

## JACOBI POLYNOMIALS FROM COMPATIBILITY CONDITIONS

YANG CHEN AND MOURAD ISMAIL

(Communicated by David R. Larson)

ABSTRACT. We revisit the ladder operators for orthogonal polynomials and re-interpret two supplementary conditions as compatibility conditions of two linear over-determined systems; one involves the variation of the polynomials with respect to the variable  $z$  (spectral parameter) and the other a recurrence relation in  $n$  (the lattice variable). For the Jacobi weight

$$w(x) = (1-x)^\alpha(1+x)^\beta, \quad x \in [-1, 1],$$

we show how to use the compatibility conditions to explicitly determine the recurrence coefficients of the monic Jacobi polynomials.

### 1. INTRODUCTION AND PRELIMINARIES

We begin with some notation. Let  $P_n(x)$  be monic polynomials of degree  $n$  in  $x$  and orthogonal, with respect to a weight,  $w(x)$ ,  $x \in [a, b]$ ;

$$(1.1) \quad \int_a^b P_m(x)P_n(x)w(x) dx = h_n\delta_{m,n}.$$

We further assume that  $v'(z) := -w'(z)/w(z)$  exists and that

$$y^n [v'(x) - v'(y)] w(y)/(x-y)$$

is integrable on  $[a, b]$  for all  $n$ ,  $n = 0, 1, \dots$ . From the orthogonality condition there follows the recurrence relation,

$$(1.2) \quad zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \quad n = 0, 1, \dots,$$

where  $\beta_0 P_{-1}(z) := 0$ ,  $\alpha_n$ ,  $n = 0, 1, 2, \dots$  is real and  $\beta_n > 0$ ,  $n = 1, 2, \dots$

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer to a future publication, the analysis in the case of the generalized Jacobi weights [8, 11]. In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at  $x = \pm 1$ . We also show that the evaluation of a Jacobi polynomial at  $x = 1$  or  $x = -1$  leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants  $h_n$  are also found.

---

Received by the editors February 21, 2003 and, in revised form, October 2, 2003.

2000 *Mathematics Subject Classification*. Primary 33C45; Secondary 42C05.

This research was supported by NSF grant DMS 99-70865 and by EPSRC grant GR/S14108.

The actions of the ladder operators on  $P_n(z)$  and  $P_{n-1}(z)$  are

$$(1.3) \quad \left( \frac{d}{dz} + B_n(z) \right) P_n(z) = \beta_n A_n(z) P_{n-1}(z),$$

$$(1.4) \quad \left( \frac{d}{dz} - B_n(z) - v'(z) \right) P_{n-1}(z) = -A_{n-1}(z) P_n(z),$$

with

$$(1.5) \quad A_n(z) := \frac{w(y) P_n^2(y)}{h_n(y-z)} \Big|_{y=a}^{y=b} + \frac{1}{h_n} \int_a^b \frac{v'(z) - v'(y)}{z-y} P_n^2(y) w(y) dy,$$

$$(1.6) \quad B_n(z) := \frac{w(y) P_n(y) P_{n-1}(y)}{h_{n-1}(y-z)} \Big|_{y=a}^{y=b} + \frac{1}{h_{n-1}} \int_a^b \frac{v'(z) - v'(y)}{z-y} P_{n-1}(y) P_n(y) w(y) dy,$$

where we have used the supplementary condition,

$$(S_1) \quad B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - v'(z),$$

to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition  $(S_1)$  were derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial  $v$ , and the authors [7] for general  $v$ . Ismail and Wimp [9] identified the additional supplementary condition,

$$(S_2) \quad B_{n+1}(z) - B_n(z) = \frac{\beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z) - 1}{z - \alpha_n}.$$

Our thesis in this work is that the supplementary conditions,  $(S_1)$  and  $(S_2)$ , being identities in  $n$  ( $n > 0$ ) and  $z \in \mathbf{C} \cup \{\infty\}$  have the information needed to determine the recurrence coefficients and other auxiliary quantities. We illustrate this by systematically using  $(S_1)$  and  $(S_2)$  to determine most of the properties of the Jacobi polynomials. See [14], [2], and [12], for detailed information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorials and hypergeometric functions in [2, 12].

