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HYPERBOLIC RANK OF PRODUCTS

THOMAS FOERTSCH AND VIKTOR SCHROEDER

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Abstract. Generalizing a result of Brady and Farb (1998), we prove the existence of a bilipschitz embedded manifold of negative curvature bounded away from zero and dimension $m_1 + m_2 - 1$ in the product $X := X_1^{m_1} \times X_2^{m_2}$ of two Hadamard manifolds $X_i^{m_i}$ of dimension m_i with negative curvature bounded away from zero.

Combining this result with a result of Buyalo and Schroeder (2002), we prove the additivity of the hyperbolic rank for products of manifolds with negative curvature bounded away from zero.

1. Introduction

In [BrFa] the authors proved that the product $X := H^{m_1} \times ... \times H^{m_k}$ of hyperbolic spaces H^{m_i} admits an embedding of $Y := H^{m_1+...+m_k-k+1}$ in X that is quasi-isometric in the sense that the Riemannian distance functions d_X and d_Y on X and Y are related via

$$d_X|_Y \ \leq d_Y \leq \ \alpha \cdot d_X|_Y \ + \ \beta,$$

with constants $\alpha, \beta \in \mathbb{R}^+$, $\alpha > 1$.

In [Foe] this was generalized to the case of Riemannian products of certain types of warped products that were shown to admit bilipschitz embeddings of warped products of negative sectional curvature.

In this paper we further generalize this observation by proving the

Theorem 1. *Let $(X_i^{m_i}, g_i)$, $i = 1, ..., k$, be Hadamard manifolds of strictly negative sectional curvature $K(X_i^{m_i}, g_i) \leq -\kappa$ (pinched negative curvature $-\kappa_2 \leq K(X_i^{m_i}, g_i) \leq -\kappa_1$, resp.). Then their Riemannian product $(X = \prod_{i=1}^k X_i^{m_i}, g_X)$ admits a bilipschitz embedding of a Hadamard manifold (Y^n, g_Y^λ) of strictly negative sectional curvature $K^{(Y,\lambda)} \leq -\kappa$ (pinched negative sectional curvature $-\kappa_2 \leq K^{(Y,\lambda)} \leq -\kappa_1$, resp.) and dimension $n := m_1 + ... + m_k - k + 1$.*

The existence result of Theorem 1 implies the additivity of the hyperbolic rank for products of Hadamard manifolds with strictly negative curvature bounded away from zero (compare [BuySch]).

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Given a metric space X consider all locally compact $CAT(-1)$ Hadamard spaces Y quasi-isometrically embedded into X, and let

$$rank_h X \ := \sup_Y \ dim\, \partial_\infty Y$$

over all such Y, where $dim \partial_\infty Y$ is the topological dimension of the boundary of Y. This invariant was introduced by Gromov ([Gr]) in a slightly stronger form, where he called it the hyperbolic corank. As explained in [BuySch] we will call it the hyperbolic rank.

Combining Theorem 1 with the result of [BuySch] we obtain the

Theorem 2. *Let $X = X_1 \times ... \times X_k$ be the product of Hadamard manifolds with strictly negative curvature bounded away from zero. Then*

$$rank_h X \ = \ \sum_{i=1}^{k} rank_h X_i.$$

2. The level set product construction

It is convenient to have the following

Definition 1. Let (X, g) be a Riemannian manifold. A function $f : X \longrightarrow \mathbb{R}$ is called a distance function on X if and only if it satisfies $||grad(f)|| \equiv 1$.

Given a Riemannian manifold (X, g) with a distance function f the level sets $H^s := \{p \in M \mid f(p) = s\}$ are hypersurfaces of X.

We fix $p \in H^s$. Note that local coordinates $\hat{x}^J$ of H^s around p and $grad(f)$ naturally induce local coordinates (t, x^J) of X around $p \in H^s \subset X$. With respect to these coordinates the metric g is of the form

$$g \ = \ dt^2 \ + \ g_{ij}(x^J)\, dx^i dx^j. \tag{1}$$

Definition 2. Given two Riemannian manifolds (X_i, g_i) with distance functions $f_i : X_i \longrightarrow \mathbb{R}$, $i = 1, 2$, their level set product $(Y, g_y) = (X_1, g_1, f_1) \times_l (M_2, g_2, f_2)$ is defined in the following way:

$$Y \ := \ \Big\{ (p, q) \in X_1 \times X_2 \ \Big| \ f_1(p) \ = \ f_2(q) \Big\},$$

with $g_Y := i^* g$ being the metric tensor on Y induced by the natural embedding $i : Y \longrightarrow X_1 \times X_2$, where g denotes the Riemannian product metric on $X_1 \times X_2$.

