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RATIONAL IRREDUCIBLE PLANE CONTINUA WITHOUT THE FIXED-POINT PROPERTY

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Abstract. We define rational irreducible continua in the plane that admit fixed-point-free maps with the condition that all of their tranches have the fixed-point property. This answers in the affirmative a question of Hagopian. The construction is based on a special class of spirals that limit on a double Warsaw circle. The closure of each of these spirals has the fixed-point property.

1. Introduction

A *continuum* is a non-void compact connected metric space. A continuum is *irreducible* between two of its points if none of its proper subcontinua contains both of these points. A continuum is *rational* if each of its points admits a base of neighborhoods with countable boundaries.

Since rational continua are hereditarily decomposable [Ku2, Thm. 5, p. 285], each rational irreducible continuum C is of *type* λ, i.e., of the order type of the unit interval $[0, 1]$. This means, following Knaster and Kuratowski [Ku1, §3, pp. 248, 262], that C admits a uniquely determined monotone upper semi-continuous decomposition to an *arc* (i.e., homeomorphic image of $[0, 1]$) with the property that each element of the decomposition has void interior relative to C [Ku2, Thm. 3, p. 216, Thm. 2, p. 215]. The continua that are elements of this decomposition are called the *tranches* of C.

A continuum has the *fixed-point property* if each map of it into itself has a fixed point.

To answer a question of Gordh [L, Prob. 43, p. 371], Hagopian [H] recently defined a non-planar irreducible rational continuum $\mathcal{M}$ such that each tranche of $\mathcal{M}$ has the fixed-point property and $\mathcal{M}$ admits a fixed-point-free map. Hagopian [H, Ques. 1] asked if there is a continuum with these properties in the plane. Here we give a positive answer to this question.

A *double Warsaw circle* is a union of two disjoint rays each of which limits on an initial arc of the other. It follows from a result of Nadler [N, Lem. 11, p. 131] that each double Warsaw circle is planar (also see Awartani's theorem [A, Thm. 4.1, p. 232]).

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Our construction involves two spirals limiting on a double Warsaw circle. One spiral lies outside and the other inside of the double Warsaw circle. The closure of one spiral has the fixed-point property, and the other does not. The closure that has the fixed-point property is approximated by a ray in such a way that the resulting irreducible rational continuum can be retracted onto the closure of the spiral without the fixed-point property.

2. PRELIMINARIES ON RAYS AND ARCS

In this paper a *ray* is a topological half-line, i.e., homeomorphic image of [0,1). Every ray R will be considered with its natural order $<_R$ inherited from $[0,1)$.

If R is a ray contained in a continuum, then ClR (the closure of R) is a continuum irreducible between the origin (initial point) of R and each limit point of R, i.e., a limit of an increasing convergent sequence of points of R [M2, Prop. 2.1]. The continuum ClR is rational if (and only if) the continuum $L(R)$ consisting of all limit points of R is rational. Obviously, $R \cap L(R) = \emptyset$ and $R \cup L(R) = ClR$. A continuum C of the form $C = ClR \cup ClS$, where R and S are rays with origins p and q, respectively, is irreducible between p and q if $R \cap ClS = \emptyset = S \cap ClR$ (see [M1, Rem. 1, p. 125] for a general statement).

A space is *(uniquely) arcwise connected* if every two of its points are end points of a (unique) arc in the space.

Lemma 2.1. *Suppose f is a map of a space L into a uniquely arcwise connected space and an arc M is contained in $f(L)$. Then for every family of arcwise connected sets $L_1, L_2, \cdots, L_\upsilon$ such that $L = \bigcup_{j=1}^{\upsilon} L_j$ and for every partition $\mathcal{P}(M)$ of M into consecutive arcs $M_1, M_2, \cdots, M_\mu$ with $\upsilon \leq \mu$, there exist L_j and M_i such that $M_i \subset f(L_j)$.*

Proof. Let $q_0, q_1, \cdots, q_\mu$ be the consecutive end points of the arcs $M_1, M_2, \cdots, M_\mu$ belonging to $\mathcal{P}(M)$. We have $M_i \cap M_{i+1} = \{q_i\}$ for each $i = 1, 2, \cdots, \mu - 1$. For each $j = 1, 2, \cdots, \upsilon$, let $\Sigma_j = \{i \mid q_i \in f(L_j)\}$. It suffices to show that some Σ_j has more than one element.

