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ASYMPTOTIC PROPERTIES OF GROUPS ACTING ON COMPLEXES

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Abstract. We examine asymptotic dimension and property A for groups acting on complexes. In particular, we prove that the fundamental group of a finite, developable complex of groups will have finite asymptotic dimension provided the geometric realization of the development has finite asymptotic dimension and the vertex groups are finitely generated and have finite asymptotic dimension. We also prove that property A is preserved by this construction provided the geometric realization of the development has finite asymptotic dimension and the vertex groups all have property A. These results naturally extend the corresponding results on preservation of these large-scale properties for fundamental groups of graphs of groups. We also use an example to show that the requirement that the development have finite asymptotic dimension cannot be relaxed.

1. Introduction

The asymptotic approach to finitely generated groups became popular following the work of Gromov [9]. In his study of asymptotic invariants of finitely generated groups, Gromov defined asymptotic dimension (asdim), the large-scale analog of Lebesgue covering dimension. G. Yu [16] applied asymptotic dimension to the Novikov higher signature conjecture for groups, showing that the conjecture holds for groups with finite asymptotic dimension. Later, Yu [17] defined another asymptotic invariant for discrete metric spaces and finitely generated groups called *property A.* This is a weak form of amenability, which also implies the Novikov conjecture for groups. (For an introduction to the Novikov and related conjectures, see [8].)

We wish to consider finitely generated groups as metric spaces. Let Γ be a finitely generated group with generating set $S = S^{-1}$. The *S-norm* on Γ is the norm given by setting $\|\gamma\|_S = 0$ precisely when γ is the group identity and otherwise taking $\|\gamma\|_S$ to be the minimal length of any S-word presenting the element γ. Then, one can define the (left-invariant) *word metric* associated to S by

$$\operatorname{dist}_S(g, h) = \|g^{-1}h\|_S.$$

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The metrics corresponding to two finite generating sets S and S' are Lipschitz equivalent. Asymptotic dimension and property A are invariants of Lipschitz equivalent metric spaces. So these properties are intrinsic to the group Γ and not the metric space associated to a specific generating set.

In view of Yu's results, [16], [17], it is important to know which groups have finite asdim or property A. Gromov [9] showed that hyperbolic groups have finite asdim. Dranishnikov and Januszkiewicz proved in [6] that Coxeter groups have finite asdim. Dranishnikov and the author proved that the finiteness of asdim is preserved by the amalgamated free product and HNN extension, and more generally by any fundamental group of a finite graph of groups with vertex groups having finite asdim; see [2], [3]. Higson and Roe [12] showed that finitely generated groups with finite asymptotic dimension have property A. In [15], J.-L. Tu proved that property A is preserved by the fundamental group of a finite graph of groups where the vertex groups all have property A. At first it was not known whether there could be a finitely generated group not having property A. A recent example of such a group due to Gromov [10] and Dranishnikov, Gong, Lafforgue and Yu [7] has made determining precisely which groups have these properties interesting.

The Bass-Serre theory of graphs of groups generalizes the constructions of amalgamated free products and HNN extensions (see [14]). There is a direct correspondence between groups acting without inversion on trees and fundamental groups of graphs of groups. Complexes of groups were introduced by Haefliger [11] in order to describe actions of groups on simply connected simplicial complexes in the same way that graphs of groups describe the action of groups on trees. The problem that arises is that the quotient of a simplicial complex by a simplicial action may identify faces of simplices. So it may not be the case that the quotient is a simplicial complex. This problem is avoided by introducing combinatorial substitutes for simplicial complexes called small categories without loops (scwols).

In the second section we develop the necessary theory of complexes of groups following [4]. We define scwols, group actions on scwols, complexes of groups, developability of complexes of groups, and the associated fundamental group of a complex of groups.

In the third section, we define the R-stabilizer that plays the role of the stabilizer in the study of the large-scale properties of groups acting on metric spaces. We also lay the ground work for the rest of the paper by proving Proposition 1, which characterizes the R-stabilizers for groups acting on scwols.

