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TIETZE EXTENSION THEOREM FOR HILBERT C^*-MODULES

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Abstract. We prove the following generalization of the noncommutative Tietze extension theorem: if V is a countably generated Hilbert C^*-module over a σ-unital C^*-algebra, then the canonical extension $\overline{\Phi}$ of a surjective morphism $\Phi : V \to W$ of Hilbert C^*-modules to extended (multiplier) modules, $\overline{\Phi} : V_d \to W_d$, is also surjective.

1. Introduction and statement of the results

Given a surjective morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$, there exists a morphism $\overline{\varphi} : M(\mathcal{A}) \to M(\mathcal{B})$, an extension of φ to the corresponding multiplier algebras. There is a purely algebraic construction of $\overline{\varphi}$ by means of double centralizers (cf. [7], Proposition 2.2.16). Another way to display $\overline{\varphi}$ is to regard $\mathcal{A}$ and $\mathcal{B}$ as Hilbert C^*-modules then $M(\mathcal{A})$ and $M(\mathcal{B})$ are interpreted as the C^*-algebras of adjointable operators on $\mathcal{A}$ and $\mathcal{B}$, respectively ([5]; see also the proof of Proposition 3 below). It turns out that $\overline{\varphi}$ is the extension of φ by strict continuity, hence $\overline{\varphi}$ is the unique (canonical) extension of φ to multiplier algebras.

The noncommutative Tietze theorem, first proved by G. Pedersen, asserts that $\overline{\varphi}$ is also a surjection whenever $\mathcal{A}$ is a σ-unital C^*-algebra; see, for instance, [5], Proposition 6.8. As noted in [5], the assertion fails, even in the commutative case, if one drops the requirement that $\mathcal{A}$ should be σ-unital. Observe that in the commutative case $\mathcal{A} = C_0(X)$, $\mathcal{B} = C_0(Y)$ with Y closed in X, we have $M(\mathcal{A}) = C_b(X)$ and $M(\mathcal{B}) = C_b(Y)$. Now the fact that any function in $M(\mathcal{B})_{\mathrm{sa}} = C_b(Y, \boldsymbol{R})$ lifts to a function in $M(\mathcal{A})_{\mathrm{sa}} = C_b(X, \boldsymbol{R})$ explains the name of the theorem.

The aim of this note is to generalize the noncommutative Tietze extension theorem to Hilbert C^*-modules. However, we note: although our Theorem 2 generalizes the noncommutative Tietze theorem, our proof uses that result. The proof of Theorem 2 also relies on our previous article [2] where the Hilbert C^*-module version of the multiplier algebra is constructed.

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module V equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ such that V is a Banach space with the norm $\|v\| = \|\langle v, v\rangle\|^{1/2}$. We assume that the linear structures on $\mathcal{A}$ and V are compatible in the sense that $(\lambda v)a = \lambda(va) = v(\lambda a)$ for all $v \in V$, $a \in \mathcal{A}$, $\lambda \in \boldsymbol{C}$. V is said to be a full Hilbert $\mathcal{A}$-module when the ideal $\langle V, V\rangle$ (= the closed linear span of all elements in $\mathcal{A}$ of the form $\langle x, y\rangle$, $x, y \in V$) coincides with $\mathcal{A}$.

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If V and W are Hilbert $\mathcal{A}$-modules, we denote by $\boldsymbol{B}(V, W)$ the Banach space of all adjointable operators from V to W. The ideal of "compact" operators (generated by all operators of the form $\theta_{x,y}$, $\theta_{x,y}(v) = x\langle y, v\rangle$) from V to W is denoted by $\boldsymbol{K}(V, W)$. When $V = W$ we write $\boldsymbol{B}(V)$ and $\boldsymbol{K}(V)$ instead of $\boldsymbol{B}(V, V)$ and $\boldsymbol{K}(V, V)$, respectively. If a C^*-algebra $\mathcal{A}$ is regarded as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = a^*b$, then $\mathcal{A} \simeq \boldsymbol{K}(\mathcal{A})$ and $M(\mathcal{A}) \simeq \boldsymbol{B}(\mathcal{A})$. The corresponding identifications $a \leftrightarrow T_a$ and $m \leftrightarrow T_m$, with T_a and T_m denoting the left translations by $a \in \mathcal{A}$ resp. $m \in M(\mathcal{A})$, will be used freely.