Suppose, on the contrary, that each Σ_j has at most one element. Then the set $\Sigma = \bigcup_{j=1}^{\upsilon} \Sigma_j$ has at most υ elements. Since $M \subset f(L) = f(\bigcup_{j=1}^{\upsilon} L_j)$, it follows that $\Sigma = \{i \mid q_i \in f(L)\}$ and $\{0, 1, 2, \cdots, \mu\} \subset \Sigma$. Hence $\mu + 1 \leq \upsilon$. This contradicts the assumption that $\upsilon \leq \mu$. □

This lemma will be used in Section 5. Given a map f, we will say sometimes that X is *mapped on* Y when $Y \subset f(X)$.

Notation. For each two points a and b of the plane E^2, we denote the straight line interval with end points a and b by $\overline{ab}$. If a and b belong to a uniquely arcwise connected set A, then $A[a, b]$ will denote the unique arc in A with end points a and b. Given a set Z in E^2 and a real number ϵ, we let $Z \uparrow \epsilon$ denote $\{(x, y) \in E^2 \mid (x, y - \epsilon) \in Z\}$. We denote the set of all positive integers $1, 2, \cdots$ by N.

3. A CLASS OF SPIRALS LIMITING ON A DOUBLE WARSAW CIRCLE

Let γ be the map of $(0, 1]$ onto $[1, 3]$ defined by $\gamma(x) = 2 + sin\ \frac{\pi}{x}$, and let Γ denote the graph of γ in E^2, i.e., $\Gamma = \{(x, \gamma(x)) \in E^2 \mid x \in (0, 1]\}$.

The set $A = \overline{(0,-1)(0,-3)} \cup \overline{(0,-3)(2,0)} \cup \overline{(2,0)(1,2)} \cup \Gamma$ is a ray with origin $(0,-1)$ that limits on the interval $\overline{(0,1)(0,3)}$. Note that ClA is the topologist's sine curve.

Let α be the map of E^2 onto E^2 defined by $\alpha(x,y) = (-x,-y)$.

Let D be the continuum $A \cup \alpha(A)$ in E^2. Note that D is a double Warsaw circle.

Consider two sequences $\{k_n\}_{n\in N}$ and $\{l_n\}_{n\in N}$ of positive integers satisfying the following conditions:

(3.1). $k_1 \le k_2 \le \cdots \le k_n \le k_{n+1} \le \cdots \to \infty$ and $l_1 \le l_2 \le \cdots \le l_n \le l_{n+1} \le \cdots \to \infty$.

Similar to T in Example 4.1 of [H–M], we define a spiral $S\{k_n, l_n\}_{n\in N}$ limiting on D.

For each positive integer n, let $\mu_n = \frac{2}{4n+1}$ and let p_n be the point $(\mu_n, 3)$ of Γ.

For each n, let

$$\begin{aligned}
&A_n = \overline{(2+n^{-1},0)(1,2+n^{-1})},\\
&B_n(k_n) = A[(1,2),p_{k_n}] \uparrow n^{-1},\\
&C_n(k_n) = \{(x,y)\in E^2 \mid 0 \le x \le \mu_{k_n} \text{ and } y = 3+n^{-1}\},\\
&D_n = \overline{(0,3+n^{-1})(-2-n^{-1},0)},\\
&E_n(l_n) = \alpha(A[(1,2),p_{l_n}]) \uparrow -n^{-1},\\
&F_n(l_n) = \{(x,y)\in E^2 \mid -\mu_{l_n} \le x \le 0 \text{ and } y = -3-n^{-1}\}, \text{ and}\\
&G_n = \overline{(0,-3-n^{-1})(2+(n+1)^{-1},0)}.
\end{aligned}$$

For each n, let

$$\Gamma_n(k_n,l_n) = A_n \cup B_n(k_n) \cup C_n(k_n) \cup D_n \cup \alpha(A_n) \cup E_n(l_n) \cup F_n(l_n) \cup G_n.$$

Let

$$S\{k_n,l_n\}_{n\in N} = \bigcup_{n=1}^{\infty} \Gamma_n(k_n,l_n).$$

Note that $S\{k_n,l_n\}_{n\in N}$ is a ray with origin $(3,0)$ that limits on the double Warsaw circle D.