In the fourth section, we define asymptotic dimension (asdim) and property A for discrete metric spaces. As mentioned above, the metric spaces we will be most interested in are finitely generated groups in the word metric. In this section we obtain our main result:

Theorem. *Let $G(\mathcal{Y})$ be a developable complex of groups over a finite scwol $\mathcal{Y}$ with development $\mathcal{X}$ such that* $\operatorname{asdim}|\mathcal{X}| < \infty$. *Let π denote the fundamental group of the complex of groups. If the local groups G_σ have finite* asdim, *then so does π; if the local groups G_σ have property A, then so does π.*

In the final section, we use an example of a finitely generated group that does not have either property A or finite asdim to show that the results we obtain here cannot be improved by relaxing the condition that the development have finite asdim.

2. Complexes of groups

Our notation and development will follow Bridson-Haefliger [4].

Definition. A *small category without loops* (abbreviated *scwol*) is a set $\mathcal{X}$ which is the disjoint union of a *vertex set* $V(\mathcal{X})$ and an *edge set* $E(\mathcal{X})$. There are maps

$$i : E(\mathcal{X}) \to V(\mathcal{X})$$

and

$$t : E(\mathcal{X}) \to V(\mathcal{X})$$

which assign to each edge a the initial vertex of a and the terminal vertex of a, respectively. Let $E^{(2)}(\mathcal{X}) = \{(a,b) \in E(\mathcal{X}) \times E(\mathcal{X}) \mid i(a) = t(b)\}$ denote the pairs of composable edges. There is also a map

$$E^{(2)}(\mathcal{X}) \to E(\mathcal{X})$$

which assigns to each pair (a,b) an edge (ab) called the composition of a and b. These maps are required to satisfy:

1. $i(ab) = i(b)$, and $t(ab) = t(a)$ for all $(a,b) \in E^{(2)}(\mathcal{X})$;
2. $a(bc) = (ab)c$ for all edges a, b and c with $i(a) = t(b)$ and $i(b) = t(c)$; and
3. $i(a) \neq t(a)$ (the no loops condition).

Let $E^{(k)}(\mathcal{X})$ denote the composable sequences of edges of length k, i.e., $E^{(k)}(\mathcal{X}) = \{(a_1, \ldots, a_k) \in (E(\mathcal{X}))^k \mid i(a_i) = t(a_{i+1}), \text{ for } i = 1, \ldots, k-1\}$. By convention, $E^{(0)}(\mathcal{X}) = V(\mathcal{X})$. We define the *dimension* of the scwol $\mathcal{X}$ to be the maximum k such that $E^{(k)}(\mathcal{X})$ is not empty.

Definition. The *geometric realization* $|\mathcal{X}|$ is a piecewise Euclidean polyhedral complex, with each k-cell isometric to the standard simplex Δ^k. There is one such k-simplex A for each $A \in E^{(k)}(\mathcal{X})$. The identifications are the obvious ones, induced by the face relation among simplices.

Observe that the geometric realization need not be a simplicial complex, since it may be the case that the intersection of two simplices is a union of faces. One can eliminate this problem by taking the barycentric subdivision, if one requires simplicial complexes. The geometric realization is a Euclidean complex and is given its intrinsic metric. The geometric dimension of $|\mathcal{X}|$ is the same as the dimension of the combinatorial object $\mathcal{X}$.

Definition. A *group action* on a scwol is a homomorphism $G \to Aut(\mathcal{X})$ satisfying

1. for every $g \in G$, and for all $a \in E(\mathcal{X})$ $g.i(a) \neq t(a)$;
2. for every $g \in G$, and for all $a \in E(\mathcal{X})$ if $g.i(a) = i(a)$, then $g.a = a$.

Notice that a group action on a scwol induces an isometric action of the group on the geometric realization $|\mathcal{X}|$. Since we are primarily concerned with isometric actions on metric spaces, this is the action that we consider.

One forms the quotient $\mathcal{Y} = G\backslash\mathcal{X}$ of the scwol $\mathcal{X}$ by the action of G by taking $V(\mathcal{Y}) = G\backslash V(\mathcal{X})$ and $E(\mathcal{Y}) = G\backslash E(\mathcal{X})$. One can verify that $\mathcal{Y}$ has the structure of a scwol.