Below, we reinterpret  $(S_2)$ . We set

$$(1.7) \quad \Phi_n(z) := \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix},$$

$$(1.8) \quad M_n(z) := \begin{pmatrix} -B_n(z) & \beta_n A_n(z) \\ -A_{n-1}(z) & B_n(z) + v'(z) \end{pmatrix},$$

$$(1.9) \quad U_n(z) := \begin{pmatrix} z - \alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix}.$$

Now equations (1.3) and (1.4) become,

$$(1.10) \quad \Phi_n'(z) = M_n(z) \Phi_n(z),$$

and the recurrence relations become

$$(1.11) \quad \Phi_{n+1}(z) = U_n(z) \Phi_n(z).$$

We find, by requiring (1.10) and (1.11) to be compatible,

$$\begin{aligned} \Phi_{n+1}'(z) &= M_{n+1}(z) \Phi_{n+1}(z) \\ &= M_{n+1}(z) U_n(z) \Phi_n(z). \end{aligned}$$

On the other hand,

$$\begin{aligned} \Phi'_{n+1}(z) &= U'_n(z)\Phi_n(z) + U_n(z)\Phi'_n(z) \\ &= U'_n(z)\Phi_n(z) + U_n(z)M_n(z)\Phi_n(z). \end{aligned}$$

We now write the above equations in matrix form as

$$(1.12) \quad S_n(z)\Phi_n(z) = 0,$$

where  $S_n(z)$  is the matrix

$$S_n(z) := U'_n(z) + U_n(z)M_n(z) - M_{n+1}(z)U_n(z),$$

whose entries  $S_{i,j}$  are

$$\begin{aligned} S_n^{11}(z) &= 1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) \\ &\quad + \beta_n A_{n-1}(z) - \beta_{n+1} A_{n+1}(z), \\ (1.13) \quad S_n^{12}(z) &= -\beta_n(B_{n+1}(z) + B_n(z) + v'(z) - (z - \alpha_n)A_n(z)), \\ S_n^{21}(z) &= S_n^{12}(z)/\beta_n, \\ S_n^{22}(z) &= 0. \end{aligned}$$

Here  $n = 1, 2, \dots$  and  $z \in \mathbf{C} \cup \{\infty\}$ . Observe that with  $(S_1)$ ,  $S_n^{12}(z) = S_n^{21}(z) = 0$ . This leaves  $S_n^{11}(z)P_n(z) = 0$ . Since  $P_n(z)$  does not vanish identically, we must have  $S_n^{11}(z) = 0$ , which is  $(S_2)$ . It is clear from (1.5) and (1.6) that, if  $v'(z)$  is a rational function, then  $A_n(z)$  and  $B_n(z)$  are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients,  $\alpha_n$  and  $\beta_n$ . In the next section, we illustrate the method by considering the Jacobi weight  $w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$  for  $x \in [-1, 1]$ .

Recall that the numerator polynomials [13], [1] are

$$(1.14) \quad Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z - y} w(y) dy,$$

and  $\{P_n(z)\}$  and  $\{Q_n(z)\}$  form a basis of solutions of the recurrence relation. We shall also use the notation

$$(1.15) \quad F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z - y} dy$$

for the Stieltjes transform of the weight function.

## 2. JACOBI WEIGHT

The Jacobi weight is  $w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$ ,  $x \in [-1, 1]$ , and for now we take  $\alpha$  and  $\beta$  to be strictly positive. It will become clear, using a real analyticity argument, the results that follow are also valid for  $\alpha, \beta > -1$ . Let  $\{\mathcal{P}_n^{(\alpha,\beta)}(x)\}$  and  $\{\mathcal{Q}_n^{(\alpha,\beta)}(x)\}$  denote the monic Jacobi polynomials, and their numerators, respectively; see (1.14). Moreover, in the present example, the Stieltjes transform of  $w^{\alpha,\beta}$  will be denoted by  $F^{(\alpha,\beta)}(z)$ .

From (1.5)–(1.6) we find

$$h_n A_n(z) = \frac{\alpha}{1-z} \int_{-1}^1 \left[ \mathcal{P}_n^{(\alpha, \beta)}(y) \right]^2 (1-y)^{\alpha-1} (1+y)^\beta dy \\ + \frac{\beta}{1+z} \int_{-1}^1 \left[ \mathcal{P}_n^{(\alpha, \beta)}(y) \right]^2 (1-y)^\alpha (1+y)^{\beta-1} dy.$$