Let X be an arc in $S\{k_n,l_n\}_{n\in N}$ that is contained in $E^2_+ = \{(x,y)\in E^2 \mid y>0\}$ or $E^2_- = \{(x,y)\in E^2 \mid y<0\}$. A subarc Y of X is called a *dip (hump)* in X if the end points of Y are relative maximum (minimum) points of X and no interior point of Y is a relative maximum (minimum) of X. Note that the n-th component of $S\{k_n,l_n\}_{n\in N} \cap E^2_+$ has k_n dips, all contained in $B_n(k_n)$, and the n-th component of $S\{k_n,l_n\}_{n\in N} \cap E^2_-$ has l_n humps, all contained in $E_n(l_n)$.

Proposition 3.2. *The continuum $D \cup S\{2n-1,2n\}_{n\in N}$ does not have the fixed-point property.*

The proof of Proposition 3.2 is given in Examples 4.1 and 4.2 of [H–M].

Theorem 3.3. *If one of the sequences $\left\{\frac{l_n-k_n}{k_n}\right\}_{n\in N}$ or $\left\{\frac{k_{n+1}-l_n}{l_n}\right\}_{n\in N}$ is unbounded, then the continuum $D\cup S\{k_n,l_n\}_{n\in N}$ has the fixed-point property.*

The proof of Theorem 3.3 will be provided in Section 5. Here, let us comment upon the assumption of Theorem 3.3.

Remark 3.4. If $k_1 \le l_1 \le k_2 \le l_2 \le \cdots \le k_n \le l_n \le k_{n+1} \le \cdots \to \infty$, then the assumption of Theorem 3.3 is equivalent to each of the following two conditions:

(1) one of the sequences $\left\{\frac{k_{n+1}}{k_n}\right\}_{n\in N}$ or $\left\{\frac{l_{n+1}}{l_n}\right\}_{n\in N}$ is unbounded;

(2) both of the sequences $\{\frac{k_{n+1}}{k_n}\}_{n\in N}$ and $\{\frac{l_{n+1}}{l_n}\}_{n\in N}$ are unbounded.

Indeed, for each positive integer n, setting $s_n = \frac{l_n - k_n}{k_n}$ and $t_n = \frac{k_{n+1}-l_n}{l_n}$, we have $l_n - k_n = s_n k_n$ and $k_{n+1} - l_n = t_n l_n$. Hence $l_n = (s_n+1)k_n$ and $k_{n+1} = (t_n+1)k_n$, and therefore $k_{n+1} - k_n = (k_{n+1} - l_n) + (l_n - k_n) = t_n l_n + s_n k_n = t_n(s_n+1)k_n + s_n k_n = (s_n + s_n t_n + t_n)k_n$. Thus $\frac{k_{n+1}}{k_n} = s_n + s_n t_n + t_n + 1$. By an analogous calculation, $\frac{l_{n+1}}{l_n} = s_{n+1} + s_{n+1}t_n + t_n + 1$.

For each n, let $\upsilon_n = \frac{2}{4n-1}$ and let q_n be the point $(\upsilon_n, 1)$ of Γ.

Proposition 3.5. *There exist a ray $S^o\{2n-1,2n\}_{n\in N}$ in E^2 inside of the double Warsaw circle D and a homeomorphism h_0 of $D \cup S\{2n-1,2n\}_{n\in N}$ onto $D \cup S^o\{2n-1,2n\}_{n\in N}$ that is the identity on D.*

Proof. For each positive integer n, let

$$A_n^o = \overline{(2-(n+1)^{-1},0)(1,2-(n+1)^{-1})},$$
$$B_n^o(2n-1) = A[(1,2),q_{2n-1}] \uparrow -(n+1)^{-1},$$
$$C_n^o(2n-1) = \{(x,y)\in E^2 \mid -(n+1)^{-1} \le x \le \upsilon_{2n-1} \text{ and } y = 1-(n+1)^{-1}\}$$
$$\cup\overline{(-(n+1)^{-1},1-(n+1)^{-1})(-(n+1)^{-1},3-(n+1)^{-1})},$$
$$D_n^o = \overline{(-(n+1)^{-1},3-(n+1)^{-1})(2-(n+1)^{-1},0)},$$
$$E_n^o(2n) = \alpha(A[(1,2),q_{2n}]) \uparrow (n+1)^{-1},$$
$$F_n^o(2n) = \{(x,y)\in E^2 \mid -\upsilon_{2n} \le x \le (n+1)^{-1} \text{ and } y = -1+(n+1)^{-1}\}$$
$$\cup\overline{((n+1)^{-1},-1+(n+1)^{-1})((n+1)^{-1},-3+(n+1)^{-1})},$$
$$G_n^o = \overline{((n+1)^{-1},-3+(n+1)^{-1})(2-(n+2)^{-1},0)}.$$

