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SELF-ADJOINTNESS OF THE PERTURBED WAVE OPERATOR ON $L^2(\mathbf{R^n}), n \geq 2$

MOHAMMED HICHEM MORTAD

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Abstract. We give classes of unbounded real-valued V for which $\Box + V$ is self-adjoint on $\mathcal{D}(\Box) \subset L^2(\mathbf{R^n})$, $n \geq 2$, where $\Box$ is the wave operator defined on $\mathbf{R^n}$.

1. Introduction

There are many classes of V for which $-\Delta + V$ is self-adjoint on $\mathcal{D}(-\Delta)$. In this paper we investigate the self-adjointness of $\Box + V$ where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_2^2} - ... - \frac{\partial^2}{\partial x_n^2}$ is the usual wave operator and V is an unbounded real-valued function defined on $\mathbf{R^n}$. By using Fourier transforms on $L^2(\mathbf{R^n})$ one can show that $\Box$ is self-adjoint on $\mathcal{D}(\Box) = \{f \in L^2(\mathbf{R^n}) : \Box f \in L^2(\mathbf{R^n})\}$ (we understand that the partial derivative is taken in the distributional sense). In the sequel $\mathcal{D}(\Box)$ will be denoted by M^n where n stands for the dimension. We assume definitions and notions about unbounded operators, Fourier transforms, BMO and interpolation theory, and for references one may consult [2], [3] or [4]. We also recall without proof the Kato-Rellich perturbation theorem (for a proof see [3], Theorem X.12), but before that we have

Definition 1. Let A and B be densely defined linear operators on a Hilbert space H. Suppose that:
i) $D(A) \subset D(B)$;
ii) for some a and b in $\mathbf{R}$ and all $\varphi \in D(A)$,

$$\|B\varphi\| \leq a\|A\varphi\| + b\|\varphi\|.$$

Then B is said to be **A-bounded**. The infimum of such an a is called the **relative bound** of B with respect to A.

Theorem 1 (Kato-Rellich theorem)**.** *Suppose that A is self-adjoint, B is symmetric, and B is A-bounded with relative bound $a < 1$. Then $A + B$ is self-adjoint on $D(A)$.*

Remark 1. Throughout this paper, $\tilde{c}$ will denote any arbitrary constant that need not be the same each time.

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2. M^2 AND THE SPACE BMO

It is known that by a simple change of variables we can change $\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$ into $\frac{\partial^2}{\partial\eta\partial\xi}$. So we will use $\frac{\partial^2}{\partial x\partial y}$ to mean $\Box$. We will prove that $M^2\subset BMO(\mathbf{R^2})$, but first we have

Lemma 1. *Fix $y_1\in\mathbf{R}$, and put $f_1(x)=\varphi(x,y_1)$ where $\varphi\in M^2$. Then $f_1\in BMO(\mathbf{R})$ with uniform BMO bound. Also $y\mapsto g_1(y)=\varphi(x_1,y)$ is in $BMO(\mathbf{R})$.*

Proof. We have to prove $\frac{1}{|I|}\int_I|f_1(x)-\overline{f_1}|dx\le c$ where c does not depend on y_1, and where $\overline{f_1}=\frac{1}{|I|}\int_I f_1(x)dx$. By putting $\rho=\frac{\partial^2\varphi}{\partial x\partial y}$ we have

$$(f_1-f)'(x)=\int_y^{y_1}\rho(x,z)dz \text{ where } f(x)=\varphi(x,y),$$

and where we have the freedom to choose the y that is convenient to us. Then we have

$$\|(f_1-f)'\|_2^2\le(y_1-y)\|\Box\varphi\|_2^2<\infty \text{ with the assumption } y<y_1\ .$$

So $(f_1-f)'\in L^2(\mathbf{R})$. Hence, f_1-f is continuous and so there exists a $c\in I$ such that

$$(f_1-f)(x)-\overline{f_1-f}=(f_1-f)(x)-(f_1-f)(c),$$

and since $(f_1-f)'\in L^1_{loc}(\mathbf{R})$ we have

$$(f_1-f)(x)-(f_1-f)(c)=\int_c^x(f_1-f)'(t)dt.$$

By taking $I=(a,b)$ and since both c and x are in I, one has

$$|(f_1-f)(x)-(f_1-f)(c)|\le(b-a)^{\frac{1}{2}}\|(f_1-f)'\|_2=|I|^{\frac{1}{2}}\|(f_1-f)'\|_2.$$

Hence

$$\frac{1}{|I|}\int_I|(f_1-f)(x)-(f_1-f)(c)|dx\le|I|^{\frac{1}{2}}(y_1-y)^{\frac{1}{2}}\|\Box\varphi\|_2.$$

