

CONVERGENCE OF PATHS
AND APPROXIMATION OF FIXED POINTS
OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

C. E. CHIDUME, JINLU LI, AND A. UDOMENE

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ABSTRACT. Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E , $T : K \rightarrow K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} t_n = 1$, $t_n k_n < 1$, and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_0 \in K$, $z_{n+1} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n z_n$, $n = 0, 1, 2, \dots$. It is proved that, for each integer $n \geq 0$, there is a unique $x_n \in K$ such that $x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n$. If, in addition, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$, then $\{z_n\}_n$ converges *strongly* to a fixed point of T .

1. INTRODUCTION

Let E be a normed linear space with dual E^* , and let K be a nonempty subset of E . Let $J : E \rightarrow 2^{E^*}$ denote the *normalized duality mapping* defined by $J(x) := \{f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|, x \in E\}$. A mapping $T : K \rightarrow K$ is called *uniformly L -Lipschitzian* ($L > 0$) if $\|T^n x - T^n y\| \leq L\|x - y\|$, $\forall x, y \in K$ and for all integers $n \geq 0$. The mapping T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all integers $n \geq 0$. It is clear that every asymptotically nonexpansive mapping is uniformly L -Lipschitzian for some constant $L > 0$. The asymptotically nonexpansive mappings are important generalizations of nonexpansive mappings, where a mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$.

The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [7], and they proved that if K is a nonempty bounded closed and convex subset of a uniformly convex Banach space E , and $T : K \rightarrow K$ is asymptotically nonexpansive, then T has a fixed point. In [8], they extended this result to the broader class of uniformly L -Lipschitzian mappings with $L < \gamma$, where γ is sufficiently near 1. Several authors have investigated iterative methods for approximating fixed points of asymptotically nonexpansive mappings (e.g., Bruck, Kuczumow and Reich [3],

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Chang [4], Lim and Xu [11], Schu [14, 15], Shioji and Takahashi [12, 13], Tan and Xu [16]). In [15], Schu proved the following theorems.

Theorem S1 ([15]). *Let $(E, \|\cdot\|)$ be a smooth reflexive Banach space possessing a duality mapping $J : E \rightarrow E^*$ that is weakly sequentially continuous at 0; $\emptyset \neq A \subset E$ closed, bounded and star-shaped with respect to 0; $T : A \rightarrow A$ asymptotically nonexpansive with sequence $\{k_n\}_n \subset [1, \infty)$ and uniformly asymptotically regular (see [15] for definition), $(id - T)$ demiclosed; $\{\lambda_n\}_n \subset (\frac{1}{2}, 1)$; $\lim_{n \rightarrow \infty} \lambda_n = 1$; $k_n \leq \frac{\lambda_n^2}{2\lambda_n - 1}$ for all integers $n \geq 0$. Then,*

- (i) for each $n \in \mathbb{N}$ there is exactly one $x_n \in A$ such that $x_n = \frac{\lambda_n}{k_n} T^n x_n$,
- (ii) $\{x_n\}_n$ converges strongly to some fixed point of T .

Theorem S2 ([15]). *Let $(E, \|\cdot\|)$ be a normed space; $\emptyset \neq A \subset E$ bounded and star-shaped with respect to 0; $T : A \rightarrow A$ asymptotically nonexpansive with sequence $\{k_n\}_n \subset [1, \infty)$; $\{\lambda_n\}_n \subset (0, 1)$ with $\lim_{n \rightarrow \infty} \lambda_n = 1$; $\{x_n\}_n \subset A$; $\mu_n = \frac{\lambda_n}{k_n}$ for all integers $n \geq 0$, $z_0 \in A$; $x_n = \frac{\lambda_n}{k_n} T^n x_n$ for all integers $n \geq 0$, and $z_{n+1} := \frac{\lambda_{n+1}}{k_{n+1}} T^n z_n$ for all integers $n \geq 0$; $\{\epsilon_n\}_n \subset (0, \infty)$ such that $(\{\epsilon_n\}_n, \{\mu_n\}_n)$ is admissible (see, e.g., [15] or [10] for a definition). Also*

- (i) $\{\frac{1-\mu_n}{1-\lambda_n}\}$ is bounded;
- (ii) $\|T^n x - T^{n+1} x\| \leq \epsilon_n$ for all integers $n \geq 0$ and all $x \in A$.

