

## HYPERBOLIC UNIT GROUPS

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ABSTRACT. In this paper we study the groups  $\mathcal{G}$  whose integral group rings have hyperbolic unit groups  $\mathcal{U}(\mathbb{Z}\mathcal{G})$ . We classify completely the torsion subgroups of  $\mathcal{U}(\mathbb{Z}\mathcal{G})$  and the polycyclic-by-finite subgroups of the group  $\mathcal{G}$ . Finally, we classify the groups for which the boundary of  $\mathcal{U}(\mathbb{Z}\mathcal{G})$  has dimension zero.

### 1. INTRODUCTION

The study of hyperbolic groups has been an active topic of research in recent years. It started with the work of M. Gromov [4] and has developed very rapidly since then.

Let  $\mathcal{G}$  be a group and  $\Gamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$  the group of normalized units of the integral group ring  $\mathbb{Z}\mathcal{G}$ . It is well known that if  $\mathcal{G}$  is finite, then  $\Gamma$  is finitely presented (see, for instance, [8]). Since almost every finitely presented group is hyperbolic [10], it is natural to investigate when  $\Gamma$  is hyperbolic. Motivated by these considerations, we are led to pose the following:

**Problem 1.** Classify the groups  $\mathcal{G}$  for which  $\Gamma$  is hyperbolic.

This paper is a contribution to the above problem. After giving in section 2 the basic facts about hyperbolic groups needed in this work, we characterize, in section 3, the torsion subgroups of  $\Gamma$  and the polycyclic-by-finite subgroups of  $\mathcal{G}$ , thus answering the problem, in particular, for torsion groups. Contrary to the theorem that almost all finitely presented groups are hyperbolic, we find that the unit group of a finite group is hyperbolic only in a very small number of cases, which we enumerate explicitly. It turns out that if  $\mathcal{G}$  is finite and  $\Gamma$  is hyperbolic, then  $\mathcal{G}$  has a normal free complement (i.e., there exists a normal free subgroup  $F$  in  $\Gamma$  such that  $\Gamma = F\mathcal{G}$ ,  $F \cap \mathcal{G} = 1$ ), and furthermore every torsion-free complement of  $\mathcal{G}$  is free.

Finally, we completely characterize the groups  $\mathcal{G}$  such that  $\Gamma$  is hyperbolic with its hyperbolic boundary having dimension zero, or equivalently that  $\Gamma$  is virtually free.

Jespers ([6], [7]) has classified those finite groups  $\mathcal{G}$  that have a normal free complement in  $\Gamma$ . This property implies that  $\Gamma$  is quasi-isometric to a free group of finite rank and hence is hyperbolic. In view of this work, our Theorems 2 and

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3 can be considered to be basically due to Jespers; however, our proofs are quite different.

## 2. PRELIMINARIES

Let  $(X, d)$  be a metric space. For  $x, y, z \in X$ , the Gromov product of  $y, z$  with respect to  $x$  is defined to be

$$(y.z)_x = \frac{1}{2}\{d(y, x) + d(z, x) - d(y, z)\}.$$

The metric space is said to be  $\delta$ -hyperbolic ( $\delta \geq 0$ ) if

$$(x.y)_w \geq \min\{(x.z)_w, (y.z)_w\} - \delta$$

for all  $w, x, y, z \in X$ . Let  $G$  be a finitely generated group and  $S$  a finite set of generators for  $G$ . The Cayley graph  $\mathcal{G}(G, S)$  of  $G$  with respect to  $S$  is the metric graph whose vertices are in one-to-one correspondence with the elements of  $G$  and which has an edge (labeled  $s$ ) of length 1 joining  $g$  to  $gs$  for each  $g \in G$  and  $s \in S$ . The group  $G$  is said to be hyperbolic (in the sense of Gromov) if its Cayley graph  $\mathcal{G}(G, S)$  is a  $\delta$ -hyperbolic metric space for some  $\delta \geq 0$ . This definition does not depend on the choice of the generating set  $S$ .

For the reader's convenience, we collect some of the facts about hyperbolic groups that we need. Let  $\mathbb{Z}^2$  denote the free Abelian group of rank two.

**Theorem 1.** *Let  $\Gamma$  be a hyperbolic group. Then*

- (a)  $\mathbb{Z}^2$  does not embed as a subgroup of  $\Gamma$ .
- (b) If  $g \in \Gamma$  has infinite order, then  $[C_\Gamma(g) : \langle g \rangle]$  is finite, where  $C_\Gamma(g)$  is the centralizer of  $g$  in  $\Gamma$ .
- (c) Torsion subgroups of  $\Gamma$  are finite of bounded order.
- (d)  $\Gamma$  is virtually free if and only if its boundary has dimension zero.
- (e) If  $\Gamma$  is quasi-isometric to a free group, then  $\Gamma$  is virtually free. If, moreover,  $\Gamma$  is torsion-free, then it is free.

