JACOBI POLYNOMIALS FROM COMPATIBILITY CONDITIONS

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Abstract. We revisit the ladder operators for orthogonal polynomials and re-interpret two supplementary conditions as compatibility conditions of two linear over-determined systems; one involves the variation of the polynomials with respect to the variable $z$ (spectral parameter) and the other a recurrence relation in $n$ (the lattice variable). For the Jacobi weight $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, $x \in [-1,1]$, we show how to use the compatibility conditions to explicitly determine the recurrence coefficients of the monic Jacobi polynomials.

1. Introduction and preliminaries

We begin with some notation. Let $P_n(x)$ be monic polynomials of degree $n$ in $x$ and orthogonal, with respect to a weight, $w(x)$, $x \in [a,b]$:

$$\int_a^b P_m(x)P_n(x)w(x)\,dx = h_n\delta_{m,n}. \quad (1.1)$$

We further assume that $\sqrt{v}'(z) := -w'(z)/w(z)$ exists and that $y^n [\sqrt{v}'(x) - \sqrt{v}'(y)] w(y)/(x - y)$ is integrable on $[a,b]$ for all $n$, $n = 0, 1, \ldots$. From the orthogonality condition there follows the recurrence relation,

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \quad n = 0, 1, \ldots, \quad (1.2)$$

where $\beta_0 P_{-1}(z) := 0$, $\alpha_n$, $n = 0, 1, 2, \ldots$ is real and $\beta_n > 0$, $n = 1, 2, \ldots$.

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer to a future publication, the analysis in the case of the generalized Jacobi weights [8, 11]. In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at $x = \pm 1$. We also show that the evaluation of a Jacobi polynomial at $x = 1$ or $x = -1$ leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants $h_n$ are also found.

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The actions of the ladder operators on $P_n(z)$ and $P_{n-1}(z)$ are
\begin{align}
(1.3) \quad \left(\frac{d}{dz} + B_n(z)\right) P_n(z) &= \beta_n A_n(z) P_{n-1}(z), \\
(1.4) \quad \left(\frac{d}{dz} - B_n(z) - \sqrt{z} \right) P_{n-1}(z) &= -A_{n-1}(z) P_n(z),
\end{align}
with
\begin{align}
A_n(z) := \frac{w(y) B_n^2(y)}{h_n(y - z)} \bigg|_{y = a}^{y = b} + \frac{1}{h_n} \int_a^b \frac{\sqrt{z} - \sqrt{y}}{z - y} P_n^2(y) w(y) \, dy, \\
B_n(z) := \frac{w(y) P_n(y) P_{n-1}(y)}{h_{n-1}(y - z)} \bigg|_{y = a}^{y = b} + \frac{1}{h_{n-1}} \int_a^b \frac{\sqrt{z} - \sqrt{y}}{z - y} P_{n-1}(y) P_n(y) w(y) \, dy,
\end{align}
where we have used the supplementary condition,
\begin{equation}
(S_1) \quad B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \sqrt{z},
\end{equation}
to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition $(S_1)$ were derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial $\nu$, and the authors [7] for general $\nu$. Ismail and Wimp [9] identified the additional supplementary condition,
\begin{equation}
(S_2) \quad B_{n+1}(z) - B_n(z) = \frac{\beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z) - 1}{z - \alpha_n}.
\end{equation}

Our thesis in this work is that the supplementary conditions, $(S_1)$ and $(S_2)$, being identities in $n$ ($n > 0$) and $z \in \mathbb{C} \cup \{\infty\}$ have the information needed to determine the recurrence coefficients and other auxiliary quantities. We illustrate this by systematically using $(S_1)$ and $(S_2)$ to determine most of the properties of the Jacobi polynomials. See [14], [2], and [12], for detailed information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorials and hypergeometric functions in [2] [12].

Below, we reinterpret $(S_2)$. We set
\begin{align}
(1.7) \quad \Phi_n(z) := \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix}, \\
(1.8) \quad M_n(z) := \begin{pmatrix} -B_n(z) & \beta_n A_n(z) \\ -A_{n-1}(z) & B_n(z) + \sqrt{z} \end{pmatrix}, \\
(1.9) \quad U_n(z) := \begin{pmatrix} z - \alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix}.
\end{align}
Now equations (1.3) and (1.4) become,
\begin{equation}
(1.10) \quad \Phi_n(z) = M_n(z) \Phi_n(z),
\end{equation}
and the recurrence relations become
\begin{equation}
(1.11) \quad \Phi_{n+1}(z) = U_n(z) \Phi_n(z).
\end{equation}
We find, by requiring (1.10) and (1.11) to be compatible,
\begin{equation}
\Phi_n'(z) = M_{n+1}(z) \Phi_{n+1}(z) = M_{n+1}(z) U_n(z) \Phi_n(z).
\end{equation}
On the other hand,
\[
\Phi'_{n+1}(z) = U'_n(z)\Phi_n(z) + U_n(z)\Phi'_n(z)
\]
\[
= U'_n(z)\Phi_n(z) + U_n(z)M_n(z)\Phi_n(z).
\]