Through integration by parts, it readily follows that

$$(2.1) \quad A_n(z) = -\frac{R_n}{z-1} + \frac{R_n}{z+1},$$

for some constant  $R_n$ . Similarly we find

$$(2.2) \quad B_n(z) = -\frac{n+r_n}{z-1} + \frac{r_n}{z+1}.$$

Here  $R_n$  and  $r_n$  are given by

$$(2.3) \quad R_n = R_n(\alpha, \beta) := \frac{\beta}{h_n} \int_{-1}^1 \frac{\left[ \mathcal{P}_n^{(\alpha, \beta)}(y) \right]^2}{1+y} w^{(\alpha, \beta)}(y) dy,$$

$$(2.4) \quad r_n = r_n(\alpha, \beta) := \frac{\beta}{h_{n-1}} \int_{-1}^1 \frac{\mathcal{P}_n^{(\alpha, \beta)}(y) \mathcal{P}_{n-1}^{(\alpha, \beta)}(y)}{1+y} w^{(\alpha, \beta)}(y) dy.$$

It is easy to see that

$$R_n(\alpha, \beta) = \frac{\beta}{h_n} \mathcal{P}_n^{(\alpha, \beta)}(-1) \left[ \mathcal{Q}_n^{(\alpha, \beta)}(-1) - F^{(\alpha, \beta)}(-1) \mathcal{P}_n^{(\alpha, \beta)}(-1) \right], \\ r_n(\alpha, \beta) = \frac{\beta}{h_{n-1}} \mathcal{P}_{n-1}^{(\alpha, \beta)}(-1) \left[ \mathcal{Q}_n^{(\alpha, \beta)}(-1) - F^{(\alpha, \beta)}(-1) \mathcal{P}_n^{(\alpha, \beta)}(-1) \right].$$

The reader may ask, “What is the point of this formalism, since in the attempt to find  $\alpha_n$  and  $\beta_n$ , two new unknown quantities,  $R_n$  and  $r_n$ , have been introduced?” However, when  $v'$  is a rational function, both sides of  $(S_1)$  and  $(S_2)$  are rational functions, and by equating coefficients and residues of both sides of  $(S_1)$  and  $(S_2)$ , we shall arrive at four equations, which should be sufficient for the determination of  $R_n$  and  $r_n$  as well as  $\alpha_n$  and  $\beta_n$ . Equating residues of both sides of  $(S_1)$ , at  $z = -1$  and  $z = +1$ , gives

$$(2.5) \quad -2n - 1 - r_n - r_{n+1} = \alpha - R_n(1 - \alpha_n),$$

$$(2.6) \quad r_n + r_{n+1} = \beta - R_n(1 + \alpha_n).$$

Similarly, from  $(S_2)$ , we obtain

$$(2.7) \quad (r_n - r_{n+1} - 1)(1 - \alpha_n) = \beta_n R_{-1} - \beta_{n+1} R_{n+1},$$

$$(2.8) \quad (r_n - r_{n+1})(1 + \alpha_n) = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}.$$

Observe that  $R_n$  can be obtained immediately by adding (2.5) and (2.6):

$$(2.9) \quad R_n = \frac{1}{2}(\alpha + \beta + 2n + 1).$$

The sum of (2.7) and (2.8) gives

$$(2.10) \quad 1 - \alpha_n = 2(r_n - r_{n+1}),$$

while (2.8) minus (2.7) and with (2.9) gives

$$(2.11) \quad \beta - \alpha - 2n - 1 - (\alpha + \beta + 2n + 1) \alpha_n = 2(r_n + r_{n+1}).$$

Now, (2.10) ± (2.11) implies

$$(2.12) \quad 4r_n = \beta - \alpha - 2n - (\alpha + \beta + 2n + 2) \alpha_n,$$

$$(2.13) \quad 4r_{n+1} = \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n) \alpha_n.$$

When (2.12) and (2.13) are made compatible, we obtain a first-order difference equation satisfied by  $\alpha_n$ :

$$(2.14) \quad \alpha_{n+1}(\alpha + \beta + 2n + 4) - \alpha_n(\alpha + \beta + 2n) = 0,$$

which has a very simple “integrating factor,”  $\alpha + \beta + 2n + 2$ . Using this, we find

$$\alpha_n = \frac{C_1}{(2R_n - 1)(2R_n + 1)},$$

where  $C_1$  is an “integration” constant, determined by the initial condition

$$\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \quad C_1 = \beta^2 - \alpha^2.$$

Here  $\mu_j := \int_{-1}^1 t^j w(t) dt$ ,  $j = 0, 1, \dots$  are the moments. Therefore we have established

$$(2.15) \quad \alpha_n = \frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}.$$

