RATIONAL IRREDUCIBLE PLANE CONTINUA
WITHOUT THE FIXED-POINT PROPERTY

CHARLES L. HAGOPIAN AND ROMAN MAŇKA

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Abstract. We define rational irreducible continua in the plane that admit fixed-point-free maps with the condition that all of their tranches have the fixed-point property. This answers in the affirmative a question of Hagopian. The construction is based on a special class of spirals that limit on a double Warsaw circle. The closure of each of these spirals has the fixed-point property.

1. Introduction

A continuum is a non-void compact connected metric space. A continuum is irreducible between two of its points if none of its proper subcontinua contains both of these points. A continuum is rational if each of its points admits a base of neighborhoods with countable boundaries.

Since rational continua are hereditarily decomposable [Kn2, Thm. 5, p. 285], each rational irreducible continuum $C$ is of type $\lambda$, i.e., of the order type of the unit interval $[0, 1]$. This means, following Knaster and Kuratowski [Kn1, §3, pp. 248, 262], that $C$ admits a uniquely determined monotone upper semi-continuous decomposition to an arc (i.e., homeomorphic image of $[0, 1]$) with the property that each element of the decomposition has void interior relative to $C$ [Kn2, Thm. 3, p. 216, Thm. 2, p. 215]. The continua that are elements of this decomposition are called the tranches of $C$.

A continuum has the fixed-point property if each map of it into itself has a fixed point.

To answer a question of Gordh [L, Prob. 43, p. 371], Hagopian [H] recently defined a non-planar irreducible rational continuum $M$ such that each tranche of $M$ has the fixed-point property and $M$ admits a fixed-point-free map. Hagopian [H, Ques. 1] asked if there is a continuum with these properties in the plane. Here we give a positive answer to this question.

A double Warsaw circle is a union of two disjoint rays each of which limits on an initial arc of the other. It follows from a result of Nadler [N, Lem. 11, p. 131] that each double Warsaw circle is planar (also see Awartani’s theorem [A, Thm. 4.1, p. 232]).

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Our construction involves two spirals limiting on a double Warsaw circle. One spiral lies outside and the other inside of the double Warsaw circle. The closure of one spiral has the fixed-point property, and the other does not. The closure that has the fixed-point property is approximated by a ray in such a way that the resulting irreducible rational continuum can be retracted onto the closure of the spiral without the fixed-point property.

2. Preliminaries on rays and arcs

In this paper a ray is a topological half-line, i.e., homeomorphic image of \([0,1)\). Every ray \(R\) will be considered with its natural order \(<_R\) inherited from \([0, 1)\).

If \(R\) is a ray contained in a continuum, then \(CIR\) (the closure of \(R\)) is a continuum irreducible between the origin (initial point) of \(R\) and each limit point of \(R\), i.e., a limit of an increasing convergent sequence of points of \(R\) \([\text{M2}]\) Prop. 2.1]. The continuum \(CIR\) is rational if (and only if) the continuum \(L(R)\) consisting of all limit points of \(R\) is rational. Obviously, \(R \cap L(R) = \emptyset\) and \(R \cup L(R) = CIR\). A continuum \(C\) of the form \(C = CIR \cup ClS\), where \(R\) and \(S\) are rays with origins \(p\) and \(q\), respectively, is irreducible between \(p\) and \(q\) if \(R \cap ClS = \emptyset = S \cap CIR\) (see \([\text{M1}]\) Rem. 1, p. 125] for a general statement).

A space is (uniquely) arcwise connected if every two of its points are end points of a (unique) arc in the space.

**Lemma 2.1.** Suppose \(f\) is a map of a space \(L\) into a uniquely arcwise connected space and an arc \(M\) is contained in \(f(L)\). Then for every family of arcwise connected sets \(L_1, L_2, \cdots, L_v\) such that \(L = \bigcup_{j=1}^v L_j\) and for every partition \(\mathcal{P}(M)\) of \(M\) into consecutive arcs \(M_1, M_2, \cdots, M_\mu\) with \(v \leq \mu\), there exist \(L_j\) and \(M_i\) such that \(M_i \subset f(L_j)\).

**Proof.** Let \(q_0, q_1, \cdots, q_\mu\) be the consecutive end points of the arcs \(M_1, M_2, \cdots, M_\mu\) belonging to \(\mathcal{P}(M)\). We have \(M_i \cap M_{i+1} = \{q_i\}\) for each \(i = 1, 2, \cdots, \mu - 1\). For each \(j = 1, 2, \cdots, v\), let \(\Sigma_j = \{i \mid q_i \in f(L_j)\}\). It suffices to show that some \(\Sigma_j\) has more than one element.