For general facts about Hilbert C^*-modules the reader is referred to [5] and [7].

Given two Hilbert C^*-modules V and W over $\mathcal{A}$ and $\mathcal{B}$, respectively, we say that a map $\Phi : V \to W$ is a morphism of Hilbert C^*-modules if there exists a morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ is satisfied for all x and y in V. When the underlying morphism φ has to be emphasized, the map Φ will be called a φ-morphism. It is known that each φ-morphism is necessarily a linear contraction satisfying $\Phi(xa) = \Phi(x)\varphi(a)$, $x \in V$, $a \in \mathcal{A}$, such that $\operatorname{Ker}\Phi = V_{\operatorname{Ker}\varphi}$, i.e. the kernel of Φ is the ideal submodule of V associated to the kernel of φ. In particular, if V is full, then Φ is an injection if and only if φ is an injection.

As a typical example, consider an ideal $\mathcal{I}$ of $\mathcal{A}$ and take the corresponding ideal submodule $V_{\mathcal{I}}$ of a Hilbert $\mathcal{A}$-module V. Then the quotient map $q : V \to V/V_{\mathcal{I}}$ is a π-morphism of Hilbert C^*-modules (with $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denoting the quotient map of the underlying C^*-algebra).

A surjective φ-morphism $\Phi : V \to W$ is a unitary operator of Hilbert C^*-modules if the underlying morphism φ is an injection. If this is the case we say that V and W are unitarily equivalent Hilbert C^*-modules. Unitary equivalence of full Hilbert C^*-modules is an equivalence relation.

For these and other facts concerning morphisms of Hilbert C^*-modules we refer the reader to [1].

In the present note we shall be concerned with extensions of morphisms of Hilbert C^*-modules. But first we need to describe the concept of the extended (multiplier) module.

Let V be a full Hilbert C^*-module over a (non-unital) C^*-algebra $\mathcal{A}$. Denote by $V_d = \boldsymbol{B}(\mathcal{A}, V)$ the Hilbert C^*-module over the multiplier algebra $M(\mathcal{A})$ consisting of all adjointable maps from $\mathcal{A}$ to V with the inner product $\langle r, s\rangle = r^*s$. Let $\Gamma : V \to V_d$ be defined by $\Gamma(v) = r_v$ where $r_v : \mathcal{A} \to V$ denotes the "multiplier" $r_v(a) = va$. Then $(V_d, M(\mathcal{A}), \Gamma)$ is an extension of V in the following sense: if we identify v in V with $\Gamma(v)$ in V_d, then V is an ideal submodule of V_d corresponding to the ideal $\mathcal{A}$ of $M(\mathcal{A})$. The extended module V_d has the following universal property: Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an injective morphism of C^*-algebras such that $\varphi(\mathcal{A})$ is an (essential) ideal in a C^*-algebra $\mathcal{B}$, and let $\lambda : \mathcal{B} \to M(\mathcal{A})$ be the resulting morphism such that $\lambda\varphi$ is the identity on $\mathcal{A}$. Let W be a Hilbert $\mathcal{B}$-module. Suppose that $\Phi : V \to W$ is a φ-morphism of Hilbert C^*-modules with $\Phi(V) = V\varphi(\mathcal{A})$, so that $(W, \mathcal{B}, \Phi)$ is another (essential) extension of V. Then there exists a λ-morphism $\Lambda : W \to V_d$ such that $\Lambda\Phi = \Gamma$. Since the maps λ and Λ are injections precisely when $\varphi(\mathcal{A})$ is an essential ideal in $\mathcal{B}$, this shows that V_d is the largest essential extension of V ([2], Theorem 1.1).

Furthermore, given an essential extension $(W, \mathcal{B}, \Phi)$ of V, we define a (variant of) strict topology τ_V on W by the family of seminorms $w \mapsto \|\langle \Phi(v), w\rangle\|$, $v \in V$, and $w \mapsto \|w\varphi(a)\|$, $a \in \mathcal{A}$. Note in passing that in the case $V = \mathcal{A}$ the extended module V_d is nothing else than the multiplier algebra $M(\mathcal{A})$ and τ_V coincides with the usual (C^*) strict topology on $M(\mathcal{A})$.