Observe that $B_n^o(2n-1)$ has $2n-2$ dips and $E_n^o(2n)$ has $2n-1$ humps.

For each $n \in N$, let

$$\Gamma_n^o(2n-1,2n) = A_n^o \cup B_n^o(2n-1) \cup C_n^o(2n-1) \cup D_n^o \cup \alpha(A_n^o) \cup E_n^o(2n) \cup F_n^o(2n) \cup G_n^o.$$

Define

$$S^o\{2n-1,2n\}_{n\in N} = \bigcup_{n=1}^{\infty} \Gamma_n^o(2n-1,2n).$$

Note that $S^o\{2n-1,2n\}_{n\in N}$ is a ray with origin $(\frac{3}{2},0)$ that limits on the double Warsaw circle D.

For each $n\in N$, let h_0 send the arcs A_n, $B_n(2n-1)\cup C_n(2n-1)$, D_n, $\alpha(A_n)$, $E_n(2n)\cup F_n(2n)$, and G_n onto the corresponding arcs A_n^o, $B_n^o(2n-1)\cup C_n^o(2n-1)$, D_n^o, $\alpha(A_n^o)$, $E_n^o(2n)\cup F_n^o(2n)$, and G_n^o. The map h_0 is affine and order-preserving on each arc. Observe that h_0 sends the final dip of $B_n(2n-1)\cup C_n(2n-1)$ onto the arc $B_n^o(2n-1)[(\mu_{2n-1}, 3-(n+1)^{-1})(\upsilon_{2n-1}, 1-(n+1)^{-1})] \cup C_n^o(2n-1)$, which indeed is a dip. Also h_0 sends the final hump of $E_n(2n)\cup F_n(2n)$ onto the arc $E_n^o(2n)[(-\mu_{2n}, -3+(n+1)^{-1})(-\upsilon_{2n}, -1+(n+1)^{-1})] \cup F_n^o(2n)$, which is a hump. The map h_0 is a one-to-one continuous extension of the identity on D. □

Proposition 3.6. *If $2n-1 \le k_n$ and $2n \le l_n$ for each $n\in N$, then there is a map ρ_1 of $D\cup S\{k_n,l_n\}_{n\in N}$ onto $D\cup S\{2n-1,2n\}_{n\in N}$ that is the identity on D.*

Proof. Observe that among the arcs $A_n, \cdots, G_n$ that define $S\{2n-1,2n\}_{n\in N}$ or $S\{k_n,l_n\}_{n\in N}$ only the arcs $B_n(2n-1), C_n(2n-1), E_n(2n)$, and $F_n(2n)$ may differ from $B_n(k_n), C_n(k_n), E_n(l_n)$, and $F_n(l_n)$, respectively. Outside of these arcs, let

ρ_1 be the identity. Also on $B_n(k_n) \cup C_n(k_n)$ (respectively, $E_n(l_n) \cup F_n(l_n)$), let ρ_1 be the identity on the first $2n-1$ dips (respectively, $2n$ humps). Let ρ_1 collapse the remaining $k_n - (2n-1)$ dips (respectively, $l_n - 2n$ humps) onto the subarc of the arc $B_n(2n-1)$ (respectively, $E_n(2n)$) that has end points $(\mu_{2n-1}, 3+n^{-1})$ and $(\upsilon_{2n-1}, 1+n^{-1})$ (respectively, $(-\mu_{2n}, -3-n^{-1})$ and $(-\upsilon_{2n}, -1-n^{-1})$. Simultaneously, ρ_1 sends the arc $C_n(k_n)$ onto $C_n(2n-1)$, and $F_n(l_n)$ onto $F_n(2n)$. □