So in order to find a uniform bound for $\|f_1-f\|_{BMO}$ for this particular I, it suffices to take y such that $(y_1-y)^{\frac{1}{2}}\le\frac{1}{|I|^{\frac{1}{2}}}$, and in such a case we will have

$$\frac{1}{|I|}\int_I|(f_1-f)(x)-\overline{(f_1-f)}|dx\le\|\Box\varphi\|_2.$$

But our interest is in the function f_1 itself, not in f_1-f. We can find a BMO bound for f_1 for this particular I if we come to show that f is BMO. We have

$$\frac{1}{|I|}\int_I|f(x)-\overline{f}|dx\le\frac{1}{|I|}\int_I|f(x)|dx+\frac{1}{|I|}\int_I\left(\frac{1}{|I|}\int_I|f(t)|dt\right)dx.$$

Then

$$\frac{1}{|I|}\int_I|f(x)-\overline{f}|dx\le\frac{2\|f\|_2}{|I|^{\frac{1}{2}}}.$$

So far we have not used the fact that $\varphi\in L^2(\mathbf{R^2})$! To have a uniform BMO bound for f we need for instance $\|f\|_2\le c|I|^{\frac{1}{2}}$ ($c>0$, to be determined). By

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applying a simple calculus argument, one can say that there exists a $y \in \mathbf{R}$, $|y_1 - y| \leq \frac{1}{(b-a)}$ such that $\|f\|_2 \leq 2\|\varphi\|_2 |I|^{\frac{1}{2}}$. So

$$\frac{1}{|I|} \int_I |f_1(x) - \overline{f_1}| dx \leq \tilde{c}\|\Box\varphi\|_2 + \tilde{c}\|\varphi\|_2.$$

The same proof works for g_1. $\Box$

Theorem 2. *Let $\varphi \in M^2$. Then $\varphi \in BMO(\mathbf{R^2})$ and*

$$\|\varphi\|_{BMO(\mathbf{R^2})} \leq a\|\Box\varphi\|_2 + b\|\varphi\|_2.$$

Proof. The proof follows from Lemma 1 and Fubini's Theorem. $\Box$

Hence we have the following corollary whose proof is an immediate consequence of an interpolation theorem ([4], Theorem 6.8).

Corollary 1. *Let $2 \leq p < \infty$. Then any $\varphi \in M^2$ will be in $L^p(\mathbf{R^2})$, and we have*

$$\|\varphi\|_p \leq a\|\Box\varphi\|_2 + b\|\varphi\|_2 \tag{1}$$

where the constants a and b depend on p.

Remark 2. The case $p = \infty$ is excluded, as will be shown below.

Lemma 2. *The constant a in Corollary 1 may be made as small as we want.*

Proof. Take $\varphi_\lambda(x, y) = \varphi(\lambda x, \lambda y) : \lambda > 0$. We get

$$\|\Box\varphi_\lambda\|_2 = \lambda\|\Box\varphi\|_2, \quad \|\varphi_\lambda\|_2 = \frac{1}{\lambda}\|\varphi\|_2 \quad \text{and} \quad \|\varphi_\lambda\|_p = \frac{1}{\lambda^{\frac{2}{p}}}\|\varphi\|_p.$$

Thus the estimate (1) applied to φ_λ instead of φ becomes

$$\|\varphi\|_p \leq a\lambda^{\frac{2}{p}+1}\|\Box\varphi\|_2 + b\lambda^{\frac{2}{p}-1}\|\varphi\|_2, \quad \lambda > 0, \ p \geq 2.$$

By taking λ small enough, the constant in front of $\|\Box\varphi\|_2$ will be arbitrarily small. $\Box$

Theorem 3. *Let $\Box$ be the wave operator on $L^2(\mathbf{R^2})$. Let $\epsilon > 0$, and let $V : \mathbf{R^2} \to \mathbf{R}$ be such that $V \in L^{2+\epsilon}(\mathbf{R^2})$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. We have by Corollary 1 that

$$\|\varphi\|_p \leq a\|\Box\varphi\|_2 + b\|\varphi\|_2, \text{ for } 2 \leq p < \infty.$$

Then by Hölder's inequality,

$$\|V\varphi\|_2 \leq \|V\|_q\|\varphi\|_p \leq a\|V\|_q\|\Box\varphi\|_2 + b\|V\|_q\|\varphi\|_2,$$

for $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ or $q = \frac{2p}{p-2}$. Since the constant in front of $\|\Box\varphi\|_2$ can be made small enough so that we have $a\|V\|_q < 1$, we conclude by the Kato-Rellich perturbation theorem that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box) = M^2$. $\Box$

Proposition 1. *Let $V \in L^{2+\epsilon}(\mathbf{R^2}) + L^\infty(\mathbf{R^2})$ be a real-valued function, $\epsilon > 0$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. The same as for Theorem 3. $\Box$

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3. A COUNTEREXAMPLE

Proposition 2. *Let* $\varphi \in L^2(\mathbf{R^2})$ *be such that* $\frac{\partial^2\varphi}{\partial x\partial y} \in L^2(\mathbf{R^2})$. *Then* φ *need not be essentially bounded on* $\mathbf{R^2}$.

