We revisit the ladder operators for orthogonal polynomials and re-interpret two supplementary conditions as compatibility conditions of two linear over-determined systems; one involves the variation of the polynomials with respect to the variable $z$ (spectral parameter) and the other a recurrence relation in $n$ (the lattice variable). For the Jacobi weight

$$w(x) = (1 - x)^\alpha (1 + x)^\beta, \quad x \in [-1, 1],$$

we show how to use the compatibility conditions to explicitly determine the recurrence coefficients of the monic Jacobi polynomials.

1. Introduction and preliminaries

We begin with some notation. Let $P_n(x)$ be monic polynomials of degree $n$ in $x$ and orthogonal, with respect to a weight, $w(x), x \in [a, b]$:

$$\int_a^b P_m(x)P_n(x)w(x)\,dx = h_n\delta_{m,n}.$$

We further assume that $\nabla(z) := -w'(z)/w(z)$ exists and that

$$y^n [\nabla(x) - \nabla(y)] w(y)/(x - y)$$

is integrable on $[a, b]$ for all $n, n = 0, 1, \ldots$. From the orthogonality condition there follows the recurrence relation,

$$z P_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z), \quad n = 0, 1, \ldots,$$

where $\beta_0 P_0(z) := 0, \alpha_n, n = 0, 1, 2, \ldots$ is real and $\beta_n > 0, n = 1, 2, \ldots$.

In this paper we describe a formalism which derives properties of orthogonal polynomials, and their recurrence coefficients, from the knowledge of the weight function. We believe this is a new and interesting approach to orthogonal polynomials. In order to keep this work accessible we will only include the example of Jacobi polynomials. We defer to a future publication, the analysis in the case of the generalized Jacobi weights [8, 11]. In the Jacobi case we find the recurrence relations in §2. In §3 we show how our approach leads to the evaluation of monic Jacobi polynomials at $x = \pm 1$. We also show that the evaluation of a Jacobi polynomial at $x = 1$ or $x = -1$ leads to explicit representations of the Jacobi polynomials. Closed form expressions for the normalization constants $h_n$ are also found.

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The actions of the ladder operators on $P_n(z)$ and $P_{n-1}(z)$ are

(1.3) \[ \left( \frac{d}{dz} + B_n(z) \right) P_n(z) = \beta_n A_n(z) P_{n-1}(z), \]

(1.4) \[ \left( \frac{d}{dz} - B_n(z) - \mathcal{V}'(z) \right) P_{n-1}(z) = -A_{n-1}(z) P_n(z), \]

with

(1.5) \[ A_n(z) := \frac{w(y) P_n^2(y)}{h_n(y-z)} \bigg|_{y=a}^{y=b} + \frac{1}{h_n} \int_a^b \frac{\mathcal{V}'(z) - \mathcal{V}'(y)}{z-y} P_n^2(y) w(y) \, dy, \]

(1.6) \[ B_n(z) := \frac{w(y) P_n(y) P_{n-1}(y)}{h_{n-1}(y-z)} \bigg|_{y=a}^{y=b} + \frac{1}{h_{n-1}} \int_a^b \frac{\mathcal{V}'(z) - \mathcal{V}'(y)}{z-y} P_{n-1}(y) P_n(y) w(y) \, dy, \]

where we have used the supplementary condition,

\( (S_1) \quad B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \mathcal{V}'(z), \)

to arrive at (1.4). The equations (1.3)–(1.6) and the supplementary condition (S1) were derived by Bonan and Clark [4], Bauldry [3], and Mhaskar [10] for polynomial $\mathcal{V}$, and the authors [7] for general $\mathcal{V}$. Ismail and Wimp [9] identified the additional supplementary condition,

\( (S_2) \quad B_{n+1}(z) - B_n(z) = \frac{\beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z) - 1}{z - \alpha_n}. \)

Our thesis in this work is that the supplementary conditions, (S1) and (S2), being identities in $n$ ($n > 0$) and $z \in \mathbb{C} \cup \{\infty\}$ have the information needed to determine the recurrence coefficients and other auxiliary quantities. We illustrate this by systematically using (S1) and (S2) to determine most of the properties of the Jacobi polynomials. See [14], [2], and [12], for detailed information concerning the Jacobi polynomials. In describing our results we shall follow the standard notation for shifted factorials and hypergeometric functions in [2], [12].