We now write the above equations in matrix form as
\[
(1.12)
S_n(z)\Phi_n(z) = 0,
\]
where $S_n(z)$ is the matrix
\[
S_n(z) := U'_n(z) + U_n(z)M_n(z) - M_{n+1}(z)U_n(z),
\]
whose entries $S_{i,j}$ are
\[
S^{11}_n(z) = 1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z))
+ \beta_nA_{n-1}(z) - \beta_{n+1}A_{n+1}(z),
\]
\[
S^{12}_n(z) = -\beta_n(B_{n+1}(z) + B_n(z) + v'(z) - (z - \alpha_n)A_n(z)),
\]
\[
S^{21}_n(z) = S^{12}_n(z)/\beta_n,
\]
\[
S^{22}_n(z) = 0.
\]

Here $n = 1, 2, \ldots$ and $z \in \mathbb{C} \cup \{\infty\}$. Observe that with $(S_1)$, $S^{12}_n(z) = S^{21}_n(z) = 0$. This leaves $S^{11}_n(z)P_n(z) = 0$. Since $P_n(z)$ does not vanish identically, we must have $S^{11}_n(z) = 0$, which is $(S_2)$. It is clear from (1.5) and (1.6) that, if $v'(z)$ is a rational function, then $A_n(z)$ and $B_n(z)$ are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients, $\alpha_n$ and $\beta_n$. In the next section, we illustrate the method by considering the Jacobi weight $w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$ for $x \in [-1, 1]$.

Recall that the numerator polynomials \[13\], \[1\] are
\[
(1.14)
Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z - y} w(y) \, dy,
\]
and $\{P_n(z)\}$ and $\{Q_n(z)\}$ form a basis of solutions of the recurrence relation. We shall also use the notation
\[
(1.15)
F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z - y} \, dy
\]
for the Stieltjes transform of the weight function.

2. Jacobi Weight

The Jacobi weight is $w^{(\alpha,\beta)}(x) = (1 - x)^\alpha(1 + x)^\beta$, $x \in [-1, 1]$, and for now we take $\alpha$ and $\beta$ to be strictly positive. It will become clear, using a real analyticity argument, the results that follow are also valid for $\alpha, \beta > -1$. Let $\{P_n^{(\alpha,\beta)}(x)\}$ and $\{Q_n^{(\alpha,\beta)}(x)\}$ denote the monic Jacobi polynomials, and their numerators, respectively; see (1.14). Moreover, in the present example, the Stieltjes transform of $w^{\alpha,\beta}$ will be denoted by $F^{(\alpha,\beta)}(z)$. 
From (1.5)–(1.6) we find
\[ h_n A_n(z) = \frac{\alpha}{1-z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1-y)^{\alpha-1}(1+y)^{\beta} \, dy \]
\[ + \frac{\beta}{1+z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1-y)^{\alpha}(1+y)^{\beta-1} \, dy. \]

Through integration by parts, it readily follows that
\[ A_n(z) = -\frac{R_n}{z-1} + \frac{R_n}{z+1}, \tag{2.1} \]
for some constant \( R_n \). Similarly we find
\[ B_n(z) = -\frac{n+r_n}{z-1} + \frac{r_n}{z+1}. \tag{2.2} \]

Here \( R_n \) and \( r_n \) are given by
\[ R_n = R_n(\alpha, \beta) := \frac{\beta}{h_n} \int_{-1}^{1} \frac{\left[ P_n^{(\alpha,\beta)}(y) \right]^2}{1+y} w^{(\alpha,\beta)}(y) \, dy, \tag{2.3} \]
\[ r_n = r_n(\alpha, \beta) := \frac{\beta}{h_{n-1}} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y) P_{n-1}^{(\alpha,\beta)}(y)}{1+y} w^{(\alpha,\beta)}(y) \, dy. \tag{2.4} \]

It is easy to see that
\[ R_n(\alpha, \beta) = \frac{\beta}{h_n} P_n^{(\alpha,\beta)}(-1) \left[ Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(-1) \right], \]
\[ r_n(\alpha, \beta) = \frac{\beta}{h_{n-1}} P_{n-1}^{(\alpha,\beta)}(-1) \left[ Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(-1) \right]. \]