Proof. We are going to build up the counterexample by using a linear interpolation. We define $(x,y) \mapsto \varphi(x,y)$ on $\mathbf{R} \times (y_n, y_{n+1})$ by

$$\varphi(x,y) = \frac{1}{y_{n+1}-y_n}[(y-y_n)f_{n+1}(x) - (y-y_{n+1})f_n(x)] \text{ where } f_n(x) = \varphi(x,y_n),$$

and the f_n and y_n are to be defined below.

Hence on $\mathbf{R} \times (y_1, \infty)$ we have

$$\|\varphi\|_2^2 = \iint\limits_{\mathbf{R}\times(y_1,\infty)} |\varphi(x,y)|^2 dxdy = \sum_1^\infty \iint\limits_{\mathbf{R}\times(y_n,y_{n+1})} |\varphi(x,y)|^2 dxdy.$$

After some calculations the condition that makes φ belong to L^2 is

$$\sum_1^\infty (y_{n+1}-y_n)(\|f_n\|_2^2 + \|f_{n+1}\|_2^2) < \infty.$$

We also have

$$\frac{\partial^2\varphi}{\partial x\partial y} = \frac{1}{y_{n+1}-y_n}(f'_{n+1} - f'_n) \in L^2(\mathbf{R}\times(y_1,\infty)) \text{ if}$$

$$\sum_1^\infty \frac{1}{y_{n+1}-y_n}\|\psi'_n\|_2^2 < \infty \text{ where } \psi_n(x) = f_{n+1}(x) - f_n(x).$$

Since $\psi_n(x) = f_{n+1}(x) - f_n(x)$, then $f_n(x) = -\sum_n^\infty \psi_k(x)$. We also want $f_n \notin L^\infty(\mathbf{R})$ so that $\varphi \notin L^\infty(\mathbf{R^2})$.

Take $\psi_n(x) = \begin{cases} \frac{e^n}{n}x + \frac{1}{n} & \text{if} \quad -e^{-n} \le x \le 0, \\ -\frac{e^n}{n}x + \frac{1}{n} & \text{if} \quad 0 \le x \le e^{-n}, \\ 0 & \text{if} \quad |x| \ge e^{-n}. \end{cases}$

Hence $\|\psi'_n\|_2^2 \sim \frac{e^n}{n^2}$ and $\|\psi_n\|_2^2 \sim \frac{e^{-n}}{n^2}$. We also have

$$\|f_n\|_2 \le \sum_{k=n}^\infty \|\psi_k\|_2 = a\sum_{k=n}^\infty \frac{e^{-\frac{n}{2}}}{n} \simeq \int_n^\infty \frac{e^{-\frac{x}{2}}}{x}dx \le \frac{1}{n}\int_n^\infty e^{-\frac{x}{2}}dx \sim \frac{e^{-\frac{n}{2}}}{n}.$$

Now if we choose $y_{n+1} - y_n = e^n$, then the series

$$\sum_1^\infty \frac{1}{y_{n+1}-y_n}\|\psi'_n\|_2^2 = \sum_1^\infty \frac{1}{e^n} \times \frac{e^n}{n^2} = \sum_1^\infty \frac{1}{n^2}$$

converges and so does

$$\sum_1^\infty (y_{n+1}-y_n)(\|f_n\|_2^2 + \|f_{n+1}\|_2^2) \le \sum_1^\infty e^n \times [\frac{e^{-(n+1)}}{(n+1)^2} + \frac{e^{-n}}{n^2}] \sim \sum_1^\infty \frac{1}{n^2}.$$

Now the φ defined on $\mathbf{R} \times (y_n, y_{n+1})$ is given by

$$\varphi(x,y) = e^{-n}[(y-y_n)(-\sum_{n+1}^\infty \psi_k(x)) - (y-y_{n+1})(-\sum_n^\infty \psi_k(x))].$$

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This φ is actually defined only for $x \in \mathbf{R}$ and $y \geq y_1$. To extend it to the case $y < y_1$ we define φ for $x \in \mathbf{R}$ and $y_1 - y_{n+1} < y < y_1 - y_n$ as follows:

$$\varphi(x,y) = \frac{1}{y_{n+1} - y_n}[(y - y_1 + y_n)f_{n+1}(x) - (y - y_1 + y_{n+1})f_n(x)].$$

This φ is in M^2 for sure. Now we need to show that φ is not in $L^\infty(\mathbf{R^2})$. Let $x > 0$ and $x \leq e^{-k}$. Then $\ln x \leq -k$ or $\ln \frac{1}{x} \geq k$. So

$$-f_n(x) = x \sum_{n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{e^k}{k} - \sum_{n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k} \geq \sum_{n}^{\lfloor \ln \frac{1}{x} \rfloor} \frac{1}{k}.$$

Now as $x \to 0$, then $\lfloor \ln \frac{1}{x} \rfloor \to \infty$. Hence $\ln \lfloor \ln \frac{1}{x} \rfloor \to \infty$. Thus $f_n(x) \to \infty$, which implies that $\varphi(x,y) \to \infty$. So $\varphi \notin L^\infty(\mathbf{R^2})$. □

Remark 3. This counterexample found is a BMO function by Theorem 2.