4. Main continuum $\mathcal{M}_0$

Suppose $\{k_n, l_n\}_{n\in N}$ satisfies the assumptions of Theorem 3.3 and Proposition 3.6, that is, (1) $\{k_n\}_{n\in N}$ and $\{l_n\}_{n\in N}$ are sequences of positive integers satisfying (3.1), (2) $\{\frac{l_n-k_n}{k_n}\}_{n\in N}$ or $\{\frac{k_{n+1}-l_n}{l_n}\}_{n\in N}$ is unbounded, and (3) $2n-1 \le k_n$ and $2n \le l_n$ for each $n \in N$. Let R be a ray in the unbounded domain of $E^2 - D$, with origin $(4,0)$, consisting of folded arcs of increasing lengths that limit on $D \cup S\{k_n, l_n\}_{n\in N}$ (taking a pattern by the graph of the map g of $(0,1]$ onto $[0,\infty)$ given by $g(x) = \frac{1}{x}(1 + sin\ \frac{\pi}{x})$, with $(0,0)$ corresponding to the origin $(3,0)$ of $S\{k_n, l_n\}_{n\in N}$). Let ρ_2 be the natural retraction of $D \cup S\{k_n, l_n\}_{n\in N} \cup R$ onto $D \cup S\{k_n, l_n\}_{n\in N}$ that collapses the folded arcs in R onto the initial arcs of $S\{k_n, l_n\}_{n\in N}$.

Define

$$\mathcal{M}_0 = S^o\{2n-1, 2n\}_{n\in N} \cup D \cup S\{k_n, l_n\}_{n\in N} \cup R.$$

For each $p \in \mathcal{M}_0$, let

$$\rho(p) = \begin{cases} h_0(\rho_1(\rho_2(p))) & \text{if } p \in R \cup S\{k_n, l_n\}_{n\in N}, \\ p & \text{if } p \in D \cup S^o\{2n-1, 2n\}_{n\in N}. \end{cases}$$

By Propositions 3.5 and 3.6, ρ is a retraction of $\mathcal{M}_0$ onto the continuum $D \cup S^o\{2n-1, 2n\}_{n\in N}$. By Propositions 3.2 and 3.5, $D \cup S^o\{2n-1, 2n\}_{n\in N}$ does not have the fixed-point property. Therefore $\mathcal{M}_0$ does not have the fixed-point property.

Obviously, $\mathcal{M}_0$ is rational. Also, $\mathcal{M}_0$ is irreducible between the origin $(\frac{3}{2}, 0)$ of $S^o\{2n-1, 2n\}_{n\in N}$ and the origin $(4,0)$ of R, since $\mathcal{M}_o = ClR \cup ClS^o\{2n-1, 2n\}_{n\in N}$ and the sets $S^o\{2n-1, 2n\}_{n\in N}$, D, $S\{k_n, l_n\}_{n\in N}$, and R are pairwise disjoint.

We will prove in Section 5 that the unique non-degenerate tranche $D \cup S\{k_n, l_n\}_{n\in N}$ of $\mathcal{M}_0$ has the fixed-point property.

Remark 4.1. Similar to the continuum $\mathcal{M}$ in E^3 defined in [H], the continuum $\mathcal{M}_0$ can be used to define another rational irreducible continuum in E^2 that admits a fixed-point-free surjection with the condition that all of its tranches have the fixed-point property.

5. Proof of Theorem 3.3

We will consider the case when the sequence $\{\frac{l_n-k_n}{k_n}\}_{n\in N}$ is unbounded.

Denoting the spiral $S\{k_n, l_n\}_{n\in N}$ by S, we assume there is a map f of $D \cup S$ into itself that moves each point.

Then

(5.1). $f(S) \subset S$ and $p <_S f(p)$ for each $p \in S$ and $f(D) = D$ [M2, Prop. 3.2].

Hence, by [M2, Prop. 3.1],

(5.2). $f(A) = \alpha(A)$ and $f(\alpha(A)) = A$, and

(5.3). $f(L(A)) = L(\alpha(A))$ and $f(L(\alpha(A))) = L(A)$.