It is proved in [2] that V is strictly dense in V_d and that V_d is complete with respect to the strict topology τ_V. Moreover, each essential extension of V complete with respect to τ_V is unitarily equivalent to V_d.

Finally, as observed in [2], V_d coincides (up to the identification $v \leftrightarrow r_v$) with V whenever either $\mathcal{A}$ or $\boldsymbol{K}(V)$ is a unital C^*-algebra.

All of this shows that V_d can be regarded as the Hilbert C^*-module version of the multiplier algebra.

Suppose now that we are given a surjective φ-morphism $\Phi : V \to W$ of full Hilbert modules. Regarding V_d and W_d as their "multiplier" modules one may ask: is there any morphism of Hilbert C^*-modules $\overline{\Phi} : V_d \to W_d$ extending Φ in the sense $\overline{\Phi}(r_v) = r_{\Phi(v)}$, $\forall v \in V$?

Observe that this condition determines uniquely the underlying morphism of C^*-algebras $\psi : M(\mathcal{A}) \to M(\mathcal{B})$. Indeed, $\langle r_v, r_v\rangle = \langle v, v\rangle$, $\forall v \in V$; hence $\langle \overline{\Phi}(r), \overline{\Phi}(r)\rangle = \psi(\langle r, r\rangle)$, $\forall r \in V_d$ implies $\psi(\langle v, v\rangle) = \psi(\langle r_v, r_v\rangle) = \langle \overline{\Phi}(r_v), \overline{\Phi}(r_v)\rangle = \langle \Phi(v), \Phi(v)\rangle = \varphi(\langle v, v\rangle)$, $\forall v \in V$; thus ψ must coincide with the canonical extension $\overline{\varphi}$ of φ.

Secondly, one would like to know when the extension $\overline{\Phi}$ of Φ is also a surjection.

Before stating main results of the paper we note that a related discussion can be found in [4]. There multiplier modules of Hilbert C^*-bimodules are defined in a purely algebraic manner and a variant of the Tietze extension theorem for homomorphisms of Hilbert C^*-bimodules is obtained.

We end this introductory section by stating two main results of this note: the first one ensures the existence of the canonical extension $\overline{\Phi}$ of a surjective morphism Φ, while the other serves as the Hilbert C^*-module version of the noncommutative Tietze extension theorem.

Proposition 1. *Let $\Phi : V \to W$ be a surjective morphism of full Hilbert C^*-modules. Then there exists a unique morphism of Hilbert C^*-modules $\overline{\Phi} : V_d \to W_d$ extending Φ in the sense that $\overline{\Phi}(r_v) = r_{\Phi(v)}$, $\forall v \in V$. The map $\overline{\Phi}$ is the continuation of Φ with respect to the pair of strict topologies on V_d and on W_d, hence uniquely determined.*

Theorem 2. *Let V be a full countably generated Hilbert C^*-module over a σ-unital C^*-algebra, and let W be another full Hilbert C^*-module. Then the canonical extension $\overline{\Phi} : V_d \to W_d$ of a surjective morphism $\Phi : V \to W$ is also a surjection.*

Observe that the above theorem covers the case of C^*-algebras (simply by taking $V = \mathcal{A}$, $W = \mathcal{B}$ and $\Phi = \varphi$), so one cannot conclude that an extension $\overline{\Phi}$ is a surjection without some additional assumptions. On the other hand, by [5], Proposition 6.7 a full Hilbert C^*-module V is countably generated if and only if $\boldsymbol{K}(V)$ is a σ-unital C^*-algebra. Hence in the basic case $V = \mathcal{A}$ the first hypothesis in Theorem 2, namely that V should be countably generated, is ensured (via the fact $\boldsymbol{K}(\mathcal{A}) = \mathcal{A}$) by the assumption that $\mathcal{A}$ is σ-unital.

2. PROOFS

Since the proof of Proposition 1 effectively requires work in the quotient $V/\operatorname{Ker}\Phi$, we first recall from [1] some facts concerned with quotients of Hilbert C^*-modules. Let V be a full $\mathcal{A}$-module, and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. The associated ideal submodule $V_{\mathcal{I}}$ is defined as the closed linear span of the set $V\mathcal{I} = \{va : v \in V,\ a \in \mathcal{I}\}$.