Assume further that $q := \lim_{n \rightarrow \infty} x_n$ exists. Then $\lim_{n \rightarrow \infty} z_n = q$.

Theorem S1 provides sufficient conditions for the path given by the sequence $\{x_n\}_n$ for an asymptotically nonexpansive mapping to converge strongly to a fixed point of T . However, since J is assumed to be weakly sequentially continuous at 0, the theorem is not applicable to L_p spaces ($p > 1$, $p \neq 2$). In [11], Lim and Xu proved a path-convergence theorem for asymptotically nonexpansive mappings in the more general *uniformly smooth Banach spaces*. These spaces include the L_p spaces, $1 < p < \infty$. They proved the following theorem.

Theorem LX ([11, Theorem 2, p. 1348]). *Let E be a uniformly smooth Banach space, K a nonempty closed convex and bounded subset of E , $T : K \rightarrow K$ an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Fix $u \in K$ and let $\{t_n\}_n \subset [0, 1)$ be chosen such that $\lim_{n \rightarrow \infty} t_n = 1$ and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Then,*

- (i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$(1) \quad x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_n.$$

Suppose, in addition, that $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. Then,

- (ii) the sequence $\{x_n\}_n$ converges strongly to a fixed point of T .

Very recently Xu [17] proved the following theorem (which is the main result of [17]) for iterative approximation of a fixed point of a *nonexpansive mapping in uniformly smooth Banach spaces*.

Theorem Xu ([17, Theorem 3.1, p. 244]). *Let X be a uniformly smooth Banach space, C be a closed convex subset of X , and $S : C \rightarrow C$ be a nonexpansive mapping with a fixed point. Let $u, x_0 \in C$ be given. Assume that $\{\alpha_n\} \subset [0, 1]$*

satisfies the following conditions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ or equivalently $\prod_{n=0}^{\infty} (1 - \alpha_n) = 0$;
- (iii) $\lim_{n \rightarrow \infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n} = 0$.

Then the sequence $\{x_n\}$ generated by

$$x_{n+1} := \alpha_n u + (1 - \alpha_n) S x_n, \quad n = 0, 1, 2, \dots,$$

converges strongly to a fixed point of S .

Our main purpose in this paper is to prove, under appropriate conditions on K and T , that a sequence defined iteratively by: $z_0 \in K$,

$$(2) \quad z_{n+1} := \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n} T^n z_n, \quad n = 0, 1, 2, \dots,$$

converges strongly to a fixed point of the asymptotically nonexpansive mapping T .

2. PRELIMINARIES

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of the Banach space E . E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$, and we call E *smooth*; and E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Furthermore, E is said to be *uniformly smooth* if the limit exists uniformly for $(x, y) \in S \times S$. It is known that if E is smooth, then any duality mapping on E is single-valued, and if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* continuous (see, e.g., [6]). Let K be a nonempty closed convex and bounded subset of the Banach space E , and let the *diameter* of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and let $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (cf. [2]) is defined by

$$N(E) := \inf\left\{\frac{d(K)}{r(K)} : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0\right\}.$$

A space E such that $N(E) > 1$ is said to have *uniform normal structure*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [1] or [11]).

We shall let LIM be a Banach limit. Recall that $\text{LIM} \in (\ell^\infty)^*$ such that $\|\text{LIM}\| = 1$, $\liminf_{n \rightarrow \infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n \rightarrow \infty} a_n$, and $\text{LIM}_n a_n = \text{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in \ell^\infty$.

The following lemmas will be needed in the sequel.

Lemma 2.1. *Let E be an arbitrary real Banach space. Then*

$$(3) \quad \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle,$$

$\forall x, y \in E$ and $\forall j(x + y) \in J(x + y)$.

Lemma 2.2 ([11]). *Suppose X is a Banach space with uniformly normal structure, C is a nonempty bounded subset of X , and $T : C \rightarrow C$ is a uniformly k -Lipschitzian mapping with $k < N(X)^{\frac{1}{2}}$. Suppose also that there exists a nonempty bounded closed convex subset A of C with the following property (P):*

$$(P) \quad x \in A \text{ implies } \omega_w(x) \subset A,$$

where $\omega_w(x)$ is the weak ω -limit set of T at x , i.e., the set

$$\{y \in X : y = \text{weak-}\lim_j T^{n_j} x \text{ for some } n_j \rightarrow \infty\}.$$

Then T has a fixed point in A .