Suppose, on the contrary, that each \(\Sigma_j\) has at most one element. Then the set \(\Sigma = \bigcup_{j=1}^v \Sigma_j\) has at most \(v\) elements. Since \(M \subset f(L) = f\left(\bigcup_{j=1}^v L_j\right)\), it follows that \(\Sigma = \{i \mid q_i \in f(L)\}\) and \(\{0, 1, 2, \cdots, \mu\} \subset \Sigma\). Hence \(\mu + 1 \leq v\). This contradicts the assumption that \(v \leq \mu\).

This lemma will be used in Section 5. Given a map \(f\), we will say sometimes that \(X\) is mapped on \(Y\) when \(Y \subset f(X)\).

**Notation.** For each two points \(a\) and \(b\) of the plane \(E^2\), we denote the straight line interval with end points \(a\) and \(b\) by \(\overline{ab}\). If \(a\) and \(b\) belong to a uniquely arcwise connected set \(A\), then \(A[a,b]\) will denote the unique arc in \(A\) with end points \(a\) and \(b\). Given a set \(Z\) in \(E^2\) and a real number \(\epsilon\), we let \(Z \uparrow \epsilon\) denote \(\{(x, y) \in E^2 \mid (x, y - \epsilon) \in Z\}\). We denote the set of all positive integers \(1, 2, \cdots\) by \(N\).

3. A class of spirals limiting on a double Warsaw circle

Let \(\gamma\) be the map of \((0, 1]\) onto \([1, 3]\) defined by \(\gamma(x) = 2 + \sin \frac{x}{\pi}\), and let \(\Gamma\) denote the graph of \(\gamma\) in \(E^2\), i.e., \(\Gamma = \{(x, \gamma(x)) \in E^2 \mid x \in (0, 1]\}\).
The set $A = (0, -1)(0, -3) \cup (0, -3)(2, 0) \cup (2, 0)(1, 2) \cup \Gamma$ is a ray with origin $(0, -1)$ that limits on the interval $(0, 1)(0, 3)$. Note that $CIA$ is the topologist’s sine curve.

Let $\alpha$ be the map of $E^2$ onto $E^2$ defined by $\alpha(x, y) = (-x, -y)$.

Let $D$ be the continuum $A \cup \alpha(A)$ in $E^2$. Note that $D$ is a double Warsaw circle.

Consider two sequences $\{k_n\}_{n \in \mathbb{N}}$ and $\{l_n\}_{n \in \mathbb{N}}$ of positive integers satisfying the following conditions:

(3.1) \quad k_1 \leq k_2 \leq \cdots \leq k_n \leq k_{n+1} \leq \cdots \to \infty \quad \text{and} \quad l_1 \leq l_2 \leq \cdots \leq l_n \leq l_{n+1} \leq \cdots \to \infty.

Similar to $T$ in Example 4.1 of [H–M], we define a spiral $S\{k_n, l_n\}_{n \in \mathbb{N}}$ limiting on $D$.

For each positive integer $n$, let $\mu_n = \frac{2}{4n+1}$ and let $p_n$ be the point $(\mu_n, 3)$ of $\Gamma$.

For each $n$, let

\[
A_n = \frac{(2 + n^{-1}, 0)(1, 2 + n^{-1})}{n^{-1}}, \quad B_n(k_n) = A((1, 2), p_k) \uparrow n^{-1},
\]

\[
C_n(k_n) = \{(x, y) \in E^2 \mid 0 \leq x \leq \mu_n \text{ and } y = 3 + n^{-1}\},
\]

\[
D_n = (0, 3 + n^{-1})(-2 - n^{-1}, 0),
\]

\[
E_n(l_n) = \alpha(A((1, 2), p_k)) \downarrow -n^{-1},
\]

\[
F_n(l_n) = \{(x, y) \in E^2 \mid -\mu_n \leq x \leq 0 \text{ and } y = -3 - n^{-1}\},
\]

\[
G_n = (0, -3 - n^{-1})(2 + (n + 1)^{-1}, 0).
\]

For each $n$, let

\[
\Gamma_n(k_n, l_n) = A_n \cup B_n(k_n) \cup C_n(k_n) \cup D_n \cup \alpha(A_n) \cup E_n(l_n) \cup F_n(l_n) \cup G_n.
\]

Let

\[
S\{k_n, l_n\}_{n \in \mathbb{N}} = \bigcup_{n=1}^{\infty} \Gamma_n(k_n, l_n).
\]

Note that $S\{k_n, l_n\}_{n \in \mathbb{N}}$ is a ray with origin $(3, 0)$ that limits on the double Warsaw circle $D$.