For the theory of hyperbolic groups, the reader may refer to [1], [3] or [4] and, for standard results and notation in group rings, to [11, 12, 13, 14].

## 3. GROUPS WITH $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$ HYPERBOLIC

Let  $\mathcal{G}$  be an arbitrary group, and let  $G \subseteq \Gamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$  be a subgroup of normalized units. In this section, unless otherwise stated, we shall always assume that  $\Gamma$  is a hyperbolic group.

We first make some easy observations:

**1<sup>0</sup>.** In case  $G$  is finite, then it constitutes a  $\mathbb{Z}$ -linearly independent subset of  $\mathbb{Z}\mathcal{G}$  and consequently  $\mathbb{Z}[G]$ , the subring of  $\mathbb{Z}\mathcal{G}$  generated by  $G$ , is isomorphic to the group ring  $\mathbb{Z}G$ . This fact shall be used freely. It is a simple consequence of a well-known theorem of Berman: If  $u$  is an element of finite order in the unit group  $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$ , then the coefficient in  $u$  of the identity element of  $\mathcal{G}$  is zero unless  $u$  equals 1.

**2<sup>0</sup>.** Let  $g \in \Gamma$  be a torsion element and  $n = o(g)$  its order. Since the torsion-free rank of  $\mathcal{U}_1(\mathbb{Z}\langle g \rangle)$  is  $\sum_{d|n, d>2} (\frac{\phi(d)}{2} - 1)$  (see [13], Theorem 3.1, p. 54) where  $\phi$  is the Euler phi-function, it follows, from Theorem 1(a), that  $n$  divides 5, 8, or 12.

**3<sup>0</sup>.** Let  $G$  be a finite subgroup of  $\Gamma$ . Since  $\mathbb{Z}^2$  does not embed in  $\Gamma$ ,  $\mathbb{Q}G$  has at most one Wedderburn component that is not a division ring, and that component must be  $M_2(\mathbb{Q})$ . (This follows by looking at the upper triangular matrices in a Wedderburn component  $M_n(D)$ , which gives rise to units of the form  $GL_n(\mathcal{O})$  for an order  $\mathcal{O}$  in  $D$ .) Hence if  $G$  is non-Abelian and none of its non-Abelian quotient embeds into a division ring, then  $\Delta(G, G')$ , the kernel of the natural projection  $\mathbb{Q}G \rightarrow \mathbb{Q}(G/G')$ , where  $G'$  is the derived group of  $G$ , is isomorphic to  $M_2(\mathbb{Q})$ , and so its dimension over  $\mathbb{Q}$  is 4.

**4<sup>0</sup>.** In view of **2<sup>0</sup>** and **3<sup>0</sup>**, it is easy to see that the groups  $C_5 \times C_3, C_5 \times C_4, C_5 \times C_5, D_5, C_5 \rtimes C_4, C_5 \rtimes C_8, C_8 \times C_2, K_8 \times C_3, (C_2 \times C_2) \rtimes C_4$  and  $Q_{16}$  cannot appear as subgroups of  $\Gamma$ , where  $Q_{16}$  is the generalized quaternion group of order 16,  $K_8$  denotes the quaternion group of order 8,  $D_n$  the dihedral group of order  $2n$  and  $C_n$  the cyclic group of order  $n$ .

**5<sup>0</sup>.** We will have occasion to use the general result that if  $\Gamma := \mathcal{U}_1(\mathbb{Z}\mathcal{G})$  is finitely generated, so is  $\mathcal{G}$ ; in particular if  $\Gamma$  is hyperbolic,  $\mathcal{G}$  is finitely generated. We supply a proof of this assertion. Suppose that  $\Gamma$  is generated by a finite set, say  $u_1, \dots, u_n$ . Let  $G_0$  be the subgroup of  $\mathcal{G}$  generated by the  $\text{supp}(u_i), \text{supp}(u_i^{-1}), i = 1, \dots, n$ . This is a finitely generated group. All the  $u_i$ 's belong to the unit group of  $\mathbb{Z}G_0$ , and hence  $\Gamma$  is contained in the unit group of  $\mathbb{Z}G_0$ . But  $\mathcal{G}$  is a subset of  $\Gamma$ , and so  $\mathcal{G}$  is contained in  $\mathbb{Z}G_0$ . This implies that  $\mathcal{G} = G_0$ , and hence  $\mathcal{G}$  is finitely generated, as  $G_0$  is.