Definition. A *complex of groups $G(\mathcal{Y})$ over a scwol $\mathcal{Y}$* is a collection $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ satisfying

1. to each $\sigma \in V(\mathcal{Y})$, there corresponds a group G_σ called the *local group* at σ;

(2) for each $a \in E(\mathcal{Y})$ there exists an injective homomorphism $\psi_a : G_{i(a)} \to G_{t(a)}$; and

(3) for each $(a, b) \in E^{(2)}(\mathcal{Y})$, there is a $g_{a,b} \in G_{t(a)}$ such that
   (i) $Ad(g_{a,b})\psi_{ab} = \psi_a\psi_b$, where $Ad(g_{a,b})$ denotes conjugation by $g_{a,b}$, and
   (ii) $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$, for all $(a, b, c) \in E^{(3)}(\mathcal{Y})$.

Given a group G and an action of G on the scwol $\mathcal{X}$, there is an explicit construction of the complex of groups over the quotient scwol which we do not describe here. However, on the other hand, it is not always the case that an arbitrary complex of groups can be associated to a group action on some scwol $\mathcal{X}$. When this occurs, we say that the complex of groups is *developable*, and we refer to the associated scwol $\mathcal{X}$ as the development of $G(\mathcal{Y})$.

It is clear that scwols of dimension 1 must have precisely two types of vertices: sources and sinks. A source is an initial vertex of every edge it is contained in and a sink is a terminal vertex of every edge it is contained in. Every one-dimensional simplicial complex (graph) can be given the structure of a one-dimensional scwol by placing a source vertex in the middle of every edge, thus giving the original vertices the structure of sinks. It is easy to verify that the theory of complexes of groups over one-dimensional scwols is precisely the same as the theory of graphs of groups. Phrased in terms of the language of complexes of groups, the Bass-Serre structure theorem for groups acting without inversion on graphs says that if $dim(\mathcal{Y}) = 1$, then $G(\mathcal{Y})$ is always developable.

When a complex of groups is developable, there is an explicit method of constructing both the scwol $\mathcal{X}$ and the group G that acts on the scwol. The scwol $\mathcal{X}$ on which the group acts is simply connected and has an explicit description in a similar way to the construction of the tree $\tilde{X}$ in the theory of graphs of groups (see [14]).

Indeed, if $G(\mathcal{Y})$ is a developable complex of groups, then we can define the development $D(\mathcal{Y})$ to be the scwol whose vertices and edges are given by $V(D(\mathcal{Y})) = \{(gG_\sigma, \sigma) \mid \sigma \in V(\mathcal{Y})\}$ and $E(D(\mathcal{Y})) = \{(gG_{i(a)}, a) \mid a \in E(\mathcal{Y})\}$. Then the group G acts on the development $D(\mathcal{Y})$ by left multiplication. The development is isomorphic to the scwol $\mathcal{X}$, mentioned above. (See [4] for more details.)

We describe the fundamental group of the complex of groups $\pi_1(G(\mathcal{Y}))$, which is the group G, up to isomorphism. As in the theory of graphs of groups, there are two equivalent descriptions of the fundamental group. Both rely on the construction of the auxiliary group $FG(\mathcal{Y})$. Let $E^{\pm}(\mathcal{Y})$ denote the collection of symbols $\{a^+, a^-\}$ where $a \in E(\mathcal{Y})$. The elements of $E^{\pm}(\mathcal{Y})$ can be thought of as *oriented edges*. If $e = a^+$, then define $i(e) = t(a)$ and $t(e) = i(a)$. Accordingly, if $e = a^-$, define $t(e) = t(a)$ and $i(e) = i(a)$. Then define $FG(\mathcal{Y})$ to be the free product of the local groups G_σ and the free group generated by the collection $E^{\pm}(\mathcal{Y})$ subject to the additional relations:

(1) $(a^+)^{-1} = a^-$, and $(a^-)^{-1} = a^+$;
(2) $a^+b^+ = g_{a,b}(ab)^+$;
(3) $\psi_a(g) = a^+ga^-$, for all $g \in G_{i(a)}$.