Since E^2_- is a neighborhood of $L(\alpha(A))$, it follows from the first equality in (5.3) that there is a neighborhood P of $L(A)$ such that $f(P \cap S) \subset E^2_- \cap S$. We can assume P is a closed rectangle lying in E^2_+ with sides perpendicular to the x-axis. By the second equality in (5.3), there is also a closed rectangular neighborhood Q of $L(\alpha(A))$ such that $f(Q \cap S) \subset P \cap S$ and $Q \subset E^2_-$ and the sides of Q are perpendicular to the x-axis. We can also assume that $P \cap (A \setminus \Gamma) = \emptyset = Q \cap (\alpha(A) \setminus \alpha(\Gamma))$.

Let n_0 be an integer such that for each $n \geq n_0$ the sets $B_n(k_n) \cap P$ and $E_n(l_n) \cap Q$ are arcs. For each $n \geq n_0$, let b_n denote the first point with respect to the order $<_S$ of $B_n(k_n) \cap P$ and let e_n denote the first point of $E_n(l_n) \cap Q$.

For each $n > n_0$, define arcs

$$I_n = S[(-\mu_{l_{n-1}}, -3 - (n-1)^{-1}), b_n] \text{ and } J_n = S[(\mu_{k_n}, 3 + n^{-1}), e_n].$$

Note that the sequences $\{I_n\}_{n>n_0}$ and $\{J_n\}_{n>n_0}$ converge to the arcs $A[(0,-3), \lim b_n]$ and $\alpha(A)[(0,3), \lim e_n]$, respectively.

Setting $E^2_r = \{(x,y) \in E^2 | \ x \geq 0\}$ and $E^2_l = \{(x,y) \in E^2 | \ x \leq 0\}$, there is an integer $n_1 \geq n_0$ such that

(5.4). $f(I_n) \subset E^2_l \cup P$ and $f(J_n) \subset E^2_r \cup Q$ for each $n > n_1$.

This follows from (5.2) and the fact that $E^2_l \cup P$ and $E^2_r \cup Q$ are neighborhoods of $f(\lim I_n) = \lim f(I_n)$ and $f(\lim J_n) = \lim f(J_n)$, respectively.

For each $n > n_1$, let $H_n = P \cap \Gamma_n(k_n, l_n)$ and $K_n = E^2_- \cap \Gamma_n(k_n, l_n)$. Observe that H_n and K_n are arc components of $P \cap S$ and $E^2_- \cap S$, respectively, and b_n is the end point of the arc H_n that belongs to E^2_r, and e_n is the end point of the arc $K_n \cap Q$ that belongs to E^2_l. Since $f(P \cap S) \subset E^2_- \cap S$, for each $n > n_1$ there is exactly one $i_n \in N$ such that $f(H_n) \subset K_{i_n}$. By (5.1), $i_n \geq n$.

We are going to prove

(5.5). $E_{i_n}(l_{i_n}) \subset f(I_n \cup B_n(k_n) \cup J_n)$ for each $n > n_1$.

The arcs I_n and J_n both meet H_n, since $b_n \in H_n \cap I_n$ and $(\mu_{k_n}, 3+n^{-1}) \in H_n \cap J_n$, and therefore

$$f(I_n) \cap K_{i_n} \neq \emptyset \neq f(J_n) \cap K_{i_n} \text{ for each } n > n_1.$$

Also, we have $f(I_n) \cap P \neq \emptyset \neq f(J_n) \cap P$ for each $n > n_1$, because $(-\mu_{l_{n-1}}, -3 - (n-1)^{-1}) \in Q \cap I_n$ implies $f((-\mu_{l_{n-1}}, -3-(n-1)^{-1})) \in f(I_n) \cap P$ and $e_n \in Q \cap J_n$ implies $f(e_n) \in f(J_n) \cap P$. Thus $f(I_n)$ meets an arc component H_m of $P \cap S$ and $f(J_n)$ meets an arc component H_l of $P \cap S$.

For each $n > n_1$,

(5.6). $f(I_n) \cap H_m \neq \emptyset$ implies $m = i_n$ and $f(J_n) \cap H_l \neq \emptyset$ implies $l = i_n + 1$.

To establish (5.6), we denote $\Gamma_n(k_n, l_n)$ by Γ_n and first observe that if $n < n'$, then $p <_S q$ for each $p \in \Gamma_n$ and $q \in \Gamma_{n'}$ such that $p \neq q$. Also, the order $<_S$ agrees with the counterclockwise orientation of E^2.