Recall that a non-Abelian group  $G$  is called Hamiltonian if all its subgroups are normal. We will abuse terminology to denote a non-Abelian, non-Hamiltonian group simply as a non-Hamiltonian group; thus in our usage, a non-Hamiltonian group is always non-Abelian. If  $G$  is a finite Hamiltonian group, then  $G \simeq K_8 \times A \times E$ , where  $K_8$  is the quaternion group of order 8,  $A$  an Abelian group of odd order and  $E$  an elementary Abelian 2-group (see [5], Theorem 7.12, p. 308). As in [6], we begin with the following result.

**Lemma 1.** *Let  $H \subseteq G \subseteq \Gamma$  be groups with  $G$  finite and non-Abelian. Then one of the following holds:*

- (a)  $H$  is Abelian.
- (b)  $H$  is a (non-Abelian) Hamiltonian 2-group.
- (c)  $\mathbb{Q}H$  contains a unique matrix Wedderburn component that is isomorphic to  $M_2(\mathbb{Q})$  and  $H = G$ .

*Proof.* Note first that if  $M_n(D)$  is a Wedderburn component of  $\mathbb{Q}H$ , then we must have  $n \leq 2$ .

Suppose  $H$  is non-Abelian. In case  $\mathbb{Q}H$  is a direct sum of division rings, then  $H \simeq K_8 \times E \times A$ , with  $E$  an elementary Abelian 2-group and  $A$  an Abelian group of odd order ([13], Theorem 1.17, p. 172). In case  $A \neq 1$ , then, because of the restriction on the orders of the elements of finite order,  $A$  is an elementary Abelian 3-group. Hence  $H_0 \simeq K_8 \times C_3$  is a subgroup of  $G$ , which is not possible by **4<sup>0</sup>**, and so  $H$  is a Hamiltonian 2-group.

Next suppose that  $\mathbb{Q}H$  is not a direct sum of division rings. Then, by **3<sup>0</sup>**,  $\mathbb{Q}H$  has a unique matrix Wedderburn component  $\mathcal{A}$  and  $\mathcal{A} \simeq M_2(\mathbb{Q})$ . Let  $e \in \mathbb{Q}H$  be the primitive central idempotent such that  $\mathbb{Q}He = \mathcal{A}$ . Let  $\{f_i \mid 1 \leq i \leq n\}$  be the

set of primitive central idempotents of  $\mathbb{Q}G$ , and so  $e = \sum_{ef_i \neq 0} ef_i$ . Since  $\mathcal{A}$  is simple,  $ef_i \neq 0$  implies that  $\mathbb{Q}Gf_i$  contains a copy of  $\mathcal{A}$  and hence, since the only matrix Wedderburn component in  $\mathbb{Q}G$  is  $M_2(\mathbb{Q})$ ,  $\mathbb{Q}Gf_i \simeq M_2(\mathbb{Q})$ . Also, there is exactly one index  $i$ , such that  $ef_i$  is non-zero; for, otherwise, there will be more than one Wedderburn component in  $\mathbb{Q}G$  which is  $M_2(\mathbb{Q})$ , which is not allowed by  $\mathfrak{3}^0$ . So  $e = ef_i$  and  $\mathbb{Q}He = \mathbb{Q}Ge$  for a central primitive idempotent  $e$  in  $H$ . We conclude from this that  $H = G$ . For this, let  $g \in G$ ; thus  $ge \in \mathbb{Q}He$ . Since the trace of a primitive central idempotent in a group ring, i.e., the coefficient of 1, is non-zero,  $g$  belongs to the support of  $ge$ , which is contained in  $H$ . Thus  $G$  is contained in  $H$ , and hence  $G = H$ .  $\square$

**Corollary 1.** *Let  $G$  be a finite non-Hamiltonian subgroup of  $\Gamma$ . Then every Wedderburn component of  $\mathbb{Q}G$  is either  $\mathbb{Q}$ , or an imaginary quadratic extension of  $\mathbb{Q}$ , or  $M_2(\mathbb{Q})$  or a totally definite quaternion algebra over  $\mathbb{Q}$ . Furthermore,  $G$  has a subgroup of index 2.*

*Proof.* By the previous lemma there exists a unique Wedderburn component of the form  $M_2(\mathbb{Q})$ . Let  $\mathcal{A}$  be any other Wedderburn component. Then  $\mathcal{A}$  is a division ring. Let  $K$  be a maximal subfield of  $\mathcal{A}$ . Then the unit group of the ring of integers of  $K$  must be finite, and hence  $K$  is at most a quadratic extension of  $\mathbb{Q}$ . So if  $\mathcal{A}$  is non-commutative, then its centre must equal  $\mathbb{Q}$ , and hence  $\dim_{\mathbb{Q}}(\mathcal{A}) = 4$ , and thus  $\mathcal{A}$  is a totally definite quaternion algebra over  $\mathbb{Q}$ .