We fix $(p, q) \in H_1^s \times H_2^s$ and take local coordinates (t_1, x^J) around p in X_1 and (t_2, x^L) around q in X_2 as above. In the following the indices i, j and J will always refer to coordinates in X_1, while the indices k, l and L refer to those in X_2. With respect to these coordinates the metrics g_1 and g_2 are of the form

$$g_1 \ = \ dt_1^2 \ + \ g_{ij}(x^J)\, dx^i dx^j \quad \text{ and } \quad g_2 \ = \ dt_2^2 \ + \ g_{kl}(x^L)\, dx^k dx^l.$$

Obviously the coordinates (t_1, x^J) and (t_2, x^L) induce coordinates around (p, q) in Y, which we again denote by (t, x^J, x^L), where no confusion about this slight abuse of notation should arise.

With respect to these coordinates, the metric g_Y is written

$$g_Y \ = \ 2dt^2 \ + \ g_{ij}(t, x^J)\, dx^i dx^j \ + \ g_{kl}(t, x^L)\, dx^k dx^l. \tag{2}$$

More generally, we will be interested in the one-parameter family g_Y^λ, $\lambda \in \mathbb{R}$, of metrics on Y given in the local coordinates on Y as above via

$$g_Y^\lambda \;=\; \lambda^2 dt^2 \;+\; g_{ij}(t,x^J)\,dx^i dx^j \;+\; g_{kl}(t,x^L)\,dx^k dx^l. \tag{3}$$

For these metrics we calculate the sectional curvatures $K(Y,\lambda)$ as follows:

In the following we do not distinguish elements in $T_pH_1^s$ or $T_qH_2^s$ from their lifts to $T_{(p,q)}(H_1^s \times H_2^s)$ or $T_{(p,q)}Y$. Thus b_1 and c_1 are either in T_pH_1, $T_{(p,q)}(H_1^s \times H_2^s)$ or $T_{(p,q)}Y$ and similarly for b_2 and c_2.

Now let $a \in span\{\frac{\partial}{\partial t}|_{(p,q)}\}$. Then the sectional curvature of a tangent plane $\Pi = span\{a+b_1+b_2, c_1+c_2\}$, generated by the g_Y^λ-orthonormal vectors $a+b_1+b_2$ and c_1+c_2, is given through

(4)
$$\begin{aligned} K^{(Y,\lambda)}(\Pi) =& K^{(Y,\lambda)}(a+b_1,c_1)\;||(a+b_1)\wedge c_1||^2_{(Y,\lambda)} \\ &+\; K^{(Y,\lambda)}(a+b_2,c_2)\;||(a+b_2)\wedge c_2||^2_{(Y,\lambda)} \\ &+\; \frac{\tilde{R}^{(Y,\lambda)}(b_1\wedge c_2 - c_1\wedge b_2, b_1\wedge c_2 - c_1\wedge b_2)}{||b_1\wedge c_2 - c_1\wedge b_2||^2_{(Y,\lambda)}}\;||b_1\wedge c_2 - c_1\wedge b_2||^2_{(Y,\lambda)}, \end{aligned}$$

where $||x\wedge y||^2_{(Y,\lambda)} = G_Y^\lambda(x\wedge y, x\wedge y)$ with

$$G_Y^\lambda(x\wedge y, v\wedge w) \;=\; g_Y^\lambda(x,v)g_Y^\lambda(y,w) - g_Y^\lambda(x,w)g_Y^\lambda(y,v) \quad \forall x,y,v,w \in T_{(p,q)}Y,$$

the usual extension of $\langle\cdot,\cdot\rangle_{(Y,\lambda)}$ to bivectors, and $\tilde{R}^{(Y,\lambda)}$ is the bilinear form on bivectors $\tilde{R}^{(Y,\lambda)} : \Lambda^2_{(p,q)}Y \times \Lambda^2_{(p,q)}Y \longrightarrow \mathbb{R}$ defined via

$$\tilde{R}^{(Y,\lambda)}(x\wedge y, v\wedge w) \;:=\; \langle R^{(Y,\lambda)}_{xy}v, w\rangle_{(Y,\lambda)} \qquad \forall x,y,v,w \in T_{p,q}(Y).$$