After applying the Hewitt-Cohen factorization ([6], Proposition 2.31) one finds

$$V_{\mathcal{I}} = V\mathcal{I} = \{v \in V : \langle v, x\rangle \in \mathcal{I},\ \forall x \in V\}. \tag{1}$$

Now consider the quotient Hilbert $\mathcal{A}/\mathcal{I}$-module $V/V_{\mathcal{I}}$ with $q : V \to V/V_{\mathcal{I}}$ denoting the quotient map, and take an arbitrary adjointable operator $T \in \boldsymbol{B}(V)$. Since $V_{\mathcal{I}}$ is obviously invariant for T (by (1)), there is a well-defined operator $\widehat{T}$ on $V/V_{\mathcal{I}}$ given by $\widehat{T}(q(v)) = q(Tv)$. Note also $\widehat{\theta_{x,y}} = \theta_{q(x),q(y)}$, $\forall x, y \in V$. It is proved in Corollary 1.18 from [1] that the map $\beta : \boldsymbol{B}(V) \to \boldsymbol{B}(V/V_{\mathcal{I}})$, $\beta(T) = \widehat{T}$ is a morphism of C^*-algebras such that $\beta(\boldsymbol{K}(V)) = \boldsymbol{K}(V/V_{\mathcal{I}})$. In particular, if V is countably generated, then $\boldsymbol{K}(V)$ is a σ-unital C^*-algebra ([5], Proposition 6.7), and by the noncommutative Tietze extension theorem, β is a surjection.

In the basic case $V = \mathcal{A}$ the corresponding map will be denoted by β_0. Observe that $\beta_0(T_a) = T_{\pi(a)}$ for $a \in \mathcal{A}$, so that β_0 restricted to "compact" operators coincides with the quotient map π (up to the identification $a \leftrightarrow T_a$). Consequently, applying β_0 to multipliers T_m one gets the canonical extension of the quotient map π.

Proposition 3. *Let V be a full $\mathcal{A}$-module, and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. The map $\tau : V_d \to (V/V_{\mathcal{I}})_d$ defined by $\tau(r) = \widehat{r}$, $\widehat{r}(\pi(a)) = q(r(a))$ is a β_0-morphism of Hilbert C^*-modules satisfying $\tau(r_x) = r_{q(x)}$, $\forall x \in V$. The map τ is continuous with respect to the pair of strict topologies on V_d and on $(V/V_{\mathcal{I}})_d$.*

Proof. First, $\widehat{r}$ is well defined: $\pi(a) = \pi(b) \Rightarrow a - b \in \mathcal{I} \Rightarrow \langle r(a - b), x\rangle = \langle a - b, r^*(x)\rangle = (a - b)^* r^*(x) \in \mathcal{I}$, $\forall x \in V$. Using (1) we now conclude that $r(a - b) \in V_{\mathcal{I}}$, hence $q(r(a)) = q(r(b))$.

Secondly, τ is well defined, i.e. $\widehat{r}$ is an adjointable map. Indeed, for an adjointable map $l : V \to \mathcal{A}$ we define $\widehat{l}: V/V_{\mathcal{I}} \to \mathcal{A}/\mathcal{I}$ by $\widehat{l}(q(v)) = \pi(l(v))$. By a straightforward calculation (which we omit) one verifies that $\widehat{l}$ is also an adjointable map and $(\widehat{r})^* = (r^*)^\wedge$.

We next show that τ is a β_0-morphism of Hilbert C^*-modules; by definition we have to check that $\langle \tau(r), \tau(s)\rangle = \beta_0(\langle r, s\rangle)$. Again, this is a routine verification: $\langle \tau(r), \tau(s)\rangle(\pi(a)) = \tau(r)^*\tau(s)(\pi(a)) = \tau(r)^*(q(s(a))) = \pi(r^*(s(a))) = \pi((r^*s)(a))$ $= \pi(\langle r, s\rangle(a)) = \beta_0(\langle r, s\rangle)(\pi(a))$.

