NON-ADDITIVITY FOR TRIPLE POINT NUMBERS
ON THE CONNECTED SUM OF SURFACE-KNOTS

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Abstract. Any surface-knot $F$ in 4-space can be projected into 3-space with a finite number of triple points, and its triple point number, $t(F)$, is defined similarly to the crossing number of a classical knot. By definition, we have $t(F_1 \# F_2) \leq t(F_1) + t(F_2)$ for the connected sum. In this paper, we give infinitely many pairs of surface-knots for which this equality does not hold.

A surface-knot $F$ is an (orientable or non-orientable) connected, closed surface smoothly embedded in Euclidean 4-space $\mathbb{R}^4$. Two surface-knots $F$ and $F'$ are equivalent, denoted by $F \cong F'$, if there is an ambient isotopy of $\mathbb{R}^4$ that maps $F$ to $F'$. For a fixed projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$, we can isotope $F$ slightly so that the projection $\pi|_F$ into $\mathbb{R}^3$ is a generic map (cf. [4]). The set of triple points of such a generic map is discrete. The triple point number of $F$, denoted by $t(F)$, is the minimal number of triple points for all possible generic projections of $F$. There are several studies on triple point numbers: [9, 10, 12, 13, 14, 16], for example.

The triple point number has an analogy to the crossing number $c(K)$ of a classical knot $K$. The connected sum $K_1 \# K_2$ of classical knots $K_1$ and $K_2$ satisfies

$$c(K_1 \# K_2) \leq c(K_1) + c(K_2),$$

and it is still an open problem whether the equality in (1) holds for any $K_1$ and $K_2$. Similarly, for the connected sum $F_1 \# F_2$ of surface-knots $F_1$ and $F_2$, we have the following by definition:

$$t(F_1 \# F_2) \leq t(F_1) + t(F_2).$$

Hence, it is natural to ask whether the equality in (2) holds for any $F_1$ and $F_2$. The aim of this paper is to give a negative answer to this question.

Let $\tau^n K$ denote the $n$-twist-spin of a classical knot $K$ (cf. [17]). Also, let $P_g(e)$ denote the non-orientable trivial surface-knots of genus $g$ specified with the normal Euler number $e$ (cf. [8]).

**Theorem 1.** Assume that $K$ is a 2-bridge knot and $n \geq 2$. Then we have

$$\tau^n K \# P_3(\pm 2) \cong \begin{cases} 
\tau^0 K \# P_3(\pm 2) & \text{if } n \text{ is even,} \\
P_3(\pm 2) & \text{if } n \text{ is odd.}
\end{cases}$$

It follows that the equality in (2) does not hold for any pair $\{F_1, F_2\} = \{\tau^n K, P_3(\pm 2)\}$. 

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Let $\sigma^n K = \tau^n K + h_1$ denote the surface-knot of a torus obtained from $\tau^n K$ by surgery along a 1-handle $h_1$ contained in the axis-plane of twisting [2].

**Lemma 2.** For any classical knot $K$, we have

\begin{equation}
\sigma^n K \# P_1(\pm 2) \cong \sigma^{n+2} K \# P_1(\pm 2).
\end{equation}

**Proof.** Recall that a projection of $\tau^n K$ is constructed by (i) making $n$ writhes on the embedded sphere in $\mathbb{R}^3$, (ii) taking a simple closed curve $L$ on the sphere that travels around all the writhes, and (iii) replacing the neighborhood of $L$ by the product of $L$ and a tangle diagram of $K$. Here, a *writhe* is regarded as a pile of motions representing Reidemeister moves I. Refer to [1] for more details. Hence, $\sigma^n K$ has a projection as shown in Figure 1.

![Figure 1](image1.png)

We will eliminate the writhes from the diagram as follows. First, we move the attaching region of the handle $h_1$ close to the feet of the tangle diagram, and then expand and extend the inside of $h_1$ along $L$ (the left of Figure 2). Next, we push the tube formed by the knot diagram of $K$ out of the writhes, and then eliminate them by an ambient isotopy of $\mathbb{R}^4$ (the right of Figure 2). Hence, $\sigma^n K$ is equivalent to the surface-knot obtained from a surface-link $S_0 \cup_n T^n K$ by surgery along the 1-handle $h_2$, where $S_0$ is the trivial sphere-knot (cf. [6]), $T^n K$ is the $n$-turned torus-knot of $K$ defined by Boyle [3], and $\cup_n$ means that $T^n K$ links $S_0$ by $n$ times. By taking the connected sum of $P_1(\pm 2)$ with $S_0$, we can regard $\sigma^n K \# P_1(\pm 2)$ as $(P_1(\pm 2) \cup_n T^n K) + h_2$.

![Figure 2](image2.png)

Since $P_1(\pm 2)$ has the fundamental group $\mathbb{Z}_2$ of the complement in $\mathbb{R}^4$, the linking number between $P_1(\pm 2)$ and $T^n K$ is changeable up to the parity and relative to the handle $h_2$. It follows that

\[
(P_1(\pm 2) \cup_n T^n K) + h_2 \cong (P_1(\pm 2) \cup_{n+2} T^n K) + h_2.
\]
This equivalence is similar to Viro’s work in [15]. Since $T^n K \cong T^{n+2} K$ relative to $S_0$ and $h_2$ (cf. [3]), we have the equivalence (4).

Proof of Theorem 7. Let $h_0$ be the trivial 1-handle on $\tau^n K$. Then it is easy to see that $\tau^n K \# P_3(\pm 2) \cong (\tau^n K + h_0) \# P_1(\pm 2)$. On the other hand, Boyle [2] proved that if $K$ is a 2-bridge knot, then $h_0$ and $h_1$ are equivalent relative to $\tau^n K$. It follows that $\tau^n K + h_0 \cong \tau^n K + h_1$, and hence, $\tau^n K \# P_3(\pm 2) \cong \sigma^n K \# P_1(\pm 2)$. Since Zeeman [17] proved that $\tau^1 K \cong S_0$, we have the equivalence (5) by Lemma 2.

The latter assertion is proved as follows. Since $\tau^n K$ is not of ribbon-type for $n \geq 2$ [5], we have $t(\tau^n K) > 0$ [16]; see also [7]. (This result has been strengthened to $t(\tau^n K) \geq 4$ in [11].) On the other hand, it follows that $t(\tau^n K \# P_3(\pm 2)) = t(P_3(\pm 2)) = 0$ by definition (cf. [3]). Note that $t(\tau^n K) = 0$ for any $K$. Hence, we have $t(\tau^n K \# P_3(\pm 2)) = 0 < t(\tau^n K) + t(P_3(\pm 2))$ by the equivalence (4).

Remark 3. It is still an open problem whether $\tau^n K \# P_1(\pm 2) \cong P_1(\pm 2)$ for any classical knot $K$ and odd integer $n > 1$. Note that they have the same fundamental group $\mathbb{Z}_2$ of the complement in $\mathbb{R}^4$.

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