The first description of the fundamental group is in terms of $G(\mathcal{Y})$-loops based at some fixed vertex σ_0. An edge path in $\mathcal{Y}$ is a sequence $(e_1, \ldots, e_k)$ with $t(e_i) = i(e_{i+1})$, for all $i = 1, \ldots, k-1$. By a $G(\mathcal{Y})$-path issuing from σ_0 we mean a sequence $(g_0, e_1, g_1, \ldots, e_k, g_k)$, where $(e_1, \ldots, e_k)$ is an edge path in $\mathcal{Y}$, and $g_0 \in G_{\sigma_0}$, $i(e_1) =$

σ_0, and $g_i \in G_{t(e_i)}$, for $i > 0$. We associate the word $g_0 e_1 \dots e_k g_k \in FG(\mathcal{Y})$ to the path described above. A $G(\mathcal{Y})$-loop based at σ_0 is a $G(\mathcal{Y})$ path with $t(e_k) = \sigma_0$. The $G(\mathcal{Y})$-loops γ and γ' are homotopic if the $FG(\mathcal{Y})$-words they represent are equal. The fundamental group $\pi_1(G(\mathcal{Y}), \sigma_0)$ is the collection of all words associated to $G(\mathcal{Y})$-loops based at σ_0, up to homotopy equivalence.

The second description is much simpler. Let T be a maximal tree in $|\mathcal{Y}|^{(1)}$. Then $\pi_1(G(\mathcal{Y}), T)$ is $FG(\mathcal{Y})$ subject to the additional relation $a^+ = 1$, for all $a \in T$. For a connected scwol, there is an isomorphism $\pi_1(G(\mathcal{Y}), \sigma_0) \to \pi_1(G(\mathcal{Y}), T)$.

3. Groups acting on metric spaces

For group actions considered on a local scale, the stabilizer plays a key role. The corresponding notion for group actions considered in the global sense is that of the R-stabilizer, which we define presently.

Definition. Let Γ be a group acting on the pointed metric space (X, x_0) by isometries. For every $R > 0$ define the *R-stabilizer* of the point x_0, denoted $W_R(x_0)$, by

$$W_R(x_0) = \{\gamma \in \Gamma \mid d(\gamma x_0, x_0) \leq R\}.$$

Let $\mathcal{X}$ denote the development of the developable complex of groups $G(\mathcal{Y})$. Then, as mentioned in section 2, the scwol $\mathcal{X}$ is given the metric it inherits as a piecewise Euclidean complex. It is sometimes convenient to work with $\mathcal{X}$ in the edge-length metric defined by its 1-skeleton. When $\mathcal{Y}$ is finite, which is the only case we consider, then these two metrics are quasi-isometric. Let $\mathcal{X}_e$ denote the scwol $\mathcal{X}$ with the edge-length metric, d_e. If no metric is specified, then we are considering $\mathcal{X}$ in the piecewise Euclidean metric.

In [3], Dranishnikov and the author characterized the R-stabilizers of the action of a fundamental group of a graph of groups on the tree corresponding to its development. That proposition was vital to the understanding of the structure of the group. The following proposition is a natural generalization of that result. As in [3], once the structure of the R-stabilizers is known, the main results follow readily.

Proposition 1. *Let $G(\mathcal{Y})$ be a developable complex of groups. Fix a vertex $\sigma_0 \in \mathcal{Y}$, and consider the action of $\pi = \pi_1(G(\mathcal{Y}), \sigma_0)$ on the simply connected scwol $\mathcal{X}_e$ induced by the complex of groups. Then for every $R > 0$, the R-stabilizer $W_R(G_{\sigma_0})$ is the set of all elements of π with associated path c of length not exceeding R.*

Proof. The proof is simply an application of the path-lifting property, [4]. Alternatively, one can prove this by working with the relations of the group $FG(\mathcal{Y})$ explicitly.

We are considering $\mathcal{X}_e$ in the edge-length metric. So we must show that the element $\gamma \in \pi$ with associated path $(g_0, e_1, \dots, e_k, g_k)$ belongs to $W_R(G_{\sigma_0})$ if and only if $k \leq R$. Clearly if $k \leq R$, then $d_e(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq R$, since the path corresponding to γ lifts to a path in $\mathcal{X}_e$, and the length of the path is no more than the distance R.