The equality $\tau(r_x) = r_{q(x)}$, $x \in V$ is evident, and the proof will be finished by showing that τ is strictly continuous. Assume $r = (\mathrm{st.}) \lim_j r_j$ in V_d. This means

$$\langle r, r_x\rangle = \lim_j \langle r_j, r_x\rangle,\ \forall x \in V \tag{2}$$

and

$$r T_a = \lim_j r_j T_a,\ \forall a \in \mathcal{A}. \tag{3}$$

We must obtain analogous relations in order to prove $\tau(r) = (\mathrm{st.}) \lim_j \tau(r_j)$. This is immediate from the properties of the map τ: $\lim_j \langle \tau(r_j), r_{q(x)}\rangle = \lim_j \langle \tau(r_j), \tau(r_x)\rangle$ $= \lim_j \beta_0(\langle r_j, r_x\rangle) =$ (by (2)) $\beta_0(\langle r, r_x\rangle) = \langle \tau(r), \tau(r_x)\rangle = \langle \tau(r), r_{q(x)}\rangle$, $\forall x \in V$. For the proof of the remaining part observe that β_0 is surjective on "compact" operators and τ is β_0-linear and norm-continuous: $\lim_j \tau(r_j)\beta_0(T_a) = \lim_j \tau(r_j T_a) =$ (by (3)) $\tau(rT_a) = \tau(r)\beta_0(T_a)$, $\forall \beta_0(T_a) \in \boldsymbol{K}(\mathcal{A}/\mathcal{I})$. ☐

Remark 4. Note that the last part of the preceding proof shows that any morphism $\tau' : V_d \to (V/V_{\mathcal{I}})_d$ satisfying $\tau'(r_x) = r_{q(x)}$, $\forall x \in V$ is strictly continuous. Since V_d and $(V/V_{\mathcal{I}})_d$ are the strict completions of the sets $\{r_x : x \in V\}$, resp. $\{r_q(x) : x \in V\}$ ([2], Theorem 1.5), we conclude that τ is uniquely determined with the property $\tau(r_x) = r_{q(x)}$, $\forall x \in V$.

Proof of Proposition 1. Suppose that V and W are full Hilbert C^*-modules over $\mathcal{A}$ and $\mathcal{B}$, respectively, and that $\Phi : V \to W$ is a surjective morphism. Note that, since V and W are full, the underlying morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ is also a surjection. Passing through the corresponding quotients we conclude: $V_{\mathrm{Ker}\varphi}$ and W are unitarily equivalent Hilbert C^*-modules; the map $q(v) \mapsto \Phi(v)$ induced by Φ serves as a unitary operator (with the underlying isomorphism of C^*-algebras given by $\pi(a) \mapsto \varphi(a)$).

It is obvious that for unitarily equivalent Hilbert C^*-modules W and W' the extended modules W_d and W'_d are also unitarily equivalent. Combining this with the preceding proposition we obtain Proposition 1. Observe additionally that $\overline{\Phi}$ is given by the formula $\overline{\Phi}(r)(\varphi(a)) = \Phi(r(a))$, $r \in V_d$. □

Remark 5. Since unitarily equivalent Hilbert C^*-modules have isomorphic C^*-algebras of "compact" and adjointable operators, we may also apply the discussion before Proposition 3 to conclude: there is a morphism of C^*-algebras $\Phi^+ : \boldsymbol{B}(V) \to \boldsymbol{B}(W)$ such that $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$, $\forall x, y \in V$ and $\Phi^+(\boldsymbol{K}(V)) = \boldsymbol{K}(W)$. The map Φ^+ is a surjection when V is countably generated.

Note that Φ is supported by Φ^+ in the same way as by the underlying map φ. Namely, if we consider the left Hilbert C^*-module structure on V over $\boldsymbol{K}(V)$ with the inner product $[x, y] = \theta_{x,y}$ then, by the above description of the action of Φ^+ on elementary operators, we have $[\Phi(x), \Phi(y)] = \Phi^+([x, y])$, $\forall x, y \in V$. As an immediate consequence we also note that $\Phi(Tx) = \Phi^+(T)\Phi(x)$, $\forall T \in \boldsymbol{K}(V), \forall x \in V$.

We now turn to the proof of Theorem 2. Our approach consists of passing to the linking algebras and making use of the noncommutative Tietze theorem for C^*-algebras.