One can prove Proposition 2 by using Fourier transforms, but it is only an existence proof. One may also wonder under what additional conditions M^2 will be in $L^\infty(\mathbf{R^2})$. The answer is given by

Proposition 3. *Let $\varphi \in L^2(\mathbf{R^2})$ be such that $\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}$ and $\frac{\partial^2 \varphi}{\partial x \partial y}$ are all in $L^2(\mathbf{R^2})$. Then $\hat{\varphi} \in L^1(\mathbf{R^2})$ and hence $\varphi \in L^\infty(\mathbf{R^2})$.*

4. CLASSES OF SELF-ADJOINT $\Box + V$ ON $L^2(\mathbf{R^n}), n \geq 3$

Remark 4. We will state theorems and propositions in the n-dimensional case, but we will only prove them for $n = 3$ since the proofs are the same for $n > 3$.

Let us first discuss the following Cauchy problem:

$$(I) \qquad \begin{cases} u_{tt} - u_{xx} - u_{yy} = f(x,y,t), & (x,y,t) \in \mathbf{R^2} \times \mathbf{R}^+, \\ u(x,y,0) = \varphi(x,y); u_t(x,y,0) = \psi(x,y). \end{cases}$$

Now let us take the Fourier transform of (I) in the (x,y)-plane only. We get

$$(\hat{I}) \qquad \begin{cases} \hat{u}_{tt} + (\eta^2 + \xi^2)\hat{u} = \hat{f}(\eta,\xi,t), \\ \hat{u}(\eta,\xi,0) = \hat{\varphi}(\eta,\xi); \hat{u}_t(\eta,\xi,0) = \hat{\psi}(\eta,\xi). \end{cases}$$

$(\hat{I})$ is a second-order ODE in t that has the following solution:

$$(2) \qquad \begin{aligned} \hat{u}_{\eta,\xi}(t) = \hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) + \frac{\hat{\psi}(\eta,\xi)}{\sqrt{\eta^2+\xi^2}}\sin(t\sqrt{\eta^2+\xi^2}) \\ + \int_0^t \hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds. \end{aligned}$$

The previous holds for $t \geq 0$. For $t < 0$, (I) becomes, after setting $\tilde{u}(x,y,t) = u(x,y,-t)$,

$$(\tilde{I}) \qquad \begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} - \tilde{u}_{yy} = f(x,y,-t), & (x,y,t) \in \mathbf{R^2} \times \mathbf{R}^-, \\ \tilde{u}(x,y,0) = \varphi(x,y); \tilde{u}_t(x,y,0) = -\psi(x,y). \end{cases}$$

Now we “fourier” everything in the (x,y)-plane to obtain

$$(\hat{\tilde{I}}) \qquad \begin{cases} \hat{\tilde{u}}_{tt} + (\eta^2 + \xi^2)\hat{\tilde{u}} = \hat{f}(\eta,\xi,-t), \\ \hat{\tilde{u}}(\eta,\xi,0) = \hat{\varphi}(\eta,\xi); \hat{\tilde{u}}_t(\eta,\xi,0) = -\hat{\psi}(\eta,\xi), \end{cases}$$

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which has the following solution:

$$\begin{aligned}\hat{u}_{\eta,\xi}(-t) = \hat{\tilde{u}}_{\eta,\xi}(t) = \hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) - \frac{\hat{\psi}(\eta,\xi)}{\sqrt{\eta^2+\xi^2}}\sin(t\sqrt{\eta^2+\xi^2})\\ + \int_0^{-t}\hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(-t-s)}{\sqrt{\eta^2+\xi^2}}ds,\end{aligned}\tag{3}$$

and this holds for $t<0$. After adding up (2) and (3), one gets for $t\in\mathbf{R}$:

$$\begin{aligned}\hat{u}_{\eta,\xi}(t) = 2\hat{\varphi}(\eta,\xi)\cos(t\sqrt{\eta^2+\xi^2}) + \int_0^{t}\hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds\\ + \int_0^{-t}\hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(-t-s)}{\sqrt{\eta^2+\xi^2}}ds - \hat{u}_{\eta,\xi}(-t).\end{aligned}\tag{4}$$

Now we can change our Cauchy problem by introducing different initial conditions mainly for the t variable; i.e., instead of working on the intervals $(0,t)$ and $(-t,0)$, we will be working on $(\frac{t+\alpha}{2},t)$ and $(\alpha,\frac{t+\alpha}{2})$ where α is to be chosen freely. Now given $u\in C_0^\infty$, then we can regard it as a solution of the Cauchy problem with $u\in C_0^\infty(\mathbf{R^3})$ where $f=\Box u$, $u(x,y,0)=\varphi(x,y)$ and $u_t(x,y,0)=\psi(x,y)$. Then (4) will be:

(5)
$$\begin{aligned}\hat{u}(\eta,\xi,t) = 2\hat{u}(\eta,\xi,\frac{t+\alpha}{2})\cos[\sqrt{\eta^2+\xi^2}(t)] - \hat{u}(\eta,\xi,\alpha)\\ + \int_{\frac{t+\alpha}{2}}^{t}\hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}ds + \int_{\frac{t+\alpha}{2}}^{\alpha}\hat{f}(\eta,\xi,s)\frac{\sin\sqrt{\eta^2+\xi^2}(\alpha-s)}{\sqrt{\eta^2+\xi^2}}ds.\end{aligned}$$

Proposition 4. *Let $n\geq 1$. Let $u\in M^{n+1}$. Then $\|u(...,t)\|_{L^2(\mathbf{R^n})}\in L^\infty(\mathbf{R})$ and we have*

$$\sup_{t\in\mathbf{R}}\|u(...,t)\|_{L^2(\mathbf{R^n})}\leq a\|\Box u\|_{L^2(\mathbf{R^{n+1}})} + b\|u\|_{L^2(\mathbf{R^{n+1}})}.$$

Proof. We shall prove this proposition for functions in $C_0^\infty(\mathbf{R^3})$ first.[1] Then the result follows for functions in M^3 since $C_0^\infty(\mathbf{R^3})$ is dense in M^3 in the graph norm of $\Box$.

We choose α such that $|t-\alpha|\leq 1$, and then we obtain:

$$|\hat{u}(\eta,\xi,t)|\leq 2|\hat{u}(\eta,\xi,\frac{t+\alpha}{2})| + \tilde{c}\int_{\frac{t+\alpha}{2}}^{t}|\hat{f}(\eta,\xi,s)|ds + \tilde{c}\int_{\frac{t+\alpha}{2}}^{\alpha}|\hat{f}(\eta,\xi,s)|ds + |\hat{u}(\eta,\xi,\alpha)|,$$

where we have used the fact that $|\frac{\sin X}{X}|\leq 1$. Using the Cauchy-Schwarz inequality and taking the squares of both sides, we will have

$$\begin{aligned}|\hat{u}(\eta,\xi,t)|^2\leq \tilde{c}|\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 + \tilde{c}\int_{\frac{t+\alpha}{2}}^{t}|\hat{f}(\eta,\xi,s)|^2ds + \tilde{c}\int_{\frac{t+\alpha}{2}}^{\alpha}|\hat{f}(\eta,\xi,s)|^2ds\\ + \tilde{c}|\hat{u}(\eta,\xi,\alpha)|^2.\end{aligned}\tag{6}$$

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[1] $C_0^\infty(\mathbf{R^3})$ is the space of infinitely differentiable functions with compact support.

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Then integrate (6) with respect to η and ξ in $\mathbf{R^2}$, and bound the integral of $|\hat{f}|^2$ by the one over $\mathbf{R^3}$ to obtain

$$\iint_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi \leq \tilde{c} \iint_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi + \tilde{c} \iiint_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds$$
$$+\tilde{c} \iiint_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds + \tilde{c} \iint_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi.$$

Now integrate everything w.r.t. α in the segment $|t-\alpha| \leq 1$ and bound again by integrals over $\mathbf{R^3}$ to get

$$\iint_{\mathbf{R^2}} |\hat{u}(\eta,\xi,t)|^2 d\eta d\xi \leq \tilde{c} \iiint_{\mathbf{R^3}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha + \tilde{c} \iiint_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds$$
$$+\tilde{c} \iiint_{\mathbf{R^3}} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds + \tilde{c} \iiint_{\mathbf{R^3}} |\hat{u}(\eta,\xi,\alpha)|^2 d\eta d\xi d\alpha.$$

Thus by taking the supremum in t over $\mathbf{R}$ and by using the Plancherel theorem, one has

$$\sup_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^2(\mathbf{R^2})} \leq \tilde{c}\|\Box u\|_{L^2(\mathbf{R^3})} + \tilde{c}\|u\|_{L^2(\mathbf{R^3})}. \tag{7}$$

The estimate for functions in M^3 then follows by the denseness of C_0^∞ in M^3. □

Theorem 4. *Let $\Box$ be the wave operator defined on $L^2(\mathbf{R^{n+1}})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(...,t)\|^2_{L^\infty(\mathbf{R^n})} dt < \infty$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. The proof is very similar to the one of Theorem 3. □

Proposition 5. *Let $V = V_1 + V_2$ be real-valued such that V_1 is as in Theorem 4 and $V_2 \in L^\infty(\mathbf{R^{n+1}})$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

By going back to equation (4), we can have a better estimate than the one in Proposition 4 and we have

Proposition 6. *Let $n \geq 2$, and let $u \in M^{n+1}$. Then $\|u(...,t)\|_{L^r(\mathbf{R^n})} \in L^\infty(\mathbf{R})$, $2 < r < \frac{2n}{n-1}$ and we have*

$$\sup_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^r(\mathbf{R^n})} \leq a\|\Box u\|_{L^2(\mathbf{R^{n+1}})} + b\|u\|_{L^2(\mathbf{R^{n+1}})}.$$

In order to do that we will need some lemmas.