Let $X$ be an arc in $S\{k_n, l_n\}_{n \in \mathbb{N}}$ that is contained in $E^2_+ = \{(x, y) \in E^2 \mid y > 0\}$ or $E^2_- = \{(x, y) \in E^2 \mid y < 0\}$. A subarc $Y$ of $X$ is called a dip (hump) in $X$ if the end points of $Y$ are relative maximum (minimum) points of $X$ and no interior point of $Y$ is a relative maximum (minimum) of $X$. Note that the $n$-th component of $S\{k_n, l_n\}_{n \in \mathbb{N}} \cap E^2_+$ has $k_n$ dips, all contained in $B_n(k_n)$, and the $n$-th component of $S\{k_n, l_n\}_{n \in \mathbb{N}} \cap E^2_-$ has $l_n$ humps, all contained in $E_n(l_n)$.

**Proposition 3.2.** The continuum $D \cup S\{2n - 1, 2n\}_{n \in \mathbb{N}}$ does not have the fixed-point property.

The proof of Proposition 3.2 is given in Examples 4.1 and 4.2 of [H–M].

**Theorem 3.3.** If one of the sequences $\{\frac{k_n - k_n}{k_n}\}_{n \in \mathbb{N}}$ or $\{\frac{k_n - l_n}{l_n}\}_{n \in \mathbb{N}}$ is unbounded, then the continuum $D \cup S\{k_n, l_n\}_{n \in \mathbb{N}}$ has the fixed-point property.

The proof of Theorem 3.3 will be provided in Section 5. Here, let us comment upon the assumption of Theorem 3.3.

**Remark 3.4.** If $k_1 \leq l_1 \leq k_2 \leq l_2 \leq \cdots \leq k_n \leq l_n \leq k_{n+1} \leq \cdots \to \infty$, then the assumption of Theorem 3.3 is equivalent to each of the following two conditions:

1. one of the sequences $\{\frac{k_n + 1}{k_n}\}_{n \in \mathbb{N}}$ or $\{\frac{l_n + 1}{l_n}\}_{n \in \mathbb{N}}$ is unbounded;
(2) both of the sequences \( \{ \frac{k_{n+1}}{k_n} \} \) and \( \{ \frac{t_{n+1}}{t_n} \} \) are unbounded.

Indeed, for each positive integer \( n \), setting \( s_n = \frac{k_n - 1}{k_n} \) and \( t_n = \frac{k_{n+1} - 1}{t_n} \), we have \( l_n - k_n = s_n k_n \) and \( k_{n+1} - l_n = t_n k_n \). Hence \( l_n = (s_n + 1)k_n \) and \( k_{n+1} = (t_n + 1)k_n \), and therefore \( k_{n+1} - k_n = (k_{n+1} - l_n) + (l_n - k_n) = t_n l_n + s_n k_n = t_n (s_n + 1)k_n + s_n k_n = (s_n + s_n t_n + t_n)k_n \). Thus \( \frac{k_{n+1}}{k_n} = s_n + s_n t_n + t_n + 1 \). By an analogous calculation, \( \frac{t_{n+1}}{t_n} = s_n + t_n + t_n + 1 \).

For each \( n \), let \( v_n = \frac{2}{2n-1} \) and let \( q_n \) be the point \( (v_n, 1) \) of \( \Gamma \).

**Proposition 3.5.** There exist a ray \( S^o \{ 2n - 1, 2n \} \) in \( E^2 \) inside of the double Warsaw circle \( D \) and a homeomorphism \( h_0 \) of \( D \cup S \{ 2n - 1, 2n \} \) onto \( D \cup S^o \{ 2n - 1, 2n \} \) that is the identity on \( D \).

**Proof.** For each positive integer \( n \), let

\[
A_n^o = \left( 2 - (n + 1)^{-1}, 0 \right], \quad B_n^o(2n - 1) = \left[ 1, 2 - (n + 1)^{-1} \right), \quad C_n^o(2n - 1) = \{(x, y) \in E^2 \mid -(n + 1)^{-1} \leq x \leq v_{2n-1} \}
\]

\[
\bigcup \left( (n + 1)^{-1}, -1 + (n + 1)^{-1} \right), \quad D_n^o = \left( -(n + 1)^{-1}, 3 - (n + 1)^{-1} \right), \quad E_n^o(2n) = \alpha \left( \left( 1, 2/q_{2n} \right) \right), \quad F_n^o(2n) = \left( -1 + (n + 1)^{-1}, 3 + (n + 1)^{-1} \right).
\]

Observe that \( B_n^o(2n - 1) \) has \( 2n - 2 \) dips and \( E_n^o(2n) \) has \( 2n - 1 \) humps.