Below, we reinterpret (S2). We set

(1.7) \[ \Phi_n(z) := \begin{pmatrix} P_n(z) \\ P_{n-1}(z) \end{pmatrix}, \]

(1.8) \[ M_n(z) := \begin{pmatrix} -B_n(z) & \beta_n A_n(z) \\ -A_{n-1}(z) & B_n(z) + \mathcal{V}'(z) \end{pmatrix}, \]

(1.9) \[ U_n(z) := \begin{pmatrix} z - \alpha_n & -\beta_n \\ 1 & 0 \end{pmatrix}. \]

Now equations (1.3) and (1.4) become

(1.10) \[ \Phi'_n(z) = M_n(z) \Phi_n(z), \]

and the recurrence relations become

(1.11) \[ \Phi_{n+1}(z) = U_n(z) \Phi_n(z). \]

We find, by requiring (1.10) and (1.11) to be compatible,

\[ \Phi'_{n+1}(z) = M_{n+1}(z) \Phi_{n+1}(z) = M_{n+1}(z) U_n(z) \Phi_n(z). \]
On the other hand,
\[ \Phi'_{n+1}(z) = U'_n(z)\Phi_n(z) + U_n(z)\Phi'_n(z) \]
\[ = U'_n(z)\Phi_n(z) + U_n(z)M_n(z)\Phi_n(z). \]

We now write the above equations in matrix form as
\[ (1.12) \quad S_n(z)\Phi_n(z) = 0, \]
where \( S_n(z) \) is the matrix
\[ S_n(z) := U'_n(z) + U_n(z)M_n(z) - M_{n+1}(z)U_n(z), \]
whose entries \( S_{i,j} \) are
\[ S^{11}_n(z) = 1 + (z - \alpha_n) (B_{n+1}(z) - B_n(z)) + \beta_n A_{n-1}(z) - \beta_{n+1} A_{n+1}(z), \]
\[ S^{12}_n(z) = -\beta_n (B_{n+1}(z) + B_n(z) + \nu'(z) - (z - \alpha_n) A_n(z)), \]
\[ S^{21}_n(z) = S^{12}_n(z)/\beta_n, \]
\[ S^{22}_n(z) = 0. \]

Here \( n = 1, 2, \ldots \) and \( z \in \mathbb{C} \cup \{ \infty \} \). Observe that with \( (S_1) \), \( S^{12}_n(z) = S^{21}_n(z) = 0. \)
This leaves \( S^{11}_n(z)P_n(z) = 0. \) Since \( P_n(z) \) does not vanish identically, we must have \( S^{11}_n(z) = 0 \), which is \( (S_2) \). It is clear from \( (1.5) \) and \( (1.6) \) that, if \( \nu'(z) \) is a rational function, then \( A_n(z) \) and \( B_n(z) \) are also rational functions. This is particularly useful for our purpose, which is to determine the recurrence coefficients, \( \alpha_n \) and \( \beta_n \). In the next section, we illustrate the method by considering the Jacobi weight \( w^{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta \) for \( x \in [-1, 1] \).

Recall that the numerator polynomials \([13], [1]\) are
\[ (1.14) \quad Q_n(z) := \int_{-\infty}^{\infty} \frac{P_n(z) - P_n(y)}{z - y} w(y) dy, \]
and \( \{P_n(z)\} \) and \( \{Q_n(z)\} \) form a basis of solutions of the recurrence relation. We shall also use the notation
\[ (1.15) \quad F(z) = \int_{-\infty}^{\infty} \frac{w(y)}{z - y} dy \]
for the Stieltjes transform of the weight function.