If $m < i_n$, then $f(I_n)$, going from K_{i_n} to H_m (clockwise), must contain $\Gamma_{i_n} \cap E^2_+$, and if $i_n < m$, then $f(I_n)$ goes from K_{i_n} to H_m counterclockwise so that $(\Gamma_{i_n+1} \cap E^2_+ \cap E^2_r) \setminus P \subset f(I_n)$; in both cases we have a contradiction of the first inclusion in (5.4).

If $i_n + 1 < l$, then $f(J_n)$ goes from K_{i_n} to H_l counterclockwise so that $\Gamma_{i_n+1} \subset f(J_n)$, and if $l < i_n + 1$, then going clockwise we obtain $\Gamma_{i_n} \cap E_+^2 \subset f(J_n)$; in both cases we have a contradiction of the second inclusion in (5.4). Hence (5.6) is true.

Thus

$$f(I_n) \cap H_{i_n} \neq \emptyset \neq f(J_n) \cap H_{i_n+1} \text{ for each } n > n_1.$$

In view of (5.4), the inequalities $f(I_n) \cap H_{i_n} \neq \emptyset \neq f(I_n) \cap K_{i_n}$ imply that $(-2 - i_n^{-1}, 0) \in f(I_n)$ and the inequalities $f(J_n) \cap K_{i_n} \neq \emptyset \neq f(J_n) \cap H_{i_n+1}$ imply that $(2 + (i_n + 1)^{-1}, 0) \in f(J_n)$. Thus both end points of the open arc K_{i_n} belong to the image $f(I_n \cup B_n(k_n) \cup J_n)$ in S. Since $b_n \in I_n \cap B_n(k_n)$ and $(\mu_{k_n}, 3+n^{-1}) \in J_n \cap B_n(k_n)$, it follows that $I_n \cup B_n(k_n) \cup J_n$ is an arc in S (with end points $(-\mu_{l_{n-1}}, -3-(n-1)^{-1})$ and e_n), and thus the image $f(I_n \cup B_n(k_n) \cup J_n)$ is an arc in S. Hence $K_{i_n} \subset f(I_n \cup B_n(k_n) \cup J_n)$. But, obviously, we have $E_{i_n}(l_{i_n}) \subset K_{i_n}$, which proves (5.5).

Claim 5.7. There exists a sequence $\{L_n\}_{n>n_1}$ of arcs in S and a sequence $\{w_n\}_{n>n_1}$ of nonnegative integers such that

(i) each L_n lies in $I_n \cup J_n$ or is a dip in $B_n(k_n)$,
(ii) each $f(L_n)$ contains w_n humps of $E_{i_n}(l_{i_n})$,
(iii) the sequence $\{L_n\}_{n>n_1}$ converges to an arc $\lim L_n \subset D$, and
(iv) the sequence $\{w_n\}_{n>n_1}$ is unbounded.

To prove Claim 5.7, for each $n > n_1$, let u_n denote the number of humps of $E_{i_n}(l_{i_n})$ that are contained in the image $f(I_n \cup J_n) = f(I_n) \cup f(J_n)$.

Case 1. Suppose the sequence $\{u_n\}_{n>n_1}$ is unbounded. Then for each $n > n_1$, let L_n be one of the arcs I_n or J_n, the image of which contains $w_n \leq u_n$ humps of $E_{i_n}(l_{i_n})$ in such a way that the sequence $\{w_n\}_{n>n_1}$ is unbounded.

Case 2. Suppose $\{u_n\}_{n>n_1}$ is bounded. Let $n > n_1$. The number of humps of $E_{i_n}(l_{i_n})$ is l_{i_n}, and the number of humps of $E_{i_n}(l_{i_n})$ that are contained in $f(I_n) \cup f(J_n)$ is u_n. By (5.4), $f(I_n)$ may contain a left subarc and $f(J_n)$ may contain a right subarc of $E_{i_n}(l_{i_n})$. Therefore, by (5.5), $f(B_n(k_n))$ contains at least $l_{i_n} - (u_n + 2)$ humps of $E_{i_n}(l_{i_n})$. Since $i_n \geq n$, it follows from (3.1) that $l_{i_n} \geq l_n$. Thus $l_{i_n} - (u_n + 2) \geq l_n - (u_n + 2)$. Setting $s_n = \frac{l_n - k_n}{k_n}$, we have