On the other hand, if $d_e(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq R$, for some $\gamma \in \pi$, then there is some $g \in G_{\sigma_0}$ so that γg lifts to a path of length $\leq R$. But, since there is a lift of g to a path of length 0, there must be a lift of γ to a path of length no more than R. $\square$

4. ASYMPTOTIC DIMENSION AND PROPERTY A

Asymptotic dimension was introduced by Gromov [9]. It is the coarse analog of Ostrand's characterization of covering dimension for metric spaces, [13].

Definition. Let X be a metric space. We define the *asymptotic dimension* (asdim) of X by the following inequality: $\operatorname{asdim} X \leq n$ if for every $D > 0$ there exist $n+1$ families of sets $\mathcal{U}_0, \dots, \mathcal{U}_n$ that are uniformly bounded, that cover X and that are D-disjoint in the sense that any two distinct sets from the same family are at a distance greater than D from each other. We define $\operatorname{asdim} X = n$ if it is the case that $\operatorname{asdim} X \leq n$, but it is not the case that $\operatorname{asdim} X \leq n-1$.

One goal of this section is to see that the finiteness of asdim is preserved by the construction of the fundamental group of a developable complex of groups. This is a natural generalization of the main theorem in [3].

Definition. Let X_α be a family of subsets of the metric space X. We say that the family satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ *uniformly* if for every $D > 0$ there is a number $R > 0$ and a collection of R-bounded, D-disjoint families $\{\mathcal{U}_i^\alpha\}$ so that for each α, $\{\mathcal{U}_i^\alpha\}$ covers X_α.

A common example of a family satisfying $\operatorname{asdim} X_\alpha \leq n$ uniformly is a family of isometric metric spaces.

Property A was introduced by G. Yu, [17]. It is a weak form of amenability which, for finitely generated groups, implies the existence of a uniform embedding into Hilbert space and hence the Novikov higher signature conjecture.

Definition. Let X be a metric space. Let $P(X)$ denote the set of probability measures on X in the l_1 metric. The metric space X has property A if there exists a sequence of maps $a^n : X \to P(X)$ satisfying the following two conditions:

1. for every n there is an R so that for every x, $\operatorname{supp} a_x^n \subset B_R(x)$, and
2. for every $K > 0$,

$$\lim_{n\to\infty} \sup_{d(x,y)<K} \|a_x^n - a_y^n\|_1 = 0.$$

As mentioned in the introduction, finitely generated groups with finite asdim have property A, [12]. Tu proved, [15], that the fundamental group of a finite graph of groups in which each vertex group has property A will have property A. In [1], the author generalized Tu's results to groups acting by isometries on metric spaces with finite asdim .

In particular, the theorem proved in [1] is the following:

Theorem. *Assume that the finitely generated group Γ acts on the metric space X by isometries. Assume that* $\operatorname{asdim} X \leq n$, *and that for every R, the R-stabilizer of a basepoint $x_0 \in X$ has property A. Then Γ has property A.*

Finite asdim and property A are closely related, and so it is not surprising that we have the following union theorem, which combines known results from [1] and [2] into a unified statement.

Theorem (Union Theorem)**.** *Let $X = \bigcup_\alpha F_\alpha$. Suppose further that for every r there is a set $Y_r \subset X$ such that $\{F_\alpha \setminus Y_r\}$ is r-disjoint. Then:*

1. *If* $\operatorname{asdim} F_\alpha \leq n$ *uniformly and if* $\operatorname{asdim} Y_r \leq n$, *then* $\operatorname{asdim} X \leq n$.

(2) *If the* F_α *are pairwise isometric and have property A, and if* Y_r *has property A, then* X *has property A.*

As a corollary, we have the following finite union theorems.

Theorem (Finite Union Theorem)**.** *Let* $X = \bigcup_{i=1}^{k} X_i$ *be a metric space. Then*

(1) $\operatorname{asdim} X \leq \max\{\operatorname{asdim} X_i \mid i = 1, \ldots, k\}$*; and*
(2) *if the* X_i $(i = 1, \ldots, k)$ *have property A, then* X *has property A.*

Let $G(\mathcal{Y})$ be a developable complex of groups, with $\mathcal{Y}$ finite and $\operatorname{asdim} |\mathcal{X}| \leq k$. Suppose further that the local groups are finitely generated. Then the finiteness of $\mathcal{Y}$ implies that the fundamental group π is finitely generated. Indeed, if S_σ denotes a finite generating set for each local group G_σ, then we can consider $FG(\mathcal{Y})$ in the metric obtained from the disjoint union of all the S_σ and the set $E^{\pm}(\mathcal{Y})$. Thus, the notion of asdim is well-defined for the fundamental group of a complex of groups.