Going back to (2.8) and using (2.10), we see that  $\beta_n$  satisfies the linear difference equation:

$$(2.16) \quad \beta_{n+1}R_{n+1} - \beta_nR_{n-1} = \frac{1 - \alpha_n^2}{2},$$

which has the “integrating factor”  $R_n$ . Therefore,

$$(2.17) \quad \begin{aligned} \beta_n R_n R_{n-1} &= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} (1 - \alpha_j^2) R_j \\ &= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left( 1 - \frac{C_1^2}{(4R_j^2 - 1)^2} \right) R_j, \end{aligned}$$

where  $C_2$  is another integration constant to be determined by the initial condition

$$\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} - \left( \frac{\mu_1}{\mu_0} \right)^2 = \frac{4(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}.$$

After some computations,

$$C_2 = \beta_1 R_0 R_1 - \frac{1}{2} (1 - \alpha_0^2) R_0 = 0.$$

Now the sum (2.17) may look complicated; however, with a partial fraction expansion, the sum can be taken and leads to

$$\beta_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2 R_n R_{n-1}}.$$

Therefore, after some simplifications we establish

$$(2.18) \quad \beta_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}.$$

## 3. EXPLICIT FORMULAS

We first determine  $\mathcal{P}_n^{(\alpha,\beta)}(\pm 1)$ . Write (1.3) as

$$\frac{d}{dz} \mathcal{P}_n^{(\alpha,\beta)}(z) = \frac{(n+r_n)\mathcal{P}_n^{(\alpha,\beta)}(z) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z)}{z-1} + \frac{\beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(z) - r_n \mathcal{P}_n^{(\alpha,\beta)}(z)}{z+1},$$

and since  $\frac{d}{dz} \mathcal{P}_n^{(\alpha,\beta)}(z)$  is regular at  $z = \pm 1$ , we arrive at

$$\begin{aligned} (n+r_n)\mathcal{P}_n^{(\alpha,\beta)}(1) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(1) &= 0, \\ \beta_n R_n \mathcal{P}_{n-1}^{(\alpha,\beta)}(-1) - r_n \mathcal{P}_n^{(\alpha,\beta)}(-1) &= 0. \end{aligned}$$

Thus we find

$$(3.1) \quad \mathcal{P}_n^{(\alpha,\beta)}(1) = \mathcal{P}_0^{(\alpha,\beta)}(1) \prod_{j=1}^n \frac{\beta_j R_j}{r_j + j}$$

and

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \mathcal{P}_0^{(\alpha,\beta)}(-1) \prod_{j=1}^n \frac{\beta_j R_j}{r_j}.$$

Substituting for  $\beta_n$ ,  $r_n$  and  $R_n$  from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

$$\mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n}{2^n} \prod_{j=1}^n \frac{(j+\beta)(j+\alpha+\beta)}{[j+(\alpha+\beta/2)][j+(\alpha+\beta-1/2)]}.$$

Using the facts  $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ ,  $(2\lambda)_{2n} = 4^n (\lambda)_n (\lambda+1/2)_n$  we rewrite the above equation as

$$(3.2) \quad \mathcal{P}_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n 2^n (\beta+1)_n}{(\alpha+\beta+n+1)_n}.$$

Similarly

$$(3.3) \quad \mathcal{P}_n^{(\alpha,\beta)}(1) = \frac{2^n (\alpha+1)_n}{(\alpha+\beta+n+1)_n}.$$

We next evaluate  $h_n$ , the squares of the  $L^2$  norms. In general (1.1) and (1.2) yield ([12])

$$(3.4) \quad h_n = h_0 \beta_1 \beta_2 \cdots \beta_n.$$

The beta integral evaluation gives

$$(3.5) \quad h_0 = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Thus

$$(3.6) \quad \int_{-1}^1 \mathcal{P}_m^{(\alpha,\beta)}(x) \mathcal{P}_n^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx = h_n \delta_{m,n},$$

with

$$(3.7) \quad h_n = \frac{2^{\alpha+\beta+n+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{(\alpha+\beta+n+1)_n \Gamma(\alpha+\beta+2n+2)} n!.$$

We now prove that

$$(3.8) \quad \frac{d}{dz} \mathcal{P}_n^{(\alpha,\beta)}(z) = n\mathcal{P}_n^{(\alpha+1,\beta+1)}(z).$$