Considering the complex group algebra  $\mathbb{C}G$  and using what we proved above about the Wedderburn components of  $\mathbb{Q}G$ , we see that  $\mathbb{C}G$  is the direct sums of copies of  $\mathbb{C}$  and two-by-two matrices over  $\mathbb{C}$ . Hence, by Corollary 12.9 of [2],  $G$  contains a subgroup of index 2.  $\square$

We note that if  $G$  is a finite non-Hamiltonian subgroup of  $\Gamma$  (and therefore has a Wedderburn component that is not a division algebra), then  $\mathbb{Z}G$  does not contain a central unit of infinite order; for, if  $\alpha$  is such an element, then for a  $\theta$  in  $\mathbb{Q}G$ , a non-zero nilpotent element with  $\theta^2 = 0$ ,  $\langle \alpha, 1 + \theta \rangle \simeq \mathbb{Z}^2$ , a contradiction to Theorem 1(a). Hence central units of  $\mathbb{Z}G$  are trivial. The following result extends this observation to arbitrary subgroups of  $\Gamma$ .

**Lemma 2.** *Let  $G$  be any subgroup of  $\Gamma$  such that  $\mathcal{U}_1(\mathbb{Z}[G])$  contains a central unit of infinite order. If  $G$  is finite, then it is Abelian; if  $G$  is infinite and contained in  $\mathcal{G}$ , then  $G = \mathcal{U}_1(\mathbb{Z}G)$  and  $G$  is centre-by-finite.*

*Proof.* If  $G$  is finite, then by Lemma 1 and the above observation  $G$  must be Abelian.

Suppose that  $G$  is an infinite subgroup of  $\mathcal{G}$  and  $\alpha$  is a central unit in  $\mathbb{Z}G$  of infinite order. Then, since  $\Gamma$  is hyperbolic, by Theorem 1(b) we have that  $[\mathcal{U}_1(\mathbb{Z}G) : \langle \alpha \rangle] < \infty$ . It follows that  $G$  is an elementary hyperbolic group with a central element  $g_0$ , say, of infinite order. If  $0 \neq \theta$  is a nilpotent element in  $\mathbb{Z}G$ , then  $\langle 1 + \theta, g_0 \rangle \simeq \mathbb{Z}^2$ , since the Kaplansky trace of a nilpotent element is zero. Hence  $\mathbb{Z}G$  has no non-zero nilpotent elements. For an element  $x$  of  $G$  of order  $d$ , and an arbitrary element  $g$  of  $G$ , the element  $(1 - x)g(1 + x + \cdots + x^{d-1})$  is nilpotent, hence is zero. From this we see that the element  $g$  normalizes the cyclic subgroup generated by  $x$  for every torsion element  $x$  in  $G$ . Thus the torsion elements  $T(G)$  of  $G$  form a subgroup of  $G$ , which is finite by Theorem 1(c). Furthermore, since all elements of  $\mathcal{U}_1(\mathbb{Z}T(G))$  will have to be of finite order,  $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$ . Since  $G/T(G)$  is ordered it follows from ([14], Proposition 45.5, p. 277) that  $\mathcal{U}_1(\mathbb{Z}G) = (\mathcal{U}_1(\mathbb{Z}T(G)))G = G$ .  $\square$

**Lemma 3.** *Let  $G$  be any group, and let  $x, y \in G$  be such that  $\langle x \rangle \cap \langle y \rangle = 1$ ,  $o(x) < \infty$ ,  $o(y) \geq 5$  and  $x^y \notin \langle x \rangle$ . Then  $\mathbb{Z}^2$  embeds into  $\mathcal{U}_1(\mathbb{Z}G)$ .*

*Proof.* Define  $\hat{x} := 1 + x + \dots + x^{n-1}$ ,  $n = o(x)$ , and  $\theta_k = (1 - x)y^k\hat{x}$ ,  $1 \leq k \leq o(y)$ . Now observe that, under the given hypothesis, it is possible to choose  $k$  such that  $\langle 1 + \theta_1, 1 + \theta_k \rangle$  is a subgroup of  $\mathcal{U}_1(\mathbb{Z}G)$  and isomorphic to  $\mathbb{Z}^2$ .  $\square$

**Lemma 4.** *If  $\Gamma$  has an element of order 5, then  $\mathcal{G} \simeq C_5$ .*

*Proof.* By general properties of elements of prime order in a group ring (see [14], Theorem 45.11, p. 278), if  $\Gamma$  has an element of order 5, so does  $\mathcal{G}$ . Let  $x \in \mathcal{G}$  be such an element.