For each \( n \in N \), let

\[
\Gamma_n^o(2n - 1, 2n) = A_n^o \cup B_n^o(2n - 1) \cup C_n^o(2n - 1) \cup D_n^o \cup E_n^o(2n) \cup F_n^o(2n) \cup G_n^o.
\]

Define

\[
S^o \{ 2n - 1, 2n \} = \bigcup_{n=1}^{\infty} \Gamma_n^o(2n - 1, 2n).
\]

Note that \( S^o \{ 2n - 1, 2n \} \) is a ray with origin \( \left( \frac{1}{2}, 0 \right) \) that limits on the double Warsaw circle \( D \).

For each \( n \in N \), let \( h_0 \) send the arcs \( A_n \), \( B_n(2n - 1) \cup C_n(2n - 1) \), \( D_n \), \( \alpha(A_n) \), \( E_n(2n) \cup F_n(2n) \), and \( G_n \) onto the corresponding arcs \( A_n^o \), \( B_n^o(2n - 1) \cup C_n^o(2n - 1) \), \( D_n^o \), \( \alpha(A_n^o) \), \( E_n^o(2n) \cup F_n^o(2n) \), and \( G_n^o \). The map \( h_0 \) is affine and order-preserving on each arc. Observe that \( h_0 \) sends the final dip of \( B_n(2n - 1) \cup C_n(2n - 1) \) onto the arc \( B_n^o(2n - 1) \) \( \bigcup \left( \mu_{2n-1}, 3 - (n + 1)^{-1} \right) \bigcup C_n^o(2n - 1) \) which indeed is a dip. Also \( h_0 \) sends the final hump of \( E_n(2n) \cup F_n(2n) \) onto the arc \( E_n^o(2n) \) \( \bigcup \left( -\mu_{2n-1}, -3 + (n + 1)^{-1} \right) \bigcup F_n^o(2n) \), which is a hump. The map \( h_0 \) is a one-to-one continuous extension of the identity on \( D \). 

**Proposition 3.6.** If \( 2n - 1 \leq k_n \) and \( 2n \leq l_n \) for each \( n \in N \), then there is a map \( \rho_1 \) of \( D \cup S \{ k_n, l_n \} \) onto \( D \cup S \{ 2n - 1, 2n \} \) that is the identity on \( D \).

**Proof.** Observe that among the arcs \( A_n, \ldots, G_n \) that define \( S \{ 2n - 1, 2n \} \) or \( S \{ k_n, l_n \} \) only the arcs \( B_n(2n - 1), C_n(2n - 1), E_n(2n), \) and \( F_n(2n) \) may differ from \( B_n(k_n), C_n(k_n), E_n(l_n), \) and \( F_n(l_n) \), respectively. Outside of these arcs, let
\( \rho_1 \) be the identity. Also on \( B_n(k_n) \cup C_n(k_n) \) (respectively, \( E_n(l_n) \cup F_n(l_n) \)), let \( \rho_1 \) be the identity on the first \( 2n - 1 \) dips (respectively, \( 2n \) humps). Let \( \rho_1 \) collapse the remaining \( k_n - (2n - 1) \) dips (respectively, \( l_n - 2n \) humps) onto the subarc of the arc \( B_n(2n - 1) \) (respectively, \( E_n(2n) \)) that has end points \( (\mu_{2n-1}, 3 + n^{-1}) \) and \( (v_{2n-1}, 1 + n^{-1}) \) (respectively, \( (-\mu_{2n}, -3 - n^{-1}) \) and \( (-v_{2n}, -1 - n^{-1}) \)). Simultaneously, \( \rho_1 \) sends the arc \( C_n(k_n) \) onto \( C_n(2n - 1) \), and \( F_n(l_n) \) onto \( F_n(2n) \). \( \square \)