The following lemma is a natural generalization of Lemma 3 from [3] and Proposition 4 from [1].

Lemma 2. *Let* π *be the fundamental group of a complex of groups* $G(\mathcal{Y})$ *where* $\mathcal{Y}$ *is finite and connected, and the local groups are finitely generated. Let* σ_0 *be a vertex. Let* $R > 0$, *and denote the* R*-stabilizer of* σ_0 *by* $W_R(\sigma_0)$*. Then*

(1) *if* $\operatorname{asdim} G_\sigma \leq n$, *for all the local groups, then* $\operatorname{asdim} W_R(G_{\sigma_0}) \leq n$*;*
(2) *if the local groups have property A, then* $W_R(\sigma_0)$ *also has property A.*

Proof. We will prove the statement in the case of asdim only. The proof of the result on property A is very similar.

In Proposition 1 we characterized $W_R(G_{\sigma_0})$ for $\mathcal{X}_e$ as the set of all elements in π with length at most R. Since the metric spaces $\mathcal{X}_e$ and $\mathcal{X}$ are coarsely isometric, there are constants λ and ϵ so that

$$\frac{1}{\lambda} d(x,y) - \epsilon \leq d_e(x,y) \leq \lambda d(x,y) + \epsilon$$

for all $x, y \in \mathcal{X}$. Then, if $d(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq R$, we have $\lambda d(\gamma G_{\sigma_0}, G_{\sigma_0}) + \epsilon \leq \lambda R + \epsilon$. Since $d_e(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq \lambda d(\gamma G_{\sigma_0}, G_{\sigma_0}) + \epsilon$, we conclude that $\{\gamma \in \pi \mid d(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq R\} \subset \{\gamma \in \pi \mid d_e(\gamma G_{\sigma_0}, G_{\sigma_0}) \leq \lambda R + \epsilon\}$. So to prove that $\operatorname{asdim} W_R(G_{\sigma_0}) \leq n$ in the natural metric on $\mathcal{X}$, it suffices to show this for elements whose paths have length not exceeding N, the greatest integer less than or equal to $\lambda R + \epsilon$.

In order to apply an inductive argument, we consider a larger set $K \subset FG(\mathcal{Y})$, which is the set of all words in $FG(\mathcal{Y})$ issuing from σ_0. The group π is a subset of K, and the set K acts on $\mathcal{X}$ by left multiplication. We show that the R-stabilizer of this action has asdim at most n. It follows then that the R-stabilizer of the action of π on $\mathcal{X}$ will also have asdim at most n.

In light of the finite union theorem, in order to show $\operatorname{asdim} W_R(G_{\sigma_0}) \leq n$, it suffices to show that the subset $K_j \subset K$ of reduced words in K with length equal to j has $\operatorname{asdim} K_j \leq n$. Indeed, $W_R(G_{\sigma_0}) \subset \bigcup_{j=0}^{N} K_j$, which is a finite union.

We proceed by induction. The case $j = 0$ is true by assumption since $K_0 = G_{\sigma_0}$. Consider the case K_{j+1} with $j \geq 0$. Observe that $K_{j+1} \subset \bigcup_{a \in E^{\pm}(\mathcal{Y})} K_j a G_{t(a)}$.

The orientation of the edge a is an issue since it determines whether the group $G_{t(a)}$ is a domain or codomain of the function ψ_a. Thus, it is necessary to consider two cases separately.

Suppose first that a has negative orientation. So, we are considering $K_j a^- G_{t(a)}$. For every $r > 0$ let $Y_r = K_j a^- N_r(\psi_a(G_{i(a)}))$, where the r-neighborhood is taken in the group $FG(\mathcal{Y})$. Then Y_r is coarsely equivalent to $K_j a^- \psi_a(G_{i(a)})$. Now we have $K_j a^- \psi_a(G_{i(a)}) = K_j G_{i(a)} a^-$, which is just $K_j a^-$. Finally, since $K_j a^-$ is coarsely equivalent to K_j, we have $\operatorname{asdim} Y_r = \operatorname{asdim} K_j$, which by the inductive hypothesis does not exceed n.