Let V be a full Hilbert $\mathcal{A}$-module. Recall from [3] (see also Lemma 2.32 and Corollary 3.21 in [6]) that the linking algebra $L(V)$ of V can be described in terms of a Hilbert $\mathcal{A}$-module $\mathcal{A} \oplus V$:

$$L(V) = \boldsymbol{K}(\mathcal{A} \oplus V) = \begin{bmatrix} \boldsymbol{K}(\mathcal{A}) & \boldsymbol{K}(V, \mathcal{A}) \\ \boldsymbol{K}(\mathcal{A}, V) & \boldsymbol{K}(V) \end{bmatrix}. \tag{4}$$

Since each operator in $\boldsymbol{K}(\mathcal{A}, V)$ is of the form r_x for some $x \in V$, we have

$$L(V) = \{ \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in V, T \in \boldsymbol{K}(V)\}. \tag{5}$$

Suppose that $\Phi : V \to W$ is a surjective φ-morphism of Hilbert C^*-modules. Then there is a map $\rho = \rho_\Phi$ of the corresponding linking algebras:

$$\rho_\Phi : L(V) \to L(W),\ \rho_\Phi(\begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix}) = \begin{bmatrix} T_{\varphi(a)} & r_{\Phi(y)}^* \\ r_{\Phi(x)} & \Phi^+(T) \end{bmatrix}. \tag{6}$$

It is proved in Theorem 2.15 in [1] that ρ_Φ is a morphism of C^*-algebras. In fact, it turns out that ρ_Φ is the restriction of the map $(\varphi \oplus \Phi)^+$ to "compact" operators, where $\varphi \oplus \Phi : \mathcal{A} \oplus V \to \mathcal{B} \oplus W$ is a φ-morphism of Hilbert C^*-modules obtained by applying φ and Φ componentwise.

Proof of Theorem 2. Let $\Phi : V \to W$ be a surjective φ-morphism of full Hilbert C^*-modules with $\varphi : \mathcal{A} \to \mathcal{B}$, and let V be countably generated and $\mathcal{A}$ σ-unital. Since V and W are full, φ is also a surjection. The map Φ^+ is surjective by Remark 5, and now the preceding paragraph shows that $\rho_\Phi : \boldsymbol{K}(\mathcal{A} \oplus V) \to \boldsymbol{K}(\mathcal{B} \oplus W)$ is also a surjective map. Denote by

$$\overline{\rho_\Phi} : \boldsymbol{B}(\mathcal{A} \oplus V) \to \boldsymbol{B}(\mathcal{B} \oplus W) \tag{7}$$

the canonical extension of ρ_Φ. Since V is by assumption a countably generated Hilbert $\mathcal{A}$-module and $\mathcal{A}$ is a σ-unital C^*-algebra, $\mathcal{A} \oplus V$ is also countably generated. By [5], Proposition 6.7, $\boldsymbol{K}(\mathcal{A} \oplus V)$ is then a σ-unital C^*-algebra. Thus, by the noncummatative Tietze extension theorem, the map $\overline{\rho_\Phi}$ is also a surjection. Now observe that (7) can be rewritten as

$$\overline{\rho_\Phi} : \begin{bmatrix} \boldsymbol{B}(\mathcal{A}) & \boldsymbol{B}(V, \mathcal{A}) \\ \boldsymbol{B}(\mathcal{A}, V) & \boldsymbol{B}(V) \end{bmatrix} \to \begin{bmatrix} \boldsymbol{B}(\mathcal{B}) & \boldsymbol{B}(W, \mathcal{B}) \\ \boldsymbol{B}(\mathcal{B}, W) & \boldsymbol{B}(W) \end{bmatrix}, \tag{8}$$

or, recognizing $\boldsymbol{B}(\mathcal{A}, V) = V_d$ and $\boldsymbol{B}(\mathcal{B}, W) = W_d$,

$$\overline{\rho_\Phi} : \begin{bmatrix} \boldsymbol{B}(\mathcal{A}) & (V_d)^* \\ V_d & \boldsymbol{B}(V) \end{bmatrix} \to \begin{bmatrix} \boldsymbol{B}(\mathcal{B}) & (W_d)^* \\ W_d & \boldsymbol{B}(W) \end{bmatrix}. \tag{9}$$

The remaining part of the proof consists of showing that the extended morphism $\overline{\rho_\Phi}$ preserves matricial corners; this will imply that $\overline{\rho_\Phi}$ is in fact equal to the map $\rho_{\overline{\Phi}}$. The argument below is similar to the proof of Theorem 2.15 in [1] and uses strict continuity of $\overline{\rho_\Phi}$ and the fact that the original map ρ_Φ preserves corners.