Note that $a+b_1+b_2$ and c_1+c_2 being g_Y^λ-orthonormal yields

$$||(a+b_1)\wedge c_1||^2_{(Y,\lambda)} \;+\; ||(a+b_2)\wedge c_2||^2_{(Y,\lambda)} \;+\; ||b_1\wedge c_2 - c_1\wedge b_2||^2_{(Y,\lambda)} \;=\; 1. \tag{5}$$

Next we examine the third term on the right-hand side of equation (4):

We write $II : T_{(p,q)}(H_1\times H_2)\times T_{(p,q)}(H_1\times H_2) \longrightarrow T_{(p,q)}(span\{\frac{\partial}{\partial t}|_{(p,q)}\})$ for the second fundamental form tensor of $H_1\times H_2$ in Y and $S : T_{(p,q)}(H_1\times H_2) \longrightarrow T_{(p,q)}(H_1\times H_2)$ for the shape operator defined via

$$\langle S(x), y\rangle_Y \;=\; \langle II(x,y), \frac{1}{\lambda}\frac{\partial}{\partial t}|_{(s,p,q)}\rangle_Y \qquad \forall\, x,y \in T_{(s,p,q)}(H_1\times H_2).$$

We note that with $II(b_1,b_2)=0$ for all $b_m \in T_pH_m$, $m=1,2$, it follows that

$$S|_{T_{(p,q)}H_m} : T_{(p,q)}H_m \longrightarrow T_{(p,q)}H_m, \qquad m=1,2.$$

Next we define the symmetric bilinear form

$$\begin{aligned} \sigma : T_{(p,q)}(H_1\times H_2)\times T_{(p,q)}(H_1\times H_2) \;&\longrightarrow\; \mathbb{R} \\ (b_1+b_2, c_1+c_2) \;&\longmapsto\; \langle S(b_1), c_1\rangle_Y \;+\; \langle S(b_2), c_2\rangle_Y, \end{aligned}$$

and denote the second fundamental form tensors of H_m in X_m by II^m and the corresponding shape operators by S^m, $m = 1, 2$. With that we find

$$\begin{aligned}
\langle S(b_m), c_m\rangle_Y &= \langle II(b_m, c_m), \frac{1}{\lambda}\frac{\partial}{\partial t}|_{(s,p,q)})\rangle_Y \\
&= \frac{1}{\lambda^2}\langle II^m(b_m, c_m), \frac{1}{\lambda}\frac{\partial}{\partial t}|_{(s,p,q)}\rangle_Y \\
&= \frac{1}{\lambda}\langle II^m(b_m, c_m), \frac{\partial}{\partial t}|_{(s,p,q)}\rangle_m \\
&= \frac{1}{\lambda}\langle S^m(b_m), c_m\rangle_m. \qquad (6)
\end{aligned}$$

Now let $\Sigma : \Lambda^2_{(p,q)}(H_1 \times H_2) \times \Lambda^2_{(p,q)}(H_1 \times H_2) \longrightarrow \mathbb{R}$ denote the extension of σ to bivectors and calculate for $\lambda = 1$,

$$\begin{aligned}
\Sigma\Big(b_1 \wedge c_2 - c_1 \wedge b_2, b_1 \wedge c_2 - c_1 \wedge b_2\Big) = \quad & \langle S(b_1), b_1\rangle_Y \langle S(c_2), c_2\rangle_Y \\
& + \langle S(c_1), c_1\rangle_Y \langle S(b_2), b_2\rangle_Y \\
& - 2\langle S(b_1), c_1\rangle_Y \langle S(b_2), c_2\rangle_Y.
\end{aligned}$$

By using the Gauss equation we thus obtain

$$\tilde{R}^Y\Big(b_1\wedge c_2 - c_1\wedge b_2, b_1\wedge c_2 - c_1\wedge b_2\Big) = -\Sigma\Big(b_1\wedge c_2 - c_1\wedge b_2, b_1\wedge c_2 - c_1\wedge b_2\Big). \qquad (7)$$

With this in mind it is easy to prove the following

Proposition 1. *Let (X_m, g_m) be Riemannian manifolds and let $f_i : X_i \longrightarrow \mathbb{R}$ be distance functions on the X_m, $m = 1, 2$. Suppose that the following hold:*