The reader may ask, “What is the point of this formalism, since in the attempt to find \( \alpha_n \) and \( \beta_n \), two new unknown quantities, \( R_n \) and \( r_n \), have been introduced?” However, when \( v' \) is a rational function, both sides of \((S_1)\) and \((S_2)\) are rational functions, and by equating coefficients and residues of both sides of \((S_1)\) and \((S_2)\), we shall arrive at four equations, which should be sufficient for the determination of \( R_n \) and \( r_n \) as well as \( \alpha_n \) and \( \beta_n \). Equating residues of both sides of \((S_1)\), at \( z = -1 \) and \( z = +1 \), gives
\[ -2n - 1 - r_n - r_{n+1} = \alpha - R_n(1 - \alpha_n), \tag{2.5} \]
\[ r_n + r_{n+1} = \beta - R_n(1 + \alpha_n). \tag{2.6} \]

Similarly, from \((S_2)\), we obtain
\[ (r_n - r_{n+1} - 1)(1 - \alpha_n) = \beta_n R_{n-1} - \beta_{n+1} R_{n+1}, \tag{2.7} \]
\[ (r_n - r_{n+1})(1 + \alpha_n) = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}. \tag{2.8} \]

Observe that \( R_n \) can be obtained immediately by adding (2.5) and (2.6):
\[ R_n = \frac{1}{2}(\alpha + \beta + 2n + 1). \tag{2.9} \]
The sum of (2.7) and (2.8) gives
\[ 1 - \alpha_n = 2\left(r_n - r_{n+1}\right), \tag{2.10} \]
while (2.8) minus (2.7) and with \( (2.9) \) gives
\[ \beta - \alpha - 2n - 1 - (\alpha + \beta + 2n + 1)\alpha_n = 2\left(r_n + r_{n+1}\right). \tag{2.11} \]
Now, (2.10) $\pm$ (2.11) implies
\begin{align}
4r_n &= \beta - \alpha - 2n - (\alpha + \beta + 2n + 2)\alpha_n, \\
4r_{n+1} &= \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n)\alpha_n.
\end{align}
When (2.12) and (2.13) are made compatible, we obtain a first-order difference equation satisfied by $\alpha_n$:
\begin{align}
\alpha_{n+1}(\alpha + \beta + 2n + 4) - \alpha_n(\alpha + \beta + 2n) &= 0,
\end{align}
which has a very simple “integrating factor,” $\alpha + \beta + 2n + 2$. Using this, we find
\begin{align}
\alpha_n = \frac{C_1}{(2R_n - 1)(2R_n + 1)},
\end{align}
where $C_1$ is an “integration” constant, determined by the initial condition
\begin{align}
\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \quad C_1 = \beta^2 - \alpha^2.
\end{align}
Here $\mu_j := \int_{-1}^1 \psi_j w(t) \, dt$, $j = 0, 1, \ldots$ are the moments. Therefore we have established
\begin{align}
\alpha_n = \frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}.
\end{align}
Going back to (2.8) and using (2.10), we see that $\beta_n$ satisfies the linear difference equation:
\begin{align}
\beta_{n+1}R_{n+1} - \beta_nR_{n-1} &= \frac{1 - \alpha_n^2}{2},
\end{align}
which has the “integrating factor” $R_n$. Therefore,
\begin{align}
\beta_nR_nR_{n-1} &= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left(1 - \alpha_j^2\right) R_j \\
&= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left(1 - \frac{C_1^2}{(4R_j^2 - 1)^2}\right) R_j,
\end{align}
where $C_2$ is another integration constant to be determined by the initial condition
\begin{align}
\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} = \left(\frac{\mu_1}{\mu_0}\right)^2 = \frac{4(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}.
\end{align}
After some computations,
\begin{align}
C_2 = \beta_1R_0R_1 - \frac{1}{2} \left(1 - \alpha_0^2\right) R_0 = 0.
\end{align}
Now the sum (2.17) may look complicated; however, with a partial fraction expansion, the sum can be taken and leads to
\begin{align}
\beta_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2R_nR_{n-1}}.
\end{align}
Therefore, after some simplifications we establish
\begin{align}
\beta_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}.
\end{align}
3. Explicit Formulas

We first determine $\mathcal{P}_n^{(\alpha, \beta)}(\pm 1)$. Write (1.3) as

$$\frac{d}{dz} \mathcal{P}_n^{(\alpha, \beta)}(z) = \frac{(n + r_n) \mathcal{P}_n^{(\alpha, \beta)}(z) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha, \beta)}(z)}{z - 1} + \frac{\beta_n R_n \mathcal{P}_{n-1}^{(\alpha, \beta)}(z) - r_n \mathcal{P}_n^{(\alpha, \beta)}(z)}{z + 1},$$

and since $\frac{d}{dz} \mathcal{P}_n^{(\alpha, \beta)}(z)$ is regular at $z = \pm 1$, we arrive at

$$(n + r_n) \mathcal{P}_n^{(\alpha, \beta)}(1) - \beta_n R_n \mathcal{P}_{n-1}^{(\alpha, \beta)}(1) = 0,$$

$$\beta_n R_n \mathcal{P}_{n-1}^{(\alpha, \beta)}(-1) - r_n \mathcal{P}_n^{(\alpha, \beta)}(-1) = 0.$$