4. MAIN CONTINUUM \( \mathcal{M}_0 \)

Suppose \( \{k_n, l_n\}_{n \in \mathbb{N}} \) satisfies the assumptions of Theorem 3.3 and Proposition 3.6, that is, (1) \( \{k_n\}_{n \in \mathbb{N}} \) and \( \{l_n\}_{n \in \mathbb{N}} \) are sequences of positive integers satisfying (3.1), (2) \( \{\frac{k_{n+1} - k_n}{k_n}\}_{n \in \mathbb{N}} \) or \( \{\frac{k_{n+1} - l_n}{l_n}\}_{n \in \mathbb{N}} \) is unbounded, and (3) \( 2n - 1 \leq k_n \) and \( 2n \leq l_n \) for each \( n \in \mathbb{N} \). Let \( R \) be a ray in the unbounded domain of \( E^2 - D \), with origin \( (4, 0) \), consisting of folded arcs of increasing lengths that limit on \( D \cup S\{k_n, l_n\}_{n \in \mathbb{N}} \) (taking a pattern by the graph of the map \( g \) of \( (0, 1) \) onto \( \mathbb{R}_1 \) given by \( g(x) = \frac{x}{2}(1 + \sin \frac{x}{2}) \), with \( 0, 0 \) corresponding to the origin \( (3, 0) \) of \( S\{k_n, l_n\}_{n \in \mathbb{N}} \). Let \( \rho_2 \) be the natural retraction of \( D \cup S\{k_n, l_n\}_{n \in \mathbb{N}} \cup R \) onto \( D \cup S\{k_n, l_n\}_{n \in \mathbb{N}} \) that collapses the folded arcs in \( R \) onto the initial arcs of \( S\{k_n, l_n\}_{n \in \mathbb{N}} \).

Define

\[
\mathcal{M}_0 = S^o\{2n - 1, 2n\}_{n \in \mathbb{N}} \cup D \cup S\{k_n, l_n\}_{n \in \mathbb{N}} \cup R.
\]

For each \( p \in \mathcal{M}_0 \), let

\[
\rho(p) = \begin{cases}
  h_0(\rho_1(\rho_2(p))) & \text{if } p \in R \cup S\{k_n, l_n\}_{n \in \mathbb{N}}, \\
  p & \text{if } p \in D \cup S^o\{2n - 1, 2n\}_{n \in \mathbb{N}}.
\end{cases}
\]

By Propositions 3.5 and 3.6, \( \rho \) is a retraction of \( \mathcal{M}_0 \) onto the continuum \( D \cup S^o\{2n - 1, 2n\}_{n \in \mathbb{N}} \). By Propositions 3.2 and 3.5, \( D \cup S^o\{2n - 1, 2n\}_{n \in \mathbb{N}} \) does not have the fixed-point property. Therefore \( \mathcal{M}_0 \) does not have the fixed-point property.

Obviously, \( \mathcal{M}_0 \) is rational. Also, \( \mathcal{M}_0 \) is irreducible between the origin \( (\frac{3}{2}, 0) \) of \( S^o\{2n - 1, 2n\}_{n \in \mathbb{N}} \) and the origin \( (4, 0) \) of \( R \), since \( \mathcal{M}_0 = CIR \cup CLS^o\{2n - 1, 2n\}_{n \in \mathbb{N}} \) and the sets \( S^o\{2n - 1, 2n\}_{n \in \mathbb{N}}, D, S\{k_n, l_n\}_{n \in \mathbb{N}} \), and \( R \) are pairwise disjoint.

We will prove in Section 5 that the unique non-degenrate tranche \( D \cup S\{k_n, l_n\}_{n \in \mathbb{N}} \) of \( \mathcal{M}_0 \) has the fixed-point property.

Remark 4.1. Similar to the continuum \( M \) in \( E^3 \) defined in [H], the continuum \( \mathcal{M}_0 \) can be used to define another rational irreducible continuum in \( E^2 \) that admits a fixed-point-free surjection with the condition that all of its tranches have the fixed-point property.

5. PROOF OF THEOREM 3.3

We will consider the case when the sequence \( \{\frac{k_n}{l_n}\}_{n \in \mathbb{N}} \) is unbounded.

Denoting the spiral \( S\{k_n, l_n\}_{n \in \mathbb{N}} \) by \( S \), we assume there is a map \( f \) of \( D \cup S \) into itself that moves each point.

Then

(5.1). \( f(S) \subset S \) and \( p <_S f(p) \) for each \( p \in S \) and \( f(D) = D \) [M2, Prop. 3.2].

Hence, by [M2, Prop. 3.1],

(5.2). \( f(A) = \alpha(A) \) and \( f(\alpha(A)) = A \), and
(5.3). \( f(L(A)) = L(\alpha(A)) \) and \( f(L(\alpha(A))) = L(A) \).

Since \( E_2^n \) is a neighborhood of \( L(\alpha(A)) \), it follows from the first equality in (5.3) that there is a neighborhood \( P \) of \( L(A) \) such that \( f(P \cap S) \subset E_2^n \cap S \). We can assume \( P \) is a closed rectangle lying in \( E_2^+ \) with sides perpendicular to the \( x \)-axis. By the second equality in (5.3), there is also a closed rectangular neighborhood \( Q \) of \( L(\alpha(A)) \) such that \( f(Q \cap S) \subset P \cap S \) and \( Q \subset E_2^n \) and the sides of \( Q \) are perpendicular to the \( x \)-axis. We can also assume that \( P \cap (A \setminus \Gamma) = \emptyset = Q \cap (\alpha(A) \setminus \alpha(\Gamma)) \).