Take an arbitrary multiplier $m \in M(\mathcal{A})$ interpreted as the left translation $T_m \in \boldsymbol{B}(\mathcal{A})$. Let (a_n) be an approximate unit for $\mathcal{A}$. We first claim that

$$\begin{bmatrix} T_m & 0 \\ 0 & 0 \end{bmatrix} = (\mathrm{st.}) \lim_n \begin{bmatrix} T_{ma_n} & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Indeed, taking arbitrary $\begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} \in \boldsymbol{K}(\mathcal{A} \oplus V)$, we must show that

$$\begin{aligned} \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} \begin{bmatrix} T_{ma_n} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} T_{ama_n} & 0 \\ r_x T_{ma_n} & 0 \end{bmatrix} \to \begin{bmatrix} T_{am} & 0 \\ r_x T_m & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} \begin{bmatrix} T_m & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} T_{ma_n} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix} &= \begin{bmatrix} T_{ma_n a} & T_{ma_n} r_y^* \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} T_{ma} & T_m r_y^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_a & r_y^* \\ r_x & T \end{bmatrix}. \end{aligned}$$

To verify the above two limit relations we need only to check that $r_x T_{ma_n} \to r_x T_m$ and $T_{ma_n} r_y^* \to T_m r_y^*$. But, this follows easily from the fact that (T_{a_n}) is an approximate unit:

$$\begin{aligned}\|r_xT_{ma_n} - r_xT_m\|^2 &= \|r_x(T_mT_{a_n} - T_m)\|^2 = \|(T_mT_{a_n} - T_m)^*r_x^*r_x(T_mT_{a_n} - T_m)\| \\ &= \|(T_mT_{a_n} - T_m)^*T_{\langle x,x\rangle}(T_mT_{a_n} - T_m)\| \\ &= \|T^{1/2}_{\langle x,x\rangle}(T_mT_{a_n} - T_m)\|^2 \to 0.\end{aligned}$$

Similarly, $\|T_mT_{a_n}r_y^* - T_mr_y^*\|^2 = \|(T_mT_{a_n} - T_m)r_y^*r_y(T_mT_{a_n} - T_m)^*\| \to 0$.

Analogously, for arbitrary $S \in \boldsymbol{B}(V)$ and an approximate unit (T_n) for $\boldsymbol{K}(V)$, we claim

$$\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix} = (\text{st.})\lim_n \begin{bmatrix} 0 & 0 \\ 0 & ST_n \end{bmatrix}. \tag{11}$$

To obtain the last equality one has only to repeat the above verification (with the final limit relations concerned with the approximate unit in $\boldsymbol{K}(V)$ and the operators $\theta_{x,x}$ and $\theta_{y,y}$). We omit the details.

Now we apply $\overline{\rho_\Phi}$ to (10) and (11). Since $\overline{\rho_\Phi}$ is strictly continuous and extends ρ_Φ, and since the canonical extensions $\overline{\varphi}$ and $\overline{\Phi^+}$ are strictly continuous, we obtain from (10),

$$\overline{\rho_\Phi}(\begin{bmatrix} T_m & 0 \\ 0 & 0 \end{bmatrix}) = (\text{st.})\lim_n \rho_\Phi(\begin{bmatrix} T_{ma_n} & 0 \\ 0 & 0 \end{bmatrix}) = (\text{st.})\lim_n \begin{bmatrix} T_{\varphi(ma_n)} & 0 \\ 0 & 0 \end{bmatrix};$$

hence

$$\overline{\rho_\Phi}(\begin{bmatrix} T_m & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} T_{\overline{\varphi}(m)} & 0 \\ 0 & 0 \end{bmatrix}, \ \forall T_m \in \boldsymbol{B}(\mathcal{A}). \tag{12}$$