Lemma 3 (Sobolev's inequality)**.** *Let $n \geq 2$. Let $f \in H^1(\mathbf{R^n})$.*[2] *Then $f \in L^q(\mathbf{R^n})$ for $2 \leq q < \frac{2n}{n-2}$ (with $n = 2$ giving $q = \infty$) and we have*

$$\|f\|_q \leq a\|\nabla f\|_2 + b\|f\|_2.$$

We recall the following interpolation result:

Proposition 7. *Let $u(t) \in L^2(\mathbf{R^n})$ be such that also $u(t) \in L^p_w(\mathbf{R^n})$.*[3] *Then $u(t) \in L^r(\mathbf{R^n})$ for $2 < r < p$.*

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[2] $H^1(\mathbf{R^n})$ is the Sobolev space $\{f \in L^2(\mathbf{R^n}) : \nabla f \in L^2(\mathbf{R^n}\}$.

[3] The L^p-weak norm of u will be denoted by $\|.\|_{p,w}$.

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Lemma 4. *Assume for all* $\epsilon > 0$, *there exist* V, g *such that* $u(t) = V + g$, $\|V\|_2 \le d\epsilon$ *and* $\|g\|_q \le \frac{c}{\epsilon}$ *where* c, d *are two constants. Then* $u(t) \in L^p_w(\mathbf{R^n})$ *where* $p = \frac{4q}{2+q}$ *and* $2 \le q < \frac{2n}{n-2}$.

Proof. We only prove the lemma for $n = 2$. Let $E_\lambda = \{x : |u(x)| \ge \lambda\}$. Then, since $u = V + g$,

$$E_\lambda \subset \{x : |V(x)| \ge \frac{\lambda}{2}\} \cup \{x : |g(x)| \ge \frac{\lambda}{2}\},$$

so that

$$|E_\lambda| \subset |\{x : |V(x)| \ge \frac{\lambda}{2}\}| + |\{x : |g(x)| \ge \frac{\lambda}{2}\}|.$$

Then by using the Chebyshev inequality, we get $|E_\lambda| \le 4\lambda^{-2}\|V\|_2^2 + 2^q\lambda^{-q}\|g\|_q^q$. But we have the freedom to choose any $\epsilon(\lambda) > 0$. So take $\epsilon(\lambda) = \lambda^b$ (b to be determined). With this choice we obtain

$$|E_\lambda| \le 4d^2\lambda^{2b-2} + 2^q c^q \lambda^{-bq-q} \text{ or } \lambda^p|E_\lambda| \le 4d^2\lambda^{p+2b-2} + 2^q c^q \lambda^{p-bq-q}.$$

Then $u(t)$ belongs to $L^p_w(\mathbf{R^2})$ if and only if $\sup_{\lambda>0}[4d^2\lambda^{p+2b-2} + 2^q c^q\lambda^{p-bq-q}] < \infty$, which is only possible if $b = \frac{2-p}{2}$ and $p = \frac{4q}{2+q}$. $\square$

Now we prove Proposition 6.

Proof. Let $u \in C_0^\infty$. We use (5) to show that $u(t)$ is in $L^r(\mathbf{R^2})$, $2 \le r < 4$ ($n = 2$ gives $q < \infty$ and hence $r < 4$). We set $u(t) = V + g$. So $\hat{u}(t) = \hat{V} + \hat{g}$,

$$\hat{V}(\eta, \xi) = 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)] - \hat{u}(\eta, \xi, \alpha)$$

and

$$\hat{g}(\eta, \xi) = \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin\sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds + \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin\sqrt{\eta^2 + \xi^2}(\alpha-s)}{\sqrt{\eta^2 + \xi^2}} ds$$

where $\hat{f} \in L^2(\mathbf{R^3})$.

We observe that both $\hat{V}$ and $\hat{g}$ depend on α. We will first show that $\nabla g \in L^2(\mathbf{R^2})$ so that $g \in L^q(\mathbf{R^2})$, $2 \le q < \infty$, by Lemma 3 since $g \in L^2(\mathbf{R^2})$. Then we show that $V \in L^2(\mathbf{R^2})$. We only do the cases $\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)]$ and $\int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin\sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}} ds$, which we also denote by $\hat{V}(\eta, \xi)$ and $\hat{g}(\eta, \xi)$. The proof of the other two cases is simply the same.