2. JACOBI WEIGHT

The Jacobi weight is \( w^{(\alpha,\beta)}(x) = (1 - x)^\alpha (1 + x)^\beta \), \( x \in [-1, 1] \), and for now we take \( \alpha \) and \( \beta \) to be strictly positive. It will become clear, using a real analyticity argument, the results that follow are also valid for \( \alpha, \beta > -1 \). Let \( \{P_n^{(\alpha,\beta)}(x)\} \) and \( \{Q_n^{(\alpha,\beta)}(x)\} \) denote the monic Jacobi polynomials, and their numerators, respectively; see \( (1.14) \). Moreover, in the present example, the Stieltjes transform of \( w^{(\alpha,\beta)} \) will be denoted by \( F^{(\alpha,\beta)}(z) \).
From (1.5)–(1.6) we find
\[
h_n A_n(z) = \frac{\alpha}{1 - z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1 - y)^{\alpha - 1}(1 + y)^{\beta} \, dy \\
+ \frac{\beta}{1 + z} \int_{-1}^{1} \left[ P_n^{(\alpha,\beta)}(y) \right]^2 (1 - y)^\alpha(1 + y)^{\beta - 1} \, dy.
\]
Through integration by parts, it readily follows that
\[
A_n(z) = -\frac{R_n}{z - 1} + \frac{R_n}{z + 1},
\]
for some constant \( R_n \). Similarly we find
\[
B_n(z) = -\frac{n + r_n}{z - 1} + \frac{r_n}{z + 1}.
\]
Here \( R_n \) and \( r_n \) are given by
\[
R_n = R_n(\alpha, \beta) := \frac{\beta}{h_n} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y)}{1 + y} w^{(\alpha,\beta)}(y) \, dy,
\]
\[
r_n = r_n(\alpha, \beta) := \frac{\beta}{h_{n-1}} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(y) P_{n-1}^{(\alpha,\beta)}(y)}{1 + y} w^{(\alpha,\beta)}(y) \, dy.
\]
It is easy to see that
\[
R_n(\alpha, \beta) = \frac{\beta}{h_n} P_n^{(\alpha,\beta)}(-1) \left[ Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(-1) \right],
\]
\[
r_n(\alpha, \beta) = \frac{\beta}{h_{n-1}} P_{n-1}^{(\alpha,\beta)}(-1) \left[ Q_n^{(\alpha,\beta)}(-1) - F^{(\alpha,\beta)}(-1) P_n^{(\alpha,\beta)}(-1) \right].
\]
The reader may ask, “What is the point of this formalism, since in the attempt to find \( \alpha_n \) and \( \beta_n \), two new unknown quantities, \( R_n \) and \( r_n \), have been introduced?” However, when \( \varphi \) is a rational function, both sides of \((S_1)\) and \((S_2)\) are rational functions, and by equating coefficients and residues of both sides of \((S_1)\) and \((S_2)\), we shall arrive at four equations, which should be sufficient for the determination of \( R_n \) and \( r_n \) as well as \( \alpha_n \) and \( \beta_n \). Equating residues of both sides of \((S_1)\) at \( z = -1 \) and \( z = +1 \), gives
\[
-2n - 1 - r_n - r_{n+1} = \alpha - R_n(1 - \alpha_n),
\]
\[
r_n + r_{n+1} = \beta - R_n(1 + \alpha_n).
\]
Similarly, from \((S_2)\), we obtain
\[
(r_n - r_{n+1} - 1)(1 - \alpha_n) = \beta_n R_{n-1} - \beta_{n+1} R_{n+1},
\]
\[
(r_n - r_{n+1})(1 + \alpha_n) = \beta_{n+1} R_{n+1} - \beta_n R_{n-1}.
\]
Observe that \( R_n \) can be obtained immediately by adding (2.5) and (2.6):
\[
R_n = \frac{1}{2}(\alpha + \beta + 2n + 1).
\]
The sum of (2.7) and (2.8) gives
\[
1 - \alpha_n = 2(r_n - r_{n+1}),
\]
while (2.8) minus (2.7) and with (2.9) gives
\[
\beta - \alpha - 2n - 1 - (\alpha + \beta + 2n + 1)\alpha_n = 2(r_n + r_{n+1}).
\]
Now, (2.10) $\pm$ (2.11) implies

\begin{align}
4r_n &= \beta - \alpha - 2n - (\alpha + \beta + 2n + 2) \alpha_n, \\
4r_{n+1} &= \beta - \alpha - 2n - 2 - (\alpha + \beta + 2n) \alpha_n.
\end{align}