1. *$K^m \leq -\kappa < 0$ ($K^m \geq -\kappa$, resp.), $m = 1, 2$, and*
2. *the level sets H_m lie κ-convex (κ-concave, resp.) in X_m, $m = 1, 2$, i.e., the second fundamental form tensors II^m of H_m in X_m satisfy*

$$\frac{g_m\Big(II^m(b_m, b_m), -grad(f)\Big)}{g_m(b_m, b_m)||grad(f)||} \geq \sqrt{\kappa} \quad (\leq \sqrt{\kappa},\ resp.) \quad \forall\, b_m \in TH_m,\ m = 1, 2.$$

Then the metric g_Y of the level set product

$$(Y, g_Y) := \Big(X_1, g_1, f_1\Big) \times_l \Big(X_2, g_2, f_2\Big)$$

is bilipschitz related to a metric g_Y^λ on Y that carries sectional curvature $K^{(Y,\lambda)} \leq -\kappa$ ($K^{(Y,\lambda)} \geq -\kappa$, resp.).

Proof. S is the endomorphism on $(T_{(p,q)}(H_1 \times H_2), g_Y)$ associated to the bilinear form σ. We denote the corresponding endomorphism on $(\Lambda^2_{(p,q)}(H_1 \times H_2), G_Y)$ associated to the bilinear form Σ by $\hat{\Sigma}$. Note that the eigenvectors u_n, $n = 1, ..., dim(X_1) + dim(X_2) - 2$, are tangential to either H_1 or H_2 and are also eigenvectors of either S^1 or S^2. One can show that for a basis of eigenvectors $\{u_n\}$ of S with eigenvalues μ_n, the vectors $\{u_n \wedge u_{\tilde{n}}\}_{n<\tilde{n}}$ form a basis of eigenvectors of $\hat{\Sigma}$ with eigenvalues $\mu_n \cdot \mu_{\tilde{n}}$.

Now consider the metric as in equation (3) with $\lambda = 1$. This metric is obviously bilipschitz related to g_Y and carries sectional curvature $\leq -\kappa$ ($\geq -\kappa$, resp.) due to the equations (4), (5), (6) and (7). $\square$

3. The proof of Theorem 1

The first step of the proof of Theorem 1 is the following corollary of Proposition 1:

Corollary 1. *Let (X_m, g_m) be Hadamard manifolds of sectional curvature $K^m \leq -\kappa$ ($-\kappa_2 \leq K^m \leq -\kappa_1$ resp.) and let $B_m : X_m \longrightarrow \mathbb{R}$ be Busemann functions on the X_m, $m = 1, 2$. Then the metric g_Y of their level distance product*

$$(Y, g_Y) \ := \ \Big(X_1, g_1, B_1\Big) \times_l \Big(X_2, g_2, B_2\Big)$$

is bilipshitz related to a metric g_Y^λ of sectional curvature $K^{(Y,\lambda)} \leq -\kappa$ ($-\kappa_2 \leq K^{(Y,\lambda)} \leq -\kappa_1$,resp.).

Proof. This immediately follows from Proposition 1 with the fact that the second fundamental forms of horospheres in Hadamard manifolds with upper curvature bound $-\kappa < 0$ are more convex than the horospheres in the real hyperbolic space of curvature $-\kappa$ and those in Hadamard manifolds with lower curvature bound $-\kappa_2$ are more concave than the corresponding horospheres in the real hyperbolic space of curvature $-\kappa_2$. □

Note that without further assumptions (such as, for instance, a lower curvature bound of the X_m), it is not even clear whether (Y, g_Y) carries curvature bounded above, while Corollary 1 tells us that (Y, g_Y) is a Gromov-hyperbolic metric space.

Since (Y, g_Y^λ) is bilipschitz to (Y, g_Y), it suffices to prove that the Riemannian embedding (Y, g_Y) is bilipschitz to (X, g_X).

Denote by d_Y the Riemannian length function on Y that is induced by the metric $g_Y := j^* g_X$ on Y, and denote by d_X that on X, induced by the Riemannian product metric g_X. With that we find the inequalities

$$d_X\Big(j(p), j(q)\Big) \ \overset{\text{a)}}{\leq} \ d_Y\Big(p, q\Big) \ \overset{\text{b)}}{\leq} \ \Big(2\sqrt{2} + 2\Big) \cdot d_X\Big(j(p), j(q)\Big) \qquad \forall\, p, q \ \in \ Y.$$

While inequality a) is merely a consequence of the fact that (Y, g_Y) is a Riemannian submanifold of (X, g_X), inequality b) requires more attention. Note that the idea for the following construction is due to Brady and Farb ([BrFa]).