Let \( n_0 \) be an integer such that for each \( n \geq n_0 \) the sets \( B_n(k_n) \cap P \) and \( E_n(l_n) \cap Q \) are arcs. For each \( n \geq n_0 \), let \( b_n \) denote the first point with respect to the order \( <_S \) of \( B_n(k_n) \cap P \) and let \( e_n \) denote the first point of \( E_n(l_n) \cap Q \).

For each \( n > n_0 \), define arcs

\[
I_n = S([\mu_{n-1}, 3 - (n-1)^{-1}], b_n] \quad \text{and} \quad J_n = S([\mu_{n-1}, 3 + n^{-1}], e_n].
\]

Note that the sequences \( \{I_n\}_{n>n_0} \) and \( \{J_n\}_{n>n_0} \) converge to the arcs \( A([0, 3], \lim b_n] \) and \( \alpha(A)([0, 3], \lim e_n] \), respectively.

Setting \( E_2 = \{(x, y) \in E_2^* \mid 0 \leq x \leq 0 \} \) and \( E_3 = \{(x, y) \in E_2^* \mid x \geq 0 \} \), there is an integer \( n_1 \geq n_0 \) such that

(5.4). \( f(I_n) \subset E_2 \cup P \) and \( f(J_n) \subset E_3 \cup Q \) for each \( n > n_1 \).

This follows from (5.2) and the fact that \( E_2^n \cup P \) and \( E_3^n \cup Q \) are neighborhoods of \( f(\lim I_n) = \lim f(I_n) \) and \( f(\lim J_n) = \lim f(J_n) \), respectively.

For each \( n > n_1 \), let \( H_n = P \cap \Gamma_n(k_n, l_n) \) and \( K_n = E_2^n \cap \Gamma_n(k_n, l_n) \). Observe that \( H_n \) and \( K_n \) are arc components of \( P \cap S \) and \( E_2^n \cap S \), respectively, and \( b_n \) is the end point of the arc \( H_n \) that belongs to \( E_2^n \), and \( e_n \) is the end point of the arc \( K_n \cap Q \) that belongs to \( E_3^n \). Since \( f(P \cap S) \subset E_2^n \cap S \), for each \( n > n_1 \) there is exactly one \( i_n \in N \) such that \( f(H_n) \subset K_i \). By (5.1), \( i_n \geq n \).

We are going to prove

(5.5). \( E_{i_n}(l_{i_n}) \subset f(I_n \cup B_n(k_n) \cup J_n) \) for each \( n > n_1 \).

The arcs \( I_n \) and \( J_n \) both meet \( H_n \), since \( b_n \in H_n \cap I_n \) and \( (\mu_{n-1}, 3n^{-1}) \in H_n \cap J_n \), and therefore

\[
f(I_n) \cap K_{i_n} \neq \emptyset \quad \text{and} \quad f(J_n) \cap K_{i_n} \neq \emptyset \quad \text{for each} \quad n > n_1.
\]

Also, we have \( f(I_n) \cap P \neq \emptyset \neq f(J_n) \cap P \) for each \( n > n_1 \), because \( (-\mu_{n-1}, 3 - (n-1)^{-1}) \in Q \cap I_n \) implies \( f((-\mu_{n-1}, 3 - (n-1)^{-1})) \in f(I_n) \cap P \) and \( e_n \in Q \cap J_n \) implies \( f(e_n) \in f(J_n) \cap P \). Thus \( f(I_n) \) meets an arc component \( H_m \) of \( P \cap S \) and \( f(J_n) \) meets an arc component \( H_l \) of \( P \cap S \).

For each \( n > n_1 \),

(5.6). \( f(I_n) \cap H_m \neq \emptyset \) implies \( m = i_n \) and \( f(J_n) \cap H_l \neq \emptyset \) implies \( l = i_n + 1 \).

To establish (5.6), we denote \( \Gamma_n(k_n, l_n) \) by \( \Gamma_n \) and first observe that if \( n < n' \), then \( p < q \) for each \( p \in \Gamma_n \) and \( q \in \Gamma_{n'} \) such that \( p \neq q \). Also, the order \( <_S \) agrees with the counterclockwise orientation of \( E_2^n \).