For  $\alpha > -1$ ,  $\beta > -1$ , and  $m < n - 1$ , integration by parts gives

$$\begin{aligned} & \int_{-1}^1 x^m \left( \frac{d}{dx} \mathcal{P}_n^{(\alpha,\beta)}(x) \right) (1-x)^{\alpha+1} (1+x)^{\beta+1} dx \\ &= - \int_{-1}^1 \mathcal{P}_n^{(\alpha,\beta)}(x) f(x) (1-x)^\alpha (1+x)^\beta dx \end{aligned}$$

where  $f(x) = x^{m-1} [m + x(\beta - \alpha) - x^2(\alpha + \beta + m + 2)]$ . Since  $f$  has degree at most  $n - 1$ , the above integral must vanish, and we conclude that  $\frac{d}{dx} \mathcal{P}_n^{(\alpha,\beta)}(x)$  is orthogonal to all polynomials of degree less than  $n - 1$  with respect to  $w^{(\alpha+1,\beta+1)}(x)$ . The uniqueness of the orthogonal polynomials and the fact that  $\mathcal{P}_n^{(\alpha,\beta)}(x)$ ,  $n \geq 0$  are monic establish (3.8). Clearly (3.8) and (3.2) give

$$(3.9) \quad \begin{aligned} \left. \frac{d^k}{dx^k} \mathcal{P}_n^{(\alpha,\beta)}(x) \right|_{x=-1} &= \frac{n!}{(n-k)!} \mathcal{P}_{n-k}^{(k+\alpha,k+\beta)}(-1) \\ &= \frac{n!}{(n-k)!} \frac{(-2)^{n-k} (\beta+k+1)_{n-k}}{(\alpha+\beta+n+k+1)_{n-k}}. \end{aligned}$$

The Taylor series about  $x = -1$  now gives the representation

$$(3.10) \quad \begin{aligned} \mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(-2)^n (\beta+1)_n}{(\alpha+\beta+n+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \beta+1; (1+x)/2), \end{aligned}$$

which we recognize to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation

$$(3.11) \quad \begin{aligned} \mathcal{P}_n^{(\alpha,\beta)}(x) &= \frac{(2)^n (\alpha+1)_n}{(\alpha+\beta+n+1)_n} \\ &\times {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2). \end{aligned}$$

REFERENCES

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, English translation, Oliver and Boyd, Edinburgh, 1965. MR 0184042 (32:1518)

[2] G. E. Andrews, R. A. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999. MR 1688958 (2000g:33001)

[3] W. Bauldry, *Estimates of asymmetric Freud polynomials on the real line*, J. Approximation Theory **63** (1990), 225–237. MR 1079852 (92c:33008)

[4] S. S. Bonan and D. S. Clark, *Estimates of the Hermite and the Freud polynomials*, J. Approximation Theory **63** (1990), 210–224. MR 1079851 (92c:33007)

[5] S. Bonan, D. S. Lubinsky, and P. Nevai, *Derivatives of Orthogonal Polynomials and characterization of weights*, in “*Approximation Theory 5*,” Academic Press, New York, 1986, pp. 271–274.

[6] S. Bonan, D. S. Lubinsky, and P. Nevai, *Orthogonal polynomials and their derivatives. II*. SIAM J. Math. Anal. **18** (1987), 1163–1176. MR 0892495 (89a:42032)

[7] Y. Chen and M. E. H. Ismail, *Ladder operators and differential equations for orthogonal polynomials*, J. Phys. A **30** (1997), 7817–7829. MR 1616931 (2000i:33011)

[8] M. E. H. Ismail, *An extremal problem for generalized Jacobi polynomials*, in “*Proc. of the Mathematics Conference, Birzeit University*,” S. Elaydi et al., eds., World Scientific, 2000, pp. 139–145. MR 1773025 (2001h:33012)

- [9] M. E. H. Ismail and J. Wimp, *On differential equations for orthogonal polynomials*, *Methods and Applications of Analysis* **5**(4) (1998), 439–452. MR 1669843 (99i:42037)
- [10] H. N. Mhaskar, *Bounds for Certain Freud-Type Orthogonal Polynomials*, *J. Approx. Theory* **63** (1990), 238–254. MR 1079853 (92c:33009)
- [11] P. Nevai, *Géza Freud, orthogonal polynomials and Christoffel functions: A case study*, *J. Approx. Theory* **48** (1986), 3–167. MR 0862231 (88b:42032)
- [12] E. D. Rainville, *Special Functions*, Chelsea, the Bronx, 1971. MR 0393590 (52:14399)
- [13] J. Shohat and J. D. Tamarkin, *The Problem of Moments*, revised edition, American Mathematical Society, Providence, 1950. MR 0008438 (5:5c)
- [14] G. Szegő, *Orthogonal Polynomials*, Fourth Edition, Amer. Math. Soc., Providence, 1975. MR 0372517 (51:8724)

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, 180 QUEEN'S GATE, LONDON SW7 2BZ,  
UNITED KINGDOM

*E-mail address:* `y.chen@imperial.ac.uk`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FLORIDA 32816

*E-mail address:* `ismail@math.ucf.edu`