Also, the assumptions we made in Lemma 4 suggested that $\|V\|_2 \le d\epsilon$ and $\|g\|_q \le \frac{c}{\epsilon}$. So here we find the constants c and d explicitly and, with no surprise, they depend on $\|u\|_{L^2(\mathbf{R^3})}$ and $\|\Box u\|_{L^2(\mathbf{R^3})}$.

We also bear in mind that we still have the freedom to choose α. We also give details of why we have $\nabla g \in L^2(\mathbf{R^2})$. We have $\hat{u} = \hat{V} + \hat{g}$ where

$$\hat{V}(\eta, \xi) = 2\hat{u}(\eta, \xi, \frac{t+\alpha}{2}) \cos[\sqrt{\eta^2 + \xi^2}(t)]$$

and

$$\hat{g}(\eta, \xi) = \int_{\frac{t+\alpha}{2}}^t \hat{f}(\eta, \xi, s) \frac{\sin\sqrt{\eta^2 + \xi^2}(t-s)}{\sqrt{\eta^2 + \xi^2}} ds \text{ where } \hat{f} \in L^2(\mathbf{R^3}).$$

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Also, $\hat{g} \in L^2(\mathbf{R^2})$ and since

$$-i\eta \frac{\sin \sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}}, -i\xi \frac{\sin \sqrt{\eta^2+\xi^2}(t-s)}{\sqrt{\eta^2+\xi^2}} \in L^\infty(\mathbf{R^2}),$$

then $-i\eta\hat{g}, -i\xi\hat{g} \in L^2(\mathbf{R^2})$, i.e., $\widehat{\nabla g} \in L^2(\mathbf{R^2})$. So by Sobolev's inequality we have $g \in L^q(\mathbf{R^2}), 2 \le q < \infty$. We also have

$$\|\nabla g\|^2_{L^2(\mathbf{R^2})} = \|\widehat{\nabla g}\|^2_{L^2(\mathbf{R^2})} = \iint\limits_{\mathbf{R^2}} |\eta\hat{g}(\eta,\xi)|^2 d\eta d\xi + \iint\limits_{\mathbf{R^2}} |\xi\hat{g}(\eta,\xi)|^2 d\eta d\xi.$$

But

$$\eta^2 |\hat{g}(\eta,\xi)|^2 \le a^2 |t-\alpha| \int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 ds$$

and

$$\xi^2 |\hat{g}(\eta,\xi)|^2 \le a^2 |t-\alpha| \int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 ds.$$

So by choosing α such that $|t-\alpha| \le \frac{1}{\epsilon^2}$ we get

$$\|\nabla g\|^2_{L^2(\mathbf{R^2})} \le \frac{2a^2}{\epsilon^2} \iint\limits_{\mathbf{R^2}} \int_{\frac{t+\alpha}{2}}^{t} |\hat{f}(\eta,\xi,s)|^2 d\eta d\xi ds.$$

Thus

$$\|\nabla g\|_{L^2(\mathbf{R^2})} \le \frac{c}{\epsilon} \|\Box u\|_{L^2(\mathbf{R^3})}.$$

In a similar way, one gets $\|g\|_{L^2(\mathbf{R^2})} \le \frac{c}{\epsilon^{\frac{3}{2}}} \|\Box u\|_{L^2(\mathbf{R^3})}$. So for $\epsilon \ge 1$ we have $\|g\|_{L^2(\mathbf{R^2})} \le \frac{c}{\epsilon} \|\Box u\|_{L^2(\mathbf{R^3})}$. Also, for $\epsilon \le 1$ one has to use $\hat{g} = \hat{u} - \hat{V}$ and the estimate we will get for $\|V\|_{L^2(\mathbf{R^2})}$ below and the fact that u belongs to $L^2(\mathbf{R^3})$.

For V we have

$$\hat{V}(\eta,\xi) = 2\hat{u}(\eta,\xi,\frac{t+\alpha}{2}) \cos[\sqrt{\eta^2+\xi^2}(t)].$$

Then

$$\|\hat{V}\|^2_{L^2(\mathbf{R^2})} \le 4 \iint\limits_{\mathbf{R^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi,$$

and then by integrating w.r.t. $\alpha \in \{\alpha : |t-\alpha| \le \frac{1}{\epsilon^2}\}$ one obtains

$$\int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} \|\hat{V}\|^2_{L^2(\mathbf{R^2})} d\alpha \le 4 \iint\limits_{\mathbf{R^2}} \int_{t-\frac{1}{\epsilon^2}}^{t+\frac{1}{\epsilon^2}} |\hat{u}(\eta,\xi,\frac{t+\alpha}{2})|^2 d\eta d\xi d\alpha \le 4\|\hat{u}\|^2_{L^2(\mathbf{R^3})}.$$