When (2.12) and (2.13) are made compatible, we obtain a first-order difference equation satisfied by $\alpha_n$:

\begin{align}
\alpha_{n+1} &= (\alpha + \beta + 2n + 4) - \alpha_n (\alpha + \beta + 2n) = 0,
\end{align}

which has a very simple “integrating factor,” $\alpha + \beta + 2n + 2$. Using this, we find

\begin{align}
\alpha_n &= \frac{C_1}{(2R_n - 1)(2R_n + 1)},
\end{align}

where $C_1$ is an “integration” constant, determined by the initial condition

\begin{align}
\alpha_0 = \frac{\mu_1}{\mu_0} = \frac{\beta - \alpha}{\alpha + \beta + 2}, \\
C_1 &= \beta^2 - \alpha^2.
\end{align}

Here $\mu_j := \int_{-1}^{1} t^j w(t) \, dt; \, j = 0, 1, \ldots$ are the moments. Therefore we have established

\begin{align}
\alpha_n = \frac{\beta^2 - \alpha^2}{(\alpha + \beta + 2n)(\alpha + \beta + 2n + 2)}.
\end{align}

Going back to (2.8) and using (2.10), we see that $\beta_n$ satisfies the linear difference equation:

\begin{align}
\beta_{n+1}R_{n+1} - \beta_n R_{n-1} = \frac{1 - \alpha_n^2}{2},
\end{align}

which has the “integrating factor” $R_n$. Therefore,

\begin{align}
\beta_nR_nR_{n-1} &= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left(1 - \alpha_j^2\right) R_j \\
&= C_2 + \frac{1}{2} \sum_{j=0}^{n-1} \left(1 - \frac{C_1^2}{(4R_j^2 - 1)^2}\right) R_j,
\end{align}

where $C_2$ is another integration constant to be determined by the initial condition

\begin{align}
\beta_1 = \frac{h_1}{h_0} = \frac{h_1}{\mu_0} = \frac{\mu_2}{\mu_0} = \left(\frac{\mu_1}{\mu_0}\right)^2 = \frac{4(\alpha + 1)(\beta + 1)}{(\alpha + \beta + 2)^2(\alpha + \beta + 3)}.
\end{align}

After some computations,

\begin{align}
C_2 = \beta_1 R_0 R_1 - \frac{1}{2} (1 - \alpha_0^2) R_0 = 0.
\end{align}

Now the sum (2.17) may look complicated; however, with a partial fraction expansion, the sum can be taken and leads to

\begin{align}
\beta_n &= \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2 R_n R_{n-1}}.
\end{align}

Therefore, after some simplifications we establish

\begin{align}
\beta_n &= \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)(2n + \alpha + \beta - 1)}.
\end{align}
3. Explicit Formulas

We first determine $P_n^{(\alpha,\beta)}(\pm 1)$. Write (1.3) as

$$
\frac{d}{dz} P_n^{(\alpha,\beta)}(z) = \left( \frac{n + r_n}{z - 1} \right) P_n^{(\alpha,\beta)}(z) - \beta_n R_n P_n^{(\alpha,\beta)}(1) + \frac{\beta_n R_n P_{n-1}^{(\alpha,\beta)}(z) - r_n P_n^{(\alpha,\beta)}(z)}{z + 1},
$$

and since $\frac{d}{dz} P_n^{(\alpha,\beta)}(z)$ is regular at $z = \pm 1$, we arrive at

$$
(n + r_n) P_n^{(\alpha,\beta)}(1) - \beta_n R_n P_{n-1}^{(\alpha,\beta)}(1) = 0,
$$

$$
\beta_n R_n P_{n-1}^{(\alpha,\beta)}(-1) - r_n P_n^{(\alpha,\beta)}(-1) = 0.
$$

Thus we find

$$
P_n^{(\alpha,\beta)}(1) = P_0^{(\alpha,\beta)}(1) \prod_{j=1}^n \frac{\alpha_j R_j}{r_j + j},
$$

and

$$
P_n^{(\alpha,\beta)}(-1) = P_0^{(\alpha,\beta)}(-1) \prod_{j=1}^n \frac{\alpha_j R_j}{r_j}.
$$