In the same way, starting from (11), we get

$$\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ 0 & S \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & \overline{\Phi^+}(S) \end{bmatrix}, \ \forall S \in \boldsymbol{B}(V). \tag{13}$$

Now take arbitrary $r \in \boldsymbol{B}(\mathcal{A}, V) = V_d$ and put

$$\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}) = \begin{bmatrix} \rho_{11}(r) & \rho_{12}(r) \\ \rho_{21}(r) & \rho_{22}(r) \end{bmatrix}. \tag{14}$$

Then

$$\begin{aligned}\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}^*)\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}) &= \begin{bmatrix} \rho_{11}(r)^* & \rho_{21}(r)^* \\ \rho_{12}(r)^* & \rho_{22}(r)^* \end{bmatrix} \begin{bmatrix} \rho_{11}(r) & \rho_{12}(r) \\ \rho_{21}(r) & \rho_{22}(r) \end{bmatrix} \\ \times &\begin{bmatrix} \rho_{11}(r)^*\rho_{11}(r) + \rho_{21}(r)^*\rho_{21}(r) & \rho_{11}(r)^*\rho_{12}(r) + \rho_{21}(r)^*\rho_{22}(r) \\ \rho_{12}(r)^*\rho_{11}(r) + \rho_{22}(r)^*\rho_{21}(r) & \rho_{12}(r)^*\rho_{12}(r) + \rho_{22}(r)^*\rho_{22}(r) \end{bmatrix},\end{aligned} \tag{15}$$

and also

$$\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}^*)\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}) = \overline{\rho_\Phi}(\begin{bmatrix} r^*r & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} \overline{\varphi}(r^*r) & 0 \\ 0 & 0 \end{bmatrix}. \tag{16}$$

Comparing (15) and (16) we conclude that $\rho_{12}(r) = 0$ and $\rho_{22}(r) = 0$. Similarly, by calculating $\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix})\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}^*)$ one finds $\rho_{11}(r) = 0$ and, after all,

$$\overline{\rho_\Phi}(\begin{bmatrix} 0 & 0 \\ r & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ \Psi(r) & 0 \end{bmatrix} \tag{17}$$

where $\Psi : V_d \to W_d$ is a well-defined map of Hilbert C^*-modules. Since $\overline{\rho_\Phi}$ extends ρ_Φ, the map Ψ is an extension of Φ. Additionally, (16) shows that Ψ is a $\overline{\varphi}$-morphism of Hilbert C^*-modules. Since by Proposition 1 $\overline{\Phi} : V_d \to W_d$ is the only morphism with these properties, we finally have $\Psi = \overline{\Phi}$.

This together with (12) and (13) gives a general formula for $\overline{\rho_\Phi}$:

$$\overline{\rho_\Phi}(\begin{bmatrix} T_m & s^* \\ r & S \end{bmatrix}) = \begin{bmatrix} T_{\overline{\varphi}(m)} & (\overline{\Phi}(s))^* \\ \overline{\Phi}(r) & \overline{\Phi^+}(S) \end{bmatrix},$$

from which the surjectivity of $\overline{\Phi}$ is evident. □

We end the paper with a result concerning strictly complete Hilbert C^*-modules. By definition, a Hilbert $\mathcal{A}$-module V is said to be strictly complete if $V_d = V$, i.e. if each adjointable map $r : \mathcal{A} \to V$ is of the form $r = r_v$, $r_v(a) = va$, for some v in V. For the properties of strictly complete modules the reader is referred to [2].

Corollary 6. *Let V be a full strictly complete countably generated Hilbert C^*-module over a σ-unital C^*-algebra. Suppose that $\Phi : V \to W$ is a surjective morphism of full Hilbert C^*-modules. Then W is also strictly complete. In particular, the quotient $V/V_{\mathcal{I}}$ over each ideal submodule $V_{\mathcal{I}}$ of V is a strictly complete Hilbert C^*-module.*

Proof. By the above theorem, $\overline{\Phi} : V_d \to W_d$ is a surjection. Thus each $r \in W_d$ has a preimage in V_d. Since $V_d = V$, there exists $x \in V$ such that $\overline{\Phi}(r_x) = r$; in other words, $r_{\Phi(x)} = r$. □

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