Suppose first that  $\mathcal{G}$  is non-Abelian. If  $x$  were central, then  $\Gamma$  would have a central element  $\alpha$  of infinite order, with support in  $\langle x \rangle$ , and hence, by Lemma 2,  $\mathcal{G}$  would have an element  $g_0$  of infinite order. Since  $\langle \alpha, g_0 \rangle \simeq \mathbb{Z}^2$ , we have a contradiction. On the other hand, if  $\langle x \rangle$  were normal but not central, then there would exist an element  $g_0 \in \mathcal{G}$  with  $o(g_0) \in \{2, 4, 8, \infty\}$  such that  $\langle x \rangle \rtimes \langle g_0 \rangle$  is a subgroup of  $G$ . In view of  $4^0$ ,  $o(g_0) = \infty$  and with  $\alpha$  as above, we have  $\langle \alpha, g_0^4 \rangle \simeq \mathbb{Z}^2$ , again a contradiction. Hence there must exist another element  $y \in \mathcal{G}$  of order 5 that does not commute with  $x$ ; but then  $x$  and  $y$  satisfy the conditions of Lemma 3, which is a contradiction.

Next suppose that  $\mathcal{G}$  is Abelian. Then, as seen above,  $\mathcal{G}$  must be torsion and hence finite. From rank considerations of  $\Gamma$ , it is clear that  $\mathcal{G} \simeq C_5$ .  $\square$

**Lemma 5.** *If  $\mathcal{G}$  is a non-torsion group, then  $T(\mathcal{G})$  is a finite Hamiltonian group and  $\mathcal{U}_1(\mathbb{Z}T(\mathcal{G})) = T(\mathcal{G})$ . Moreover, the primitive central idempotents of  $\mathbb{Q}T(\mathcal{G})$  are central in  $\mathbb{Q}\mathcal{G}$ .*

*Proof.* Let  $x, y \in \mathcal{G}$ , with  $o(x) < \infty = o(y)$ . Then, by Lemma 3, we must have that  $x^y \in \langle x \rangle$ . Since the orders of the torsion elements of  $\mathcal{G}$  divide 8 or 12, it follows that  $y^4$  must centralize  $T(\mathcal{G})$ , the set of torsion elements of  $\mathcal{G}$ . Let  $y_0 = y^4$ , and let  $z \in \mathcal{G}$  be any other torsion element. If  $z$  does not normalize  $\langle x \rangle$ , then  $(1 - x)z\hat{x}$  and  $y_0(1 - x)z\hat{x}$  are  $\mathbb{Q}$ -linearly independent commuting nilpotent elements, and so  $\mathbb{Z}^2$  embeds into  $\Gamma$ , a contradiction. It follows that  $T(\mathcal{G})$  is a subgroup, is locally finite, and hence, since  $\Gamma$  is hyperbolic, it is finite. If  $\mathcal{U}_1(\mathbb{Z}T(\mathcal{G}))$  is not trivial, then, since  $y_0 \in \mathcal{G}$  has infinite order, we can embed  $\mathbb{Z}^2$  into  $\Gamma$ . Finally, let  $e \in \mathbb{Q}T(\mathcal{G})$  be a central primitive idempotent that is not fixed by  $g \in \mathcal{G}$ . Then  $o(g) = \infty$  and so  $\langle 1 + eg, 1 + y_0eg \rangle \simeq \mathbb{Z}^2$ , a contradiction.  $\square$

**Lemma 6.** *Let  $G$  be a finite non-Abelian subgroup of  $\Gamma$ . Then  $\exp(G)$  divides 12 and  $G = \langle H, x \rangle$ , where  $H$  is a subgroup of index 2 and  $x$  is a 2-element. Furthermore, (i) if 3 divides  $|G|$ , then  $G$  is isomorphic either to  $S_3$  or to  $Q_{12}$ ; (ii) if  $G$  is a 2-group having a non-central element of order 2, then  $G \simeq D_4$ .*

*Proof.* In view of Corollary 1, observe that  $G$  has a subgroup  $H$  of index 2 that is either Abelian or a Hamiltonian 2-group and its order is not divisible by 5. So we may choose a 2-element  $x \in G$  such that  $x^2 \in H$  and  $G = \langle H, x \rangle$ .