$$l_n - (u_n + 2) = k_n s_n + k_n - (u_n + 2).$$

However $k_n \to \infty$ and, in the case being considered, the sequence $\{u_n + 2\}_{n>n_1}$ is bounded. Hence there exists an integer $n_2 \geq n_1$ such that $k_n - (u_n + 2) \geq 0$ for each $n > n_2$. It follows that $l_n - (u_n + 2) \geq k_n s_n$ for each $n > n_2$. Thus, for each $n > n_2$, the series of k_n dips of $B_n(k_n)$ is mapped by f on a series of $k_n v_n$ consecutive humps of $E_{i_n}(l_{i_n})$, where v_n is the integer part of s_n.

Applying Lemma 2.1 (with $\upsilon = \mu = k_n$) we obtain a dip V_n of $B_n(k_n)$ that is mapped on a series of v_n consecutive humps of $E_{i_n}(l_{i_n})$. Let $\{V_{m_n}\}_{n>n_2}$ be a convergent subsequence of $\{V_n\}_{n>n_2}$ such that $v_{m_n} \to \infty$. For each $n > n_2$, let $L_n = V_{m_n}$ and $w_n = v_{m_n}$. Let $L_n = I_n$ and $w_n = 0$ whenever $n_1 \leq n \leq n_2$. Note that $\lim L_n$ is either a dip in Γ or the interval $\overline{(0,1)(0,3)} = L(\Gamma) = L(A)$. This concludes the proof of Claim 5.7.

Let $\{L_n\}_{n>n_1}$ and $\{w_n\}_{n>n_1}$ be sequences that satisfy conditions (i) - (iv) of Claim 5.7. For each $n > n_1$, let $\mathcal{P}_{w_n}(L_n)$ be the partition of the arc L_n into w_n consecutive subarcs with the same length. By (iv), the sequence $\{w_n\}_{n>n_1}$ is unbounded. Instead of taking a strictly increasing subsequence of $\{w_n\}_{n>n_1}$, we assume that $\{w_n\}_{n>n_1}$ itself is strictly increasing. Hence $w_n \to \infty$.

By (iii), the sequence $\{\mathcal{P}_{w_n}(L_n)\}_{n>n_1}$ of partitions converges to a decomposition $\mathcal{P}$ of the arc $\lim L_n \subset D$. Since $w_n \to \infty$, it follows from (i) that each element of $\mathcal{P}$ consists of a single point. By (ii), each $f(L_n)$ contains w_n humps of $E_{i_n}(l_{i_n})$. For each $n > n_1$, it follows from Lemma 2.1 (with $\upsilon = \mu = w_n$) that the partition $\mathcal{P}_{w_n}(L_n)$ has an element $L_{j_n,n}$, the image of which contains an entire hump of $E_{i_n}(l_{i_n})$. Passing to the limit with a convergent subsequence of the sequence $\{L_{j_n,n}\}_{n>n_1}$, we obtain a point (of the limit arc $\lim L_n$), the image of which contains the limit of the corresponding humps, i.e., either a hump of $\alpha(A)$ or the interval $L(\alpha(A))$, a contradiction.

By a dual argument, the assumption that $\left\{\frac{k_{n+1}-l_n}{l_n}\right\}_{n\in N}$ is unbounded also leads to a contradiction.

6. Related questions

The tranche $D \cup S\{k_n, l_n\}_{n\in N}$ of $\mathcal{M}_0$ separates E^2.

Question 6.1. *Must a plane continuum of type λ have the fixed-point property if none of its tranches separates the plane?*

A plane continuum of type λ separates the plane if and only if at least one of its tranches separates the plane [Mr]. Since it is not known if every nonseparating plane continuum has the fixed-point property [Bi, Ques. 3], [K–W, p. 66, p. 145], Question 6.1 should also be considered with the additional assumption that each tranche has the fixed-point property. A counterexample must contain an invariant indecomposable continuum in one of its tranches [B], [I], [S].

Question 6.2. *Must a plane continuum of type λ have the fixed-point property if each of its tranches has the fixed-point property and its decomposition is continuous?*

A counterexample to Question 6.2 cannot be rational since it must have an indecomposable tranche [D].

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