Lemma 2.3 (see, e.g., [17]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n, n \geq 0,$$

where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose, either (a) $\sigma_n = o(\alpha_n)$, or (b) $\limsup \sigma_n \leq 0$. Then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

3. MAIN RESULTS

In the sequel $F(T)$ will denote the fixed point set of T .

Theorem 3.1. *Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K a nonempty closed convex and bounded subset of E , $T : K \rightarrow K$ an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} t_n = 1$, and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Then,*

(i) *for each integer $n \geq 0$, there is a unique $x_n \in K$ such that*

$$x_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}T^n x_n;$$

and if, in addition, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then,

(ii) *the sequence $\{x_n\}_n$ converges strongly to a fixed point of T .*

Remark 1. We remark that Theorem 2 of [11] (cited here as Theorem LX) actually works in a Banach space that has a uniformly Gâteaux differentiable norm and has a uniformly normal structure. Consequently, following the lines of the proof of Theorem LX, Theorem 3.1 is easily proved.

Remark 2. Lim and Xu [11] have shown that a sequence $\{t_n\}_n \subset (0, 1)$ satisfying the condition of Theorem 3.1 always exists. They gave, for example, $t_n := \min\{1 - (k_n - 1)^{\frac{1}{2}}, 1 - n^{-1}\}$.

Corollary 3.2. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, K a nonempty closed convex and bounded subset of E , $T : K \rightarrow K$ a completely continuous, asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} t_n = 1$,*

and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Then,

(i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$x_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}T^n x_n;$$

and if, in addition, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then,

(ii) the sequence $\{x_n\}_n$ converges strongly to a fixed point of T .

Proof. For each integer $n \geq 0$, the mapping $f_n : K \rightarrow K$ defined for each $x \in K$ by $f_n x := \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}T^n x$ is a contraction. It follows that there exists a unique $x_n \in K$ such that $f_n x_n = x_n$. Since T is completely continuous there is a subsequence $\{Tx_{n_i}\}_i$ of $\{Tx_n\}_n$ that converges strongly to some $y^* \in K$, and since $\|x_{n_i} - Tx_{n_i}\| \rightarrow 0$ as $i \rightarrow \infty$, we have that $y^* = Ty^*$. The rest of the proof follows as in the proof of Theorem 3.1. \square

Theorem 3.3. Let E be a real Banach space with a uniformly Gâteaux differentiable norm possessing uniform normal structure, K be a nonempty closed convex and bounded subset of E , $T : K \rightarrow K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} t_n = 1$, $t_n k_n < 1$, and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_0 \in K$,

$$(4) \quad z_{n+1} = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}T^n z_n, \quad n = 0, 1, 2, \dots$$

Then,

(i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$(5) \quad x_n = \left(1 - \frac{t_n}{k_n}\right)u + \frac{t_n}{k_n}T^n x_n;$$

and if, in addition, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, $\lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0$, then

(ii) $\{z_n\}_n$ converges strongly to a fixed point of T .

Proof. From (5), $x_m - z_n = \left(1 - \frac{t_m}{k_m}\right)(u - z_n) + \frac{t_m}{k_m}(T^m x_m - z_n)$. Applying inequality (3), we estimate as follows:

$$\begin{aligned} \|x_m - z_n\|^2 &\leq \frac{t_m^2}{k_m^2} \|T^m x_m - z_n\|^2 + 2\left(1 - \frac{t_m}{k_m}\right) \langle u - z_n, j(x_m - z_n) \rangle \\ &\leq \frac{t_m^2}{k_m^2} [\|T^m x_m - T^m z_n\| + \|T^m z_n - z_n\|]^2 \\ &\quad + 2\left(1 - \frac{t_m}{k_m}\right) [\langle u - x_m, j(x_m - z_n) \rangle + \|x_m - z_n\|^2] \\ &\leq \frac{t_m^2}{k_m^2} [k_m \|x_m - z_n\| + \|T^m z_n - z_n\|]^2 \\ &\quad + 2\left(1 - \frac{t_m}{k_m}\right) [\langle u - x_m, j(x_m - z_n) \rangle + k_m^2 \|x_m - z_n\|^2] \\ &= \frac{t_m^2}{k_m^2} [k_m^2 \|x_m - z_n\|^2 + 2k_m \|x_m - z_n\| \|T^m z_n - z_n\| + \|T^m z_n - z_n\|^2] \\ &\quad + 2\left(1 - \frac{t_m}{k_m}\right) [\langle u - x_m, j(x_m - z_n) \rangle + k_m^2 \|x_m - z_n\|^2] \end{aligned}$$