Suppose that 3 divides  $|G|$ ; then  $H$  must be Abelian. If  $Syl_3(G)$ , the Sylow 3-subgroup of  $G$ , were central, then, by Lemma 1,  $G = Syl_3(G) \times Syl_2(G)$  with  $Syl_2(G)$  a Hamiltonian 2-group, and so  $G \simeq C_3 \times K_8$ , a contradiction to  $4^0$ . Hence there exists  $a \in H$  of order 3 such that  $a^x \neq a$ . Then, since clearly  $(a^{-1}a^x)^x = (a^{-1}a^x)^{-1}$ , we have that  $G \simeq C_3 \rtimes \langle x \rangle$ . We only need to rule out the case when

$o(x) = 8$ . In this case  $\mathbb{Q}(G/G') \simeq \mathbb{Q}C_8$ , and so  $\mathcal{U}_1(\mathbb{Z}G)$  has a central element of infinite order. Therefore, by Lemma 2,  $G$  cannot be a subgroup of  $\Gamma$ .

Suppose next that  $G$  is a 2-group and  $y \in G$  is a non-central element of order 2. Suppose first that  $y \in H$ . Then, by Lemma 1,  $G = \langle x, y \rangle$ . Since  $x^2 \in H$ , it follows that  $[y, x^2] = 1$  and so  $G' = \langle yy^x \rangle$ . Since such a group does not possess a non-Abelian quotient that embeds in a non-commutative division ring, it follows by  $\mathbf{3}^0$  that  $\dim_{\mathbb{Q}}(\Delta(G, G')) = 4$  and so  $|G| = 8$ ; it thus follows that  $G \simeq D_4$ . Suppose that  $y \notin H$ . Then  $G = \langle H, y \rangle$ . Choose  $a \in H$  such that  $[a, y] \neq 1$ . Then  $G = \langle a, y \rangle$ . Note that  $z := a^{-1}a^y$  is inverted by  $y$ . If it is not fixed, then  $G = \langle z, y \rangle = \langle z \rangle \rtimes \langle y \rangle$ . On the other hand, if  $z$  is fixed, then  $G' = \langle z \rangle$  and the same argument as given above shows that  $|G| = 8$ . Thus in any case it follows, by  $\mathbf{3}^0$ , that  $G \simeq D_4$ .

It remains to show that  $G$  has no element of order 8. Suppose  $y \in G$  is an element of order 8. Then, in view of the previous paragraph, elements of order two are central. Let  $z$  be such an element. If  $z \notin \langle y \rangle$ , then  $C_8 \times C_2$  would embed in  $G$ , which is not the case. Hence  $z \in \langle y \rangle$ , and thus  $z = y^4$ . Hence  $G$  has a unique element of order 2. Since  $G$  is not Abelian, it follows that  $G$  is isomorphic to  $Q_{16}$ , which is ruled out by  $\mathbf{4}^0$ .  $\square$

We are now ready to present our main results.

**Theorem 2.** *Let  $G$  be a finite non-Hamiltonian group. Then the following are equivalent:*

- (1) *Exactly one Wedderburn component of  $\mathbb{Q}G$  is  $M_2(\mathbb{Q})$ , and any other component is either  $\mathbb{Q}$ , or an imaginary quadratic extension of  $\mathbb{Q}$  or a totally definite quaternion algebra over  $\mathbb{Q}$ .*
- (2)  *$G$  has a normal free complement in  $\mathcal{U}_1(\mathbb{Z}G)$ .*
- (3)  *$\mathcal{U}_1(\mathbb{Z}G)$  is virtually free.*
- (4)  *$\mathcal{U}_1(\mathbb{Z}G)$  is hyperbolic.*

*Moreover, if one of the above conditions holds, then every finitely generated torsion-free subgroup of  $\mathcal{U}_1(\mathbb{Z}G)$  is free. In particular, any normal torsion-free complement of  $G$  in  $\mathcal{U}_1(\mathbb{Z}G)$  is free.*

*Proof.* (1)  $\Rightarrow$  (2) : From (1) it easily follows that  $c.d.(G)$ , the set of complex character degrees of  $G$ , is  $\{1, 2\}$ ; hence, by Corollary 12.9 of [2],  $G$  is metabelian. Furthermore, (1) also implies that  $\mathbb{Q}(G/G')$  is a direct sum of copies of  $\mathbb{Q}$  and imaginary quadratic fields. Hence the exponent of  $G/G'$  divides 4 or 6, and so  $\mathcal{U}_1(\mathbb{Z}(G/G'))$  is trivial. Therefore  $F := \mathcal{U}_1(\mathbb{Z}G) \cap (1 + \Delta(G)\Delta(G'))$ , which is known to be torsion-free (see [14]; for a more general result see [9]), is a complement of  $G$  in  $\mathcal{U}_1(\mathbb{Z}G)$ . Since  $SL(2, \mathbb{Z})$  contains a free group of rank 2 as a subgroup of finite index, it easily follows that  $F$  is quasi-isometric to a free group and so, by Theorem 1(e),  $F$  is a free group.