If \( m < i_n \), then \( f(I_n) \) going from \( K_{i_n} \) to \( H_m \) (clockwise), must contain \( \Gamma_{i_n} \cap E_2^n \), and if \( i_n < m \), then \( f(I_n) \) goes from \( K_{i_n} \) to \( H_m \) counterclockwise so that \( (\Gamma_{i_n} \cap E_2^n \cap E_2^n) \setminus P \subset f(I_n) \); in both cases we have a contradiction of the first inclusion in (5.4).
If \( i_n + 1 < l \), then \( f(J_n) \) goes from \( K_{i_n} \) to \( H_1 \) counterclockwise so that \( \Gamma_{i_n+1} \subset f(J_n) \), and if \( l < i_n + 1 \), then going clockwise we obtain \( \Gamma_{i_n} \cap E_2^L \subset f(J_n) \); in both cases we have a contradiction of the second inclusion in (5.4). Hence (5.6) is true.

Thus

\[
f(I_n) \cap H_{i_n} \neq \emptyset \neq f(J_n) \cap H_{i_n+1}
\]

for each \( n > n_1 \).

In view of (5.4), the inequalities \( f(I_n) \cap H_{i_n} \neq \emptyset \neq f(I_n) \cap K_{i_n} \) imply that \((-2 - i_n^{-1}, 0) \in f(I_n)\) and the inequalities \( f(J_n) \cap K_{i_n} \neq \emptyset \neq f(J_n) \cap H_{i_n+1} \) imply that \((2 + (i_n + 1)^{-1}, 0) \in f(J_n)\). Thus both end points of the open arc \( K_{i_n} \) belong to the image \( f(I_n \cup B_n(k_n) \cup J_n) \) in \( S \). Since \( b_n \in I_n \cap B_n(k_n) \) and \((\mu_{i_n}, 3+n^{-1}) \in J_n \cap B_n(k_n)\), it follows that \( I_n \cup B_n(k_n) \cup J_n \) is an arc in \( S \) (with end points \((-\mu_{i_n-1}, -3-(n-1)^{-1}) \) and \( e_n \)), and thus the image \( f(I_n \cup B_n(k_n) \cup J_n) \) is an arc in \( S \). Hence \( K_{i_n} \subset f(I_n \cup B_n(k_n) \cup J_n) \). But, obviously, we have \( E_{i_n}(l_n) \subset K_{i_n} \), which proves (5.5).

**Claim 5.7.** There exists a sequence \( \{L_n\}_{n>n_1} \) of arcs in \( S \) and a sequence \( \{w_n\}_{n>n_1} \) of nonnegative integers such that

(i) each \( L_n \) lies in \( I_n \cup J_n \) or is a dip in \( B_n(k_n) \),
(ii) each \( f(L_n) \) contains \( w_n \) humps of \( E_{i_n}(l_n) \),
(iii) the sequence \( \{L_n\}_{n>n_1} \) converges to an arc \( \lim L_n \subset D \), and
(iv) the sequence \( \{w_n\}_{n>n_1} \) is unbounded.

To prove Claim 5.7, for each \( n > n_1 \), let \( u_n \) denote the number of humps of \( E_{i_n}(l_n) \) that are contained in the image \( f(I_n \cup J_n) = f(I_n) \cup f(J_n) \).

Case 1. Suppose the sequence \( \{u_n\}_{n>n_1} \) is unbounded. Then for each \( n > n_1 \), let \( L_n \) be one of the arcs \( I_n \) or \( J_n \), the image of which contains \( w_n \) humps of \( E_{i_n}(l_n) \) in such a way that the sequence \( \{w_n\}_{n>n_1} \) is unbounded.

Case 2. Suppose \( \{u_n\}_{n>n_1} \) is bounded. Let \( n > n_1 \). The number of humps of \( E_{i_n}(l_n) \) is \( l_n \), and the number of humps of \( E_{i_n}(l_n) \) that are contained in \( f(I_n) \cup f(J_n) \) is \( u_n \). By (5.4), \( f(I_n) \) may contain a left subarc and \( f(J_n) \) may contain a right subarc of \( E_{i_n}(l_n) \). Therefore, by (5.5), \( f(B_n(k_n)) \) contains at least \( l_n - (u_n + 2) \) humps of \( E_{i_n}(l_n) \). Since \( i_n \geq n \), it follows from (3.1) that \( l_n \geq l_n \). Thus \( l_n - (u_n + 2) \geq l_n - (u_n + 2) \). Setting \( s_n = \frac{L_n - k_n}{k_n} \), we have

\[
l_n - (u_n + 2) = k_n s_n + k_n - (u_n + 2).
\]

However \( k_n \to \infty \) and, in the case being considered, the sequence \( \{u_n + 2\}_{n>n_1} \) is unbounded. Hence there exists an integer \( n_2 \geq n_1 \) such that \( k_n - (u_n + 2) \geq 0 \) for each \( n > n_2 \). It follows that \( l_n - (u_n + 2) \geq k_n s_n \) for each \( n > n_2 \). Thus, for each \( n > n_2 \), the series of \( k_n \) dips of \( B_n(k_n) \) is mapped by \( f \) on a series of \( k_n v_n \) consecutive humps of \( E_{i_n}(l_n) \), where \( v_n \) is the integer part of \( s_n \).