$$\begin{aligned}
&= \left(1 - \left(1 - \frac{t_m}{k_m}\right)\right)^2 k_m^2 \|x_m - z_n\|^2 + \|T^m z_n - z_n\| [2k_m \|x_m - z_n\| \\
&\quad + \|T^m z_n - z_n\|] + 2\left(1 - \frac{t_m}{k_m}\right) [\langle u - x_m, j(x_m - z_n) \rangle + k_m^2 \|x_m - z_n\|^2] \\
&\leq \left(1 + \left(1 - \frac{t_m}{k_m}\right)^2\right) k_m^2 \|x_m - z_n\|^2 + \|T^m z_n - z_n\| M \\
&\quad + 2\left(1 - \frac{t_m}{k_m}\right) \langle u - x_m, j(x_m - z_n) \rangle,
\end{aligned}$$

for some constant $M > 0$. It follows that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle u - x_m, j(z_n - x_m) \rangle &\leq \frac{[k_m^2 - 1 + k_m^2 (1 - \frac{t_m}{k_m})^2]}{2(1 - \frac{t_m}{k_m})} \limsup_{n \rightarrow \infty} \|x_m - z_n\|, \\
&\quad + \limsup_{n \rightarrow \infty} \frac{M \|z_n - T^m z_n\|}{(1 - \frac{t_m}{k_m})}.
\end{aligned}$$

Observe that $\frac{[k_m^2 - 1 + k_m^2 (1 - \frac{t_m}{k_m})^2]}{2(1 - \frac{t_m}{k_m})} = \frac{k_m(k_m + 1)}{2} \left[\frac{k_m - 1}{k_m - t_m}\right] + \frac{k_m^2}{2} (1 - \frac{t_m}{k_m}) \rightarrow 0$ as $m \rightarrow \infty$. Since $\{z_n\}$ and $\{x_m\}$ are bounded, $\{T^m z_n\}$ is bounded and $\|z_n - T z_n\| \rightarrow 0$ as $n \rightarrow \infty$, it follows from the last inequality that

$$(6) \quad \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \langle u - x_m, j(z_n - x_m) \rangle \leq 0.$$

But by Theorem 3.1 we have that $x_m \rightarrow x^* \in F(T)$ as $m \rightarrow \infty$. Moreover, j is norm to weak* uniformly continuous on bounded sets. Therefore, there exists $N > 0$ such that

$$|\langle x^* - x_m, j(z_n - x_m) \rangle| < \frac{\epsilon}{2} \quad \text{and} \quad |\langle u - x^*, j(z_n - x_m) - j(z_n - x^*) \rangle| < \frac{\epsilon}{2},$$

for all $n, m \geq N$. This implies

$$\begin{aligned}
&|\langle u - x_m, j(z_n - x_m) \rangle - \langle u - x^*, j(z_n - x^*) \rangle| \\
&\leq |\langle u - x_m, j(z_n - x_m) \rangle - \langle u - x^*, j(z_n - x_m) \rangle| \\
&\quad + |\langle u - x^*, j(z_n - x_m) \rangle - \langle u - x^*, j(z_n - x^*) \rangle| \\
&= |\langle x^* - x_m, j(z_n - x_m) \rangle| + |\langle u - x^*, j(z_n - x_m) - j(z_n - x^*) \rangle| \\
(7) \quad &\leq \epsilon \quad \text{for all } n, m \geq N.
\end{aligned}$$