(2)  $\Rightarrow$  (3) : This implication is trivial, since  $G$  is finite.

(3)  $\Rightarrow$  (4) : This is a consequence of the fact that a free group is hyperbolic and hyperbolicity is stable under quasi-isometry.

(4)  $\Rightarrow$  (1) : This follows from Corollary 1.

Finally, if  $H$  is a finitely generated torsion-free subgroup of  $\mathcal{U}_1(\mathbb{Z}G)$  and  $F$  is a free subgroup of finite index in  $\mathcal{U}_1(\mathbb{Z}G)$ , then  $H \cap F$  is a finitely generated free subgroup of finite index in  $H$ . Hence, once again by Theorem 1(e), we have that  $H$  is free.  $\square$

The following result characterizes the torsion groups that can occur as subgroups of hyperbolic unit groups.

**Theorem 3.** *If a torsion group  $G$  embeds into a hyperbolic unit group, then  $G$  must be finite and isomorphic to one of the following groups:*

- (1)  $C_5, C_8, C_{12}$ , an Abelian group of exponent dividing 4 or 6;
- (2) a Hamiltonian 2-group;
- (3)  $S_3, D_4, Q_{12}, C_4 \rtimes C_4$ .

*Conversely, all of the groups listed above have hyperbolic unit groups.*

*Proof.* Since a torsion subgroup of a hyperbolic group is finite,  $G$  must be so.

Suppose first that  $G$  is Abelian. Write

$$\mathbb{Q}G = \bigoplus a_d \mathbb{Q}(\xi_d),$$

where  $a_d = \frac{n_d}{\phi(d)}$ ,  $n_d =$  the number of elements of order  $d$ , and  $\phi$  is the Euler phi-function. Then  $\mathcal{U}_1(\mathbb{Z}G)$  is hyperbolic if and only if its torsion-free rank is at most one, i.e.,

$$\sum_d a_d \left( \frac{\phi(d)}{2} - 1 \right) \leq 1.$$

Hence either  $\mathcal{U}_1(\mathbb{Z}G)$  is finite, and so  $G$  has exponent dividing 4 or 6, or there exists a unique integer  $d$  such that  $\phi(d) = 4$  and  $a_d = 1$ . It then follows that  $G \in \{C_5, C_8, C_{12}\}$ .

Clearly, if  $G$  is one of these groups, then the unit group of  $\mathcal{U}_1(\mathbb{Z}G)$  is either trivial or has torsion-free rank equal to one and so is hyperbolic.

Suppose  $G$  is non-Hamiltonian. In view of Lemma 6, we can suppose that  $G$  is a 2-group of order at least 16 in which all elements of order 2 are central.

Since  $G$  is non-Hamiltonian, we may choose  $a, x \in G$  such that  $a^x \notin \langle a \rangle$  and so, by Lemma 1,  $G = \langle a, x \rangle$  and  $[a, x] \neq a^2$ . If  $x^a \in \langle x \rangle$ , then  $G \simeq C_4 \rtimes C_4$ . So we also suppose that  $x^a \notin \langle x \rangle$ . Let  $\overline{G} = G/\langle a^2 \rangle$ ; then the Wedderburn components of  $\mathbb{Q}\overline{G}$  are among those of  $\mathbb{Q}G$ . Hence, by Theorem 2,  $\overline{G}$  embeds into a hyperbolic unit group. Since the image of  $a$  in  $\overline{G}$  is not central, it follows, by Lemma 6, that  $\overline{G} \simeq D_4$ . So  $G$  has order 16 and we may choose a non-central element  $y \in G$  whose image has order 4 in  $\overline{G}$ . We still have that  $G = \langle y, x \rangle$  and either  $[y, x] = y^2$  or  $[y, x] = y^2 x^2$ . In particular,  $[y, x] = [x, y]$ . If  $[y, x] = y^2$ , then  $G \simeq C_4 \rtimes C_4$ . On the other hand, if  $[y, x] = y^2 x^2$ , then  $xy$  would be a non-central element of order 2 and so, by Lemma 6,  $G$  would have order 8, which is not the case.

If  $G$  is Hamiltonian, then, by Lemma 6,  $G$  is a 2-group.