Applying Lemma 2.1 (with \( v = \mu = k_n \)) we obtain a dip \( V_n \) of \( B_n(k_n) \) that is mapped on a series of \( v_n \) consecutive humps of \( E_{i_n}(l_n) \). Let \( \{V_{m_n}\}_{m>n_2} \) be a convergent subsequence of \( \{V_n\}_{n>n_2} \) such that \( v_{m_n} \to \infty \). For each \( n > n_2 \), let \( L_n = V_{m_n} \) and \( w_n = v_{m_n} \). Let \( L_n = I_n \) and \( w_n = 0 \) whenever \( n \leq n \leq n_2 \). Note that \( \lim L_n \) is either a dip in \( \Gamma \) or the interval \( [0, 1] = L(\Gamma) = L(A) \). This proves the conclusion of Claim 5.7.

Let \( \{L_n\}_{n>n_1} \) and \( \{w_n\}_{n>n_1} \) be sequences that satisfy conditions (i)-(iv) of Claim 5.7. For each \( n > n_1 \), let \( P_{w_n}(L_n) \) be the partition of the arc \( L_n \) into \( w_n \) consecutive subarcs with the same length. By (iv), the sequence \( \{w_n\}_{n>n_1} \) is unbounded. Instead of taking a strictly increasing subsequence of \( \{w_n\}_{n>n_1} \), we assume that \( \{w_n\}_{n>n_1} \) itself is strictly increasing. Hence \( w_n \to \infty \).
By (iii), the sequence \( \{ P_{w_n} (L_n) \}_{n>n_1} \) of partitions converges to a decomposition \( P \) of the arc \( \lim L_n \subset D \). Since \( w_n \to \infty \), it follows from (i) that each element of \( P \) consists of a single point. By (ii), each \( f(L_n) \) contains \( w_n \) humps of \( E_{i_n} (l_{i_n}) \). For each \( n > n_1 \), it follows from Lemma 2.1 (with \( v = \mu = w_n \)) that the partition \( P_{w_n} (L_n) \) has an element \( L_{j_n,n} \), the image of which contains an entire hump of \( E_{i_n} (l_{i_n}) \). Passing to the limit with a convergent subsequence of the sequence \( \{ L_{j_n,n} \}_{n>n_1} \), we obtain a point (of the limit arc \( \lim L_n \)), the image of which contains the limit of the corresponding humps, i.e., either a hump of \( \alpha(A) \) or the interval \( L(\alpha(A)) \), a contradiction.

By a dual argument, the assumption that \( \{ k_{n+1} - 1 \}_{n} \in N \) is unbounded also leads to a contradiction.

6. Related questions

The tranche \( D \cup S \{k_n, l_n \}_{n \in N} \) of \( M_0 \) separates \( E^2 \).

**Question 6.1.** Must a plane continuum of type \( \lambda \) have the fixed-point property if none of its tranches separates the plane?

A plane continuum of type \( \lambda \) separates the plane if and only if at least one of its tranches separates the plane [Mr]. Since it is not known if every nonseparating plane continuum has the fixed-point property [Bi, Ques. 3], [K-W, p. 66, p. 145], Question 6.1 should also be considered with the additional assumption that each tranche has the fixed-point property. A counterexample must contain an invariant indecomposable continuum in one of its tranches [H], [I], [S].

**Question 6.2.** Must a plane continuum of type \( \lambda \) have the fixed-point property if each of its tranches has the fixed-point property and its decomposition is continuous?

A counterexample to Question 6.2 cannot be rational since it must have an indecomposable tranche [D].

**References**


[N] S. B. Nadler, *Continua which are a one-to-one continuous image of [0,∞)*, Fund. Math. 75 (1972), 123–133. MR 47:5848


Department of Mathematics, California State University, Sacramento, Sacramento, California 95819-6051

E-mail address: hagopian@csus.edu

Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-956 Warsaw, Poland

E-mail address: manka@impan.gov.pl