Then there exists $\alpha \in (t-\frac{1}{\epsilon^2}, t+\frac{1}{\epsilon^2})$ s.t. $\|\hat{V}\|^2_{L^2(\mathbf{R^2})} \le \frac{4\|\hat{u}\|^2_{L^2(\mathbf{R^3})}}{\frac{2}{\epsilon^2}}$. So $\|V\|_{L^2(\mathbf{R^2})} \le d\epsilon\|u\|_{L^2(\mathbf{R^3})}$. So one has

$$\sup_{\lambda>0}[4d^2\lambda^{p-2+2b} + 2^q c^q \lambda^{p-q-bq}] = \tilde{c}\|u\|^2_{L^2(\mathbf{R^3})} + \tilde{c}\|\Box u\|^q_{L^2(\mathbf{R^3})}. \tag{8}$$

We have by Proposition 7 that $\|u(t)\|_{L^r(\mathbf{R^2})} \le \tilde{c}\|u(t)\|_2 + \tilde{c}\|u(t)\|_{p,w}$. So by using (8), Proposition 4, scaling by a suitable constant and taking the supremum in t over $\mathbf{R}$, one obtains

$$\sup_{t\in\mathbf{R}} \|u(.,.,t)\|_{L^r(\mathbf{R^2})} \le a\|\Box u\|_{L^2(\mathbf{R^3})} + b\|u\|_{L^2(\mathbf{R^3})}. \tag{9}$$

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Remark 5. The constant a in (9) may be made as small as we would like.

Theorem 5. *Let $n \geq 2$. Let $\Box$ be the wave operator defined on $L^2(\mathbf{R^{n+1}})$. Let V be a real-valued function such that $\int_{-\infty}^{\infty} \|V(...,t)\|^2_{L^s(\mathbf{R^n})}dt < \infty$ for $\frac{1}{s} = \frac{1}{2} - \frac{1}{r}$ and $2 \leq r < \frac{2n}{n-1}$. Then $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$.*

Proof. Again we only prove the theorem for $n = 2$. We have by Hölder's inequality, for $\frac{1}{2} = \frac{1}{r} + \frac{1}{s}$, $2 \leq r < 4$:

$$\iint\limits_{\mathbf{R^2}} |V(x,y,t)u(x,y,t)|^2 dxdy \leq \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})}.\|u(.,.,t)\|^2_{L^r(\mathbf{R^2})}$$

$$\leq \sup_{t\in\mathbf{R}} \|u(.,.,t)\|^2_{L^r(\mathbf{R^2})}(\|V(.,.,t)\|^2_{L^s(\mathbf{R^2})}).$$

Then by using Proposition 6 and by integrating w.r.t. t over $\mathbf{R}$ one gets

$$\|Vu\|_2^2 \leq a\left(\int\limits_{\mathbf{R}} \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})}dt\right)\|\Box u\|_2^2 + b\left(\int\limits_{\mathbf{R}} \|V(.,.,t)\|^2_{L^s(\mathbf{R^2})}dt\right)\|u\|_2^2.$$

Since we can make a small enough to have $a\int\limits_{\mathbf{R}} \|V(.,.,t)\|^2_{L^\infty(\mathbf{R^2})}dt < 1$ we conclude by the Kato-Rellich perturbation theorem that $\Box + V$ is self-adjoint on $\mathcal{D}(\Box)$. ☐

5. OPEN PROBLEMS

This paper contains results from my Ph.D. thesis. I have a series of questions that I would like to know the answers to:

1) If V is real-valued and $V \in L^2(\mathbf{R^n})$, then will $\Box + V$ be self-adjoint?

2) If $V \geq 0$ is real-valued and $V \in L^2_{loc}(\mathbf{R^n})$, then will $\Box + V$ be self-adjoint? (We insist on V being positive since if it has no sign.) Then there are examples that show that this may fail, and they exploit the non-self-adjointness of the one-dimensional Hamiltonian for some classes of potentials [3].

3) In Section 4, can we have larger spaces than the ones found in Section 2 and 4 and which include M^n, $n \geq 3$?

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REFERENCES

1. E. H. Lieb, M. Loss, *Analysis*, Vol. 14, Graduate Studies in Mathematics, American Mathematical Society, 2001 (2nd edition). MR 2001i:00001
2. M. Reed, B. Simon, *Methods of Modern Mathematical Physics,* Vol.1, Functional Analysis, Academic Press, 1972. MR 58:12429a
3. M. Reed, B. Simon, *Methods of Modern Mathematical Physics,* Vol.2, Fourier Analysis, Self-adjointness, Academic Press, 1975. MR 58:12429b
4. J. Duoandikoetxea, *Fourier Analysis,* Graduate Studies in Mathematics Vol. 29, American Mathematical Society, 2001. MR 2001k:42001

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, JCMB, MAYFIELD ROAD, EDINBURGH, EH9 3JZ, UNITED KINGDOM

E-mail address: `mortad@maths.ed.ac.uk`

E-mail address: `hichem1978@yahoo.fr`

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