We denote the various canonical projections as follows:

$$\begin{array}{ll} \pi_i : X \longrightarrow X_i, & t : X_i \longrightarrow \mathbb{R}, \\ \eta : Y \longrightarrow N, & \eta_i : X \longrightarrow N_i. \end{array}$$

Consider an arbitrary differentiable curve $c : [t_p, t_q] \longrightarrow X$ connecting $j(p) \in X$ with $j(q) \in X$. The idea is to construct a curve $\tilde{c} : [\alpha, \omega] \longrightarrow Y$ that connects $p \in Y$ with $q \in Y$, whose Riemannian length $L_Y(\tilde{c})$ in (Y, g_Y) is bounded by a constant times the Riemannian length $L_X(c)$ of c in (X, g_X).

Therefore, we consider the projections $c_i := \pi_i \circ c$ of c to the factors X_i, $i = 1, 2$, that connect $p_i := \pi_i(j(p))$ with $q_i := \pi_i(j(q))$. The further projections $t_i \circ c_i$ are continuous. Thus, the set $\mathcal{K} := \bigcup_{i=1,2}(t_i \circ c_i)([t_p, t_q]) \subset \mathbb{R}$ is compact and therefore takes its maximum $(t_{i_0} \circ c_{i_0})(b)$ for some $b \in [t_p, t_q]$, $i_0 \in \{1, 2\}$.

The continuous and piecewise differentiable curve $\tilde{c}$ in Y that we are going to follow from p to q consists of three differentiable segments v_1, γ and v_2 as follows:

- v_1 has constant projection $\eta \circ v_1$ to $N_1 \times N_2$ that is given through $\eta \circ v_1 \equiv (\eta_1(p_1), \eta_2(p_2))$, while its projection to $\mathbb{R}$ is $t \circ v_1 = (t_{i_0} \circ c_{i_0})|_{[t_p, b]}$.

- γ is the curve keeping its projection to $\mathbb{R}$ constant: $(t \circ \gamma) \equiv (t_{i_0} \circ c_{i_0})(b)$, while varying along $N_1 \times N_2$ with $\eta(\gamma) = (\eta_1(c_1), \eta_2(c_2))$.
- v_2 again has constant projection $\eta \circ v_2$ to $N_1 \times N_2$, that is, $\eta \circ v_2 \equiv (\eta_1(q_1), \eta_2(q_2))$. Its projection to $\mathbb{R}$ is $t \circ v_2 = (t_{i_0} \circ c_{i_0})|_{[b,t_q]}$.

The length of $\tilde{c} := v_2 * \gamma * v_1$ is the sum

$$L_Y(\tilde{c}) \;=\; L_Y(v_1) \;+\; L_Y(\gamma) \;+\; L_Y(v_2).$$

From (2) it directly follows that

$$\begin{aligned} L_Y(v_m) &= \sqrt{2} \cdot L_{(\mathbb{R},dt^2)}((t_{i_0} \circ c_{i_0})|_{I_m}) \\ &\leq \sqrt{2} \cdot L_{(\mathbb{R},dt^2)}(t_{i_0} \circ c_{i_0}) \\ &\leq \sqrt{2} \cdot L_X(c), \end{aligned} \tag{8}$$

where $I_1 = [t_p, b]$ and $I_2 = [b, t_q]$.

Now we know that the natural diffeomorphisms $\phi_i^{tt'} : N_i^t \longrightarrow N_i^{t'}$ are length contracting for $t \leq t'$. With that and the particular choice of b we have

$$\begin{aligned} L_Y(\gamma) &= \int_{t_p}^{t_q} \sqrt{g_Y(\gamma', \gamma')}\, dt \;=\; \int_{t_p}^{t_q} \Big[g_Y\Big|_{N^{t_0}} \Big((\eta \circ \gamma)', (\eta \circ \gamma)' \Big) \Big]^{\frac{1}{2}}\, dt \\ &= \int_{t_p}^{t_q} \Big[g_Y\Big|_{N_1^{t_0}} \Big((\eta_1 \circ c)', (\eta_1 \circ c)' \Big) \;+\; g_Y\Big|_{N_2^{t_0}} \Big((\eta_2 \circ c)', (\eta_2 \circ c)' \Big) \Big]^{\frac{1}{2}}\, dt \\ &\leq \int_{t_p}^{t_q} \Big[g_Y\Big|_{N_1^{t}} \Big((\eta_1 \circ c)', (\eta_1 \circ c)' \Big) \;+\; g_Y\Big|_{N_2^{t}} \Big((\eta_2 \circ c)', (\eta_2 \circ c)' \Big) \Big]^{\frac{1}{2}}\, dt \\ &\leq 2 \cdot L_X(c), \end{aligned} \tag{9}$$

where $t_0 := (t_{i_0} \circ c_{i_0})(b)$.