Substituting for $\beta_n$, $r_n$ and $R_n$ from (2.9), (2.12), and (2.18), and applying (2.15) we prove that

$$
P_n^{(\alpha,\beta)}(-1) = \left( \frac{-1)^n}{n!} \prod_{j=1}^n \frac{(j + \beta)(j + \alpha + \beta)}{[j + (\alpha + \beta/2)] [j + (\alpha + \beta + 1/2)]} \right).
$$

Using the facts $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda)$, $(2\lambda)_{2n} = 4^n(\lambda)_n(\lambda + 1/2)_n$ we rewrite the above equation as

$$
P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n \prod_{j=1}^n (j + \beta + 1)n_n}{(\alpha + \beta + n + 1)n_n}.
$$

Similarly

$$
P_n^{(\alpha,\beta)}(1) = \frac{2^n (\alpha + 1)n_n}{(\alpha + \beta + n + 1)n_n}.
$$

We next evaluate $h_n$, the squares of the $L^2$ norms. In general (1.1) and (1.2) yield (12)

$$
h_n = h_0 \beta_1 \beta_2 \cdots \beta_n.
$$

The beta integral evaluation gives

$$
h_0 = 2^{\alpha + \beta + 1} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)}.
$$

Thus

$$
\int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1 - x)^\alpha (1 + x)^\beta \, dx = h_n \delta_{m,n},
$$

with

$$
h_n = \frac{2^{\alpha + \beta + n + 1} \Gamma(\alpha + n + 1) \Gamma(\beta + n + 1)}{(\alpha + \beta + n + 1)n_n \Gamma(\alpha + \beta + 2n + 2)} n!.
$$
We now prove that
\begin{equation}
\frac{d}{dz} \mathcal{P}_n^{(\alpha,\beta)}(z) = n \mathcal{P}_n^{(\alpha+1,\beta+1)}(z).
\end{equation}

For \(\alpha > -1, \beta > -1,\) and \(m < n - 1,\) integration by parts gives
\begin{align*}
\int_{-1}^{1} x^m \left( \frac{d}{dx} \mathcal{P}_n^{(\alpha,\beta)}(x) \right) (1 - x)^{\alpha + 1}(1 + x)^{\beta + 1} dx & = -\int_{-1}^{1} \mathcal{P}_n^{(\alpha,\beta)}(x) f(x)(1 - x)^\alpha(1 + x)^\beta dx
\end{align*}
where \(f(x) = x^{m-1} [m + x(\beta - \alpha) - x^2(\alpha + \beta + m + 2)].\) Since \(f\) has degree at most \(n - 1,\) the above integral must vanish, and we conclude that \(\frac{d}{dz} \mathcal{P}_n^{(\alpha,\beta)}(x)\) is orthogonal to all polynomials of degree less than \(n - 1\) with respect to \(w_n^{(\alpha+1,\beta+1)}(x).\)

The uniqueness of the orthogonal polynomials and the fact that \(\mathcal{P}_n^{(\alpha,\beta)}(x), n \geq 0\) are monic establish (3.8). Clearly (3.8) and (3.2) give
\begin{equation}
\frac{d^k}{dx^k} \mathcal{P}_n^{(\alpha,\beta)}(x) \bigg|_{x=-1} = \frac{n!}{(n-k)!} \mathcal{P}_{n-k}^{(k+\alpha,k+\beta)}(-1)
\end{equation}
\begin{align*}
\frac{n!}{(n-k)!} \frac{(-2)^{n-k}(\beta + k + 1)_{n-k}}{\alpha + \beta + n + 1}_{n-k}.
\end{align*}

The Taylor series about \(x = -1\) now gives the representation
\begin{equation}
\mathcal{P}_n^{(\alpha,\beta)}(x) = \frac{(-2)^n(\beta + 1)_n}{(\alpha + \beta + n + 1)_n} 	imes 2F_1(-n,n + \alpha + \beta + 1;\beta + 1;1 + x)/2,
\end{equation}
which we recognize to be the monic Jacobi polynomials. Similarly (3.3) and (3.8) give the alternate representation
\begin{equation}
\mathcal{P}_n^{(\alpha,\beta)}(x) = \frac{(2)^n(\alpha + 1)_n}{(\alpha + \beta + n + 1)_n} 	imes 2F_1(-n,n + \alpha + \beta + 1;\alpha + 1;1 - x)/2.
\end{equation}

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