 \cong \tau^n K + h_1$, and hence, $\tau^n K \# P_3(\pm 2) \cong \sigma^n K \# P_1(\pm 2)$. Since Zeeman [17] proved that $\tau^1 K \cong S_0$, we have the equivalence (3) by Lemma 2.

The latter assertion is proved as follows. Since $\tau^n K$ is not of ribbon-type for $n \geq 2$ [5], we have $\mathrm{t}(\tau^n K) > 0$ [16]; see also [7]. (This result has been strengthened to $\mathrm{t}(\tau^n K) \geq 4$ in [11].) On the other hand, it follows that $\mathrm{t}\big(\tau^0 K \# P_3(\pm 2)\big) = \mathrm{t}\big(P_3(\pm 2)\big) = 0$ by definition (cf. [4]). Note that $\mathrm{t}(\tau^0 K) = 0$ for any K. Hence, we have $\mathrm{t}\big(\tau^n K \# P_3(\pm 2)\big) = 0 < \mathrm{t}(\tau^n K) + \mathrm{t}\big(P_3(\pm 2)\big)$ by the equivalence (3). □

Remark 3. It is still an open problem whether $\tau^n K \# P_1(\pm 2) \cong P_1(\pm 2)$ for any classical knot K and odd integer $n > 1$. Note that they have the same fundamental group $\mathbb{Z}_2$ of the complement in $\mathbb{R}^4$.

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