Thus with (8) and (9) we can conclude that for an arbitrary curve c in X connecting two points $j(p)$, $j(q) \in j(Y) \subset X$ there exists a curve $\tilde{c}$ in Y connecting p and q with

$$L_Y(\tilde{c}) \;\leq\; \Big(2\sqrt{2} \;+\; 2 \Big)\, L_X(c).$$

Thus the required inequality b) follows by the definitions of the Riemannian length functions.

Finally, note that the generalization to products of finitely many Hadamard manifolds of pinched negative sectional curvature is straightforward, using once again that products of quasi-isometric maps are quasi-isometric. □

4. The proof of Theorem 2

Recall the definition of the hyperbolic rank as given in [BuySch]:

Definition 3. Let X be a metric space. Then the hyperbolic rank, $rank_h X$, of X is defined via

$$rank_h X \;:=\; \sup_Y \; dim\, \partial_\infty Y,$$

where the supremum is taken over all locally compact $CAT(-1)$ Hadamard spaces Y quasi-isometrically embedded into X, and $\partial_\infty Y$ denotes the topological dimension of the boundary of Y.

In [BuySch] the authors proved the subadditivity of this hyperbolic rank with respect to products of Hadamard manifolds of strictly negative sectional curvature bounded away from zero. This together with the following lemma, which states the corresponding superadditivity, yields the proof of Theorem 2.

Lemma 1. *The hyperbolic rank, $rank_h$, is superadditive with respect to Riemannian products of Hadamard manifolds (X_i, g_i) of strictly negative sectional curvature bounded away from zero, i.e., for the Riemannian product (X, g) of the (X_i, g_i), $i = 1, ..., k$, one has*

$$rank_h\Big(X, g_X\Big) \ \geq \ \sum_{i=1}^{k} rank_h\Big(X_i, g_i\Big).$$

Proof of Lemma 1. From the Morse quasi-isometric lemma it easily follows that $rank_h(X_i) = dimX_i - 1$. From Theorem 2 we further conclude that $rank_h X \geq -k + \sum_{i=1}^{k} dimX_i$ and thus

$$rank_h X \ \geq \ -k + \sum_{i=1}^{k} dimX_i \ = \ \sum_{i=1}^{k} (dimX_i \ - \ 1) \ = \ \sum_{i=1}^{k} rank_h\Big(X_i, g_i\Big).$$

□

References

[BuySch] S.Buyalo & V.Schroeder, *Hyperbolic rank and subexponential corank of metric spaces*, Geom. Funct. Anal., vol. 12, no.2 (2002), 293-306. MR 2003e:53045

[BrFa] N.Brady & B. Farb, *Filling-Invariants at Infinity for Manifolds of Nonpositive Curvature*, Trans. Amer. Math. Soc., vol. 350, no. 8, (1998), 3393-3405. MR 99c:53039

[Foe] T.Foertsch, *Bilipschitz Embeddings of Negative Sectional Curvature in Products of Warped Product Manifolds*, Proc. Amer. Math. Soc., vol.130, no.7 (2002), 2089-2096. MR 2003b:53041

[Gr] M.Gromov, *Asymptotic invariants of infinite groups*, Geometric Group theory. Editors: Niblo, Roller, London Math. Soc., Lecture Notes Series 182 Vol.2, Cambridge University Press, 1993. MR 95m:20041

[Leu] E.Leuzinger, *Corank and Asymptotic Filling-Invariants for Symmetric Spaces*, Geom. Funct. Anal., vol.10, no.4 (2000), 863-873. MR 2001k:53074

Universität Zürich, Mathematisches Institut, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: `foertsch@math.unizh.ch`

Current address: Department of Mathematics, University of Michigan, 525 E University Avenue, East Hall, Ann Arbor, Michigan 48109-1109

Universität Zürich, Mathematisches Institut, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: `vschroed@math.unizh.ch`