Thus, from (6) and (7),

$$\limsup_{n \rightarrow \infty} \langle u - x^*, j(z_n - x^*) \rangle \leq \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \langle u - x_m, j(z_n - x_m) \rangle + \epsilon \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain that

$$\limsup_{n \rightarrow \infty} \langle u - x^*, j(z_n - x^*) \rangle \leq 0.$$

Now from the iteration procedure (4) and inequality (3) we have that

$$\begin{aligned}
\|z_{n+1} - x^*\|^2 &\leq \frac{t_n^2}{k_n^2} \|T^n z_n - x^*\|^2 + 2\left(1 - \frac{t_n}{k_n}\right) \langle u - x^*, j(z_{n+1} - x^*) \rangle \\
&\leq \frac{t_n}{k_n} \|z_n - x^*\|^2 + 2\left(1 - \frac{t_n}{k_n}\right) \langle u - x^*, j(z_{n+1} - x^*) \rangle, \\
&\leq (1 - \alpha_n) \|z_n - x^*\|^2 + 2\alpha_n \beta_n,
\end{aligned}$$

where $\alpha_n := (1 - \frac{t_n}{k_n})$. So $\beta_n := \langle u - x^*, j(z_{n+1} - x^*) \rangle$, so that $\limsup \alpha_n \beta_n \leq 0$. It now follows from Lemma 2.3 that $z_n \rightarrow x^*$ as $n \rightarrow \infty$, completing the proof. \square

Corollary 3.4. *Let E be a real reflexive Banach space with a uniformly Gâteaux differentiable norm, let K be a nonempty closed convex and bounded subset of E , and let $T : K \rightarrow K$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}_n \subset [1, \infty)$. Let $u \in K$ be fixed, $\{t_n\}_n \subset (0, 1)$ be such that $\lim_{n \rightarrow \infty} t_n = 1$, $t_n k_n < 1$ and $\lim_{n \rightarrow \infty} \frac{k_n - 1}{k_n - t_n} = 0$. Define the sequence $\{z_n\}_n$ iteratively by $z_0 \in K$,*

$$z_{n+1} = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n z_n, \quad n = 0, 1, 2, \dots$$

Then,

- (i) for each integer $n \geq 0$, there is a unique $x_n \in K$ such that

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n;$$

and if, in addition, $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ and $\lim_n \|z_n - Tz_n\| = 0$, then,

- (ii) $\{z_n\}_n$ converges strongly to a fixed point of T .

Proof. As in the proof of Corollary 3.2 there exists $y^* \in K$ such that $Ty^* = y^*$. The rest of the proof follows as in Theorem 3.3. \square

Remark 3. An example of a sequence $\{t_n\}_n \subset (0, 1)$ satisfying the conditions of Theorem 3.3 and Corollary 3.4 is given by

$$t_n := \min\{k_n^{-1} - k_n^{-1}(k_n - 1)^{\frac{1}{2}}, 1 - n^{-1}\}, \quad n = 0, 1, 2, \dots$$

Remark 4. We observe that if we set $\alpha_n = 1 - \frac{t_n}{k_n}$ and T is nonexpansive, in which case $k_n \equiv 1 \forall$ integers $n \geq 0$, then, replacing T^n by T , our iteration process (2) reduces to the one studied by Xu [17]. Moreover, the boundedness of K in Theorem 3.3 is no longer required. Only the boundedness of the sequence $\{z_n\}$ is needed, and this follows as in Xu [17]. Furthermore, in this case the condition $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ is satisfied. The choice $t_n := \min\{k_n^{-1} - k_n^{-1}(k_n - 1)^{\frac{1}{2}}, 1 - n^{-1}\}$ reduces to $\alpha_n = \frac{1}{n}$, which is the natural choice that is part of the motivation of the paper of Xu [17]. Finally, it is well known (see, e.g., [6] and [9]) that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm and possesses uniform normal structure.

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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS, TRIESTE, ITALY
E-mail address: `chidume@ictp.trieste.it`

DEPARTMENT OF MATHEMATICS, SHAWNEE STATE UNIVERSITY, PORTSMOUTH, OHIO 45662
E-mail address: `jli@shawnee.edu`

DEPARTMENT OF MATHEMATICS, STATISTICS, COMPUTER SCIENCE, UNIVERSITY OF PORT HARCOURT, PORT HARCOURT, NIGERIA
E-mail address: `EpsilonAni@aol.com`