Thus we find

$$\mathcal{P}_n^{(\alpha, \beta)}(1) = \mathcal{P}_0^{(\alpha, \beta)}(1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j + j}$$

and

$$\mathcal{P}_n^{(\alpha, \beta)}(-1) = \mathcal{P}_0^{(\alpha, \beta)}(-1) \prod_{j=1}^{n} \frac{\beta_j R_j}{r_j}.$$

Substituting for $\beta_n$, $r_n$ and $R_n$ from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

$$\mathcal{P}_n^{(\alpha, \beta)}(-1) = (-1)^n \frac{2^n}{\prod_{j=1}^{n} \frac{(j + \alpha + \beta)(j + \alpha + \beta + 1)}{j + (\alpha + \beta + 1/2)}}.$$

Using the facts $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, $(2\lambda)_{2n} = 4^n (\lambda)_n (\lambda + 1/2)_n$ we rewrite the above equation as

$$\mathcal{P}_n^{(\alpha, \beta)}(-1) = (-1)^n \frac{2^n \prod_{j=1}^{n} \frac{(\beta + 1)_n}{(\beta + n + 1)_n}}.$$

Similarly

$$\mathcal{P}_n^{(\alpha, \beta)}(1) = \frac{2^n \prod_{j=1}^{n} \frac{(\alpha + 1)_n}{(\alpha + \beta + n + 1)_n}}.$$

We next evaluate $h_n$, the squares of the $L^2$ norms. In general (1.1) and (1.2) yield (12)

$$h_n = h_0 \beta_1 \beta_2 \cdots \beta_n.$$

The beta integral evaluation gives

$$h_0 = 2^{\alpha + \beta + 1} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}.$$

Thus

$$\int_{-1}^{1} \mathcal{P}_m^{(\alpha, \beta)}(x) \mathcal{P}_n^{(\alpha, \beta)}(x)(1-x)\alpha(1+x)\beta \, dx = h_n \delta_{m,n},$$

with

$$h_n = \frac{2^{\alpha + \beta + n + 1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(\alpha + \beta + n + 1)_n \Gamma(\alpha + \beta + 2n + 2)} n!.$$
We now prove that

\[ \frac{d}{dz} P_n^{(\alpha, \beta)}(z) = n P_{n+1}^{(\alpha+1, \beta+1)}(z). \]

For \( \alpha > -1, \beta > -1, \) and \( m < n - 1, \) integration by parts gives

\[
\int_{-1}^{1} x^m \left( \frac{d}{dx} P_n^{(\alpha, \beta)}(x) \right) (1 - x)^{\alpha+1} (1 + x)^{\beta+1} \, dx \\
= - \int_{-1}^{1} P_n^{(\alpha, \beta)}(x) f(x) (1 - x)^{\alpha} (1 + x)^{\beta} \, dx
\]

where \( f(x) = x^{m-1} \left[ m + x(\beta - \alpha) - x^2 (\alpha + \beta + m + 2) \right] \). Since \( f \) has degree at most \( n - 1 \), the above integral must vanish, and we conclude that \( \frac{d}{dx} P_n^{(\alpha, \beta)}(x) \) is orthogonal to all polynomials of degree less than \( n - 1 \) with respect to \( w_{n+1}^{(\alpha+1, \beta+1)}(x) \).

The uniqueness of the orthogonal polynomials and the fact that \( P_n^{(\alpha, \beta)}(x), n \geq 0 \) are monic establish (3.8). Clearly (3.8) and (3.2) give

\[ \frac{d^k}{dx^k} P_n^{(\alpha, \beta)}(x) \bigg|_{x=-1} = \frac{n!}{(n-k)!} P_{n-k}^{(k+\alpha,k+\beta)}(-1) \]

(3.9)

The Taylor series about \( x = -1 \) now gives the representation

\[
P_n^{(\alpha, \beta)}(x) = \frac{(-2)^n(\beta+1)_n}{(\alpha + \beta + n + 1)_n} \times 2F1(-n, n + \alpha + \beta + 1; \beta + 1; (1 + x)/2),
\]

which we recognize to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation

\[ P_n^{(\alpha, \beta)}(x) = \frac{(2)^n(\alpha + 1)_n}{(\alpha + \beta + n + 1)_n} \times 2F1(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2). \]

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