For the converse, since the unit group of a Hamiltonian 2-group is trivial, it only remains to show that the groups in (3) have hyperbolic unit groups. To do so, we prove that for all these groups Theorem 2 (1) is satisfied. Indeed, for all of them we have that  $G/G'$  has exponent dividing 4, and so  $\mathbb{Q}(G/G')$  is a direct sum of copies of  $\mathbb{Q}$  and  $\mathbb{Q}(\sqrt{-1})$ . Also, for all of them,  $\dim_{\mathbb{Q}}(\Delta(G, G')) \leq 8$ . Each of the groups  $Q_{12}$  and  $C_4 \rtimes C_4$  embeds into a division ring (see [15], Theorem 2.1.5, p. 47). Because of the limitation on the dimension, these division rings are four dimensional over  $\mathbb{Q}$  and hence are totally definite quaternion algebras, and the proof is complete by Theorem 2. □

We next characterize the infinite polycyclic-by-finite groups  $G$  that embed in a group  $\mathcal{G}$  whose unit group is hyperbolic.

**Theorem 4.** *An infinite polycyclic-by-finite group  $G$  embeds into a group  $\mathcal{G}$  whose unit group  $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$  is hyperbolic if and only if*

- (1)  $T(G)$ , the set of elements of finite order in  $G$ , is a subgroup of  $G$ ;
- (2)  $G \simeq T(G) \rtimes \mathbb{Z}$ ;
- (3)  $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$ .

*Proof.* Let  $G$  be infinite and polycyclic-by-finite; then  $G$  has an element of infinite order and so, by Lemma 5,  $T(G)$  is a finite subgroup and  $\mathcal{U}_1(\mathbb{Z}(T(G))) = T(G)$ . Since  $G$  is polycyclic-by-finite, it contains a normal free Abelian subgroup  $A$ , say. Since  $\mathcal{U}_1(\mathbb{Z}\mathcal{G})$  is hyperbolic, it follows that  $A = \langle x \rangle$  is cyclic. Applying Theorem 1 (b), it is easy to see that  $[G : \langle x \rangle] < \infty$  and so  $G$  has Hirsch length one and thus,  $G \simeq T(G) \rtimes \mathbb{Z}$ . Since the hypothesis of ([14], Proposition 45.5, p. 277) is satisfied, we conclude that  $\mathcal{U}_1(\mathbb{Z}\mathcal{G}) = (\mathcal{U}_1(\mathbb{Z}(T(G))))G = (T(G))G = G$ . The converse being trivial, the proof is complete.  $\square$

Finally, we give a complete characterization of the groups  $\mathcal{G}$  whose unit group  $\Gamma$  is finitely generated virtually free, or equivalently the boundary  $\partial(\Gamma)$  has dimension zero [4]. For convenience we adopt the following:

**Definition 1.** A group  $\mathcal{G}$  is called a  $*$ -group if one of the following conditions holds.

- (1)  $\mathcal{G}$  is a finite Abelian group of exponent dividing 4 or 6.
- (2)  $\mathcal{G}$  is a finite Hamiltonian 2-group.
- (3)  $\mathcal{G} \in \{C_5, C_8, C_{12}, S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$ .
- (4)  $\mathcal{G} = H \rtimes F$ , where  $H$  is of type (1) or (2) above and  $F$  is a finitely generated free group.

**Theorem 5.** *The unit group  $\Gamma = \mathcal{U}_1(\mathbb{Z}G)$  of a group  $G$  is finitely generated virtually free if and only if  $G$  is a  $*$ -group. Furthermore, in case  $G$  is infinite,  $\Gamma = G$ .*

*Proof.* Suppose that  $\Gamma$  is finitely generated virtually free, i.e., there exist a finitely generated free group  $F$  contained in  $\Gamma$  of finite index. Then  $F$  is finitely generated and  $\Gamma$  is hyperbolic. Hence if  $G$  is a finite group, then by Theorem 3,  $G$  is of type (1), (2) or (3). So suppose that  $G$  is infinite. By  $\mathfrak{S}^0$ , it follows that  $G \cap F$  is a finitely generated free subgroup of  $G$  of finite index in  $G$ , and thus  $G$  is virtually free and so is hyperbolic. Since  $G$  must necessarily be non-torsion, therefore, by Lemma 5, we have that  $T(G)$  is finite and  $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$ . The quotient group  $H := G/(T(G))$  is quasi-isometric to the free group  $G \cap F$ , both being quasi-isometric to  $G$ ; therefore, by Theorem 1 (e),  $H$  is virtually free. Since  $H$  is torsion-free, it follows that  $H$  is free. Hence  $G \simeq T(G) \rtimes H$ . Since  $H$  is ordered and  $\mathcal{U}_1(\mathbb{Z}T(G)) = T(G)$ , it follows from Proposition 45.5 of [14] that  $\Gamma = G$ . So in any case  $G$  is a  $*$ -group.

The converse follows by the previous results and ([14], Prop. 45.5).  $\square$

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