Next, decompose the set $K_j a^- G_{t(a)}$ into sets of the form $\{x a^- G_{t(a)}\}$, where the index runs over all $x \in K_j$ that do not end with an element $g \in G_{i(a)}$. One can still obtain these elements through the relations of $FG(\mathcal{Y})$. For instance, to obtain $x g a^- g'$, with x of the required form, $g \in G_{i(a)}$ and g' in $G_{t(a)}$, simply take the word $x a^- \psi_a(g) g'$, which is of the required form. Next, observe that the map $G_{t(a)} \mapsto x a^- G_{t(a)}$ is an isometry in the (left-invariant) word metric. So the family $\{x a^- G_{t(a)}\}$ has asdim $\leq n$, uniformly.

In order to apply the union theorem to this family, it remains to show only that the family $\{x a^- G_{t(a)} \setminus Y_r\}$ is r-disjoint. To this end, let $x a^- z$ and $x' a^- z'$ be given in different families. Then we compute $d(x a^- z, x' a^- z') = \|z^{-1} a^+ x^{-1} x' a^- z'\|$. Since z and z' lie outside of $N_r(\psi_a(G_{i(a)}))$, take $z = \psi_a(g)s$ and $z' = \psi_a(g')s'$, where $\|s\| > r$, $\|s'\| > r$, and $\psi_a(g)$ and $\psi_a(g')$ are in $\psi_a(G_{t(a)})$. Then,

$$\|z^{-1} a^+ x^{-1} x' a^- z'\| = \|s^{-1} a^+ g^{-1} x^{-1} x' g' a^- s'\|.$$

Now, in order for this length to be less than r, a reduction must occur in the middle, so that a^+ and a^- annihilate each other. In order for this to occur, we must have $g^{-1} x^{-1} x' g' \in G_{i(a)}$. Thus, $x^{-1} x' \in G_{i(a)}$. But, this means that $x a^- G_{t(a)}$ and $x' a^- G_{t(a)}$ define the same set. Thus, in the case that the edge has negative orientation, we have $\operatorname{asdim} K_j a^- G_{t(a)} \leq n$.

Next, we consider the case where the edge a has positive orientation. In this case, $K_j a^+ G_{i(a)} = K_j \psi_a(G_{i(a)}) a^+$, which is coarsely equivalent to K_j. We conclude that $\operatorname{asdim} K_j a^+ G_{i(a)} = \operatorname{asdim} K_j \leq n$. □

The following result appears as Theorem 2 from [2].

Theorem. *Assume that a finitely generated group* Γ *acts by isometries on a metric space* X *with a base point* x_0 *and with* $\operatorname{asdim} X \leq k$. *Suppose that* $\operatorname{asdim} W_R(x_0) \leq n$ *for all* R. *Then* $\operatorname{asdim} \Gamma \leq (n+1)(k+1) - 1$.

This estimate on the dimension is far from sharp. It is useful only as a means to prove that $\operatorname{asdim} \Gamma < \infty$. The exact estimate should be $n + k$. (See [3] for the proof of the exact formula in the case of groups acting on trees by isometries.)

As a consequence of the preceding theorem, we have our main result on asdim.

Theorem 3. *Let* Γ *be the fundamental group of a finite developable complex of groups* $G(\mathcal{Y})$ *corresponding to an action by isometries on the geometric realization of the scwol* $\mathcal{X}$. *Suppose that the local groups are finitely generated and that* $\operatorname{asdim} G_\sigma \leq n$. *Assume additionally that* $\operatorname{asdim} |\mathcal{X}| \leq k$. *Then* $\operatorname{asdim} \Gamma \leq (n+1)(k+1) - 1$.

The important result is summarized as a corollary:

Corollary. *Let* Γ *be the fundamental group of a finite developable complex of groups* $G(\mathcal{Y})$ *such that the development* $\mathcal{X}$ *has* $\operatorname{asdim} |\mathcal{X}| < \infty$, *and such that every base group* G_σ *has* $\operatorname{asdim} G_\sigma < \infty$. *Then,* $\operatorname{asdim} \Gamma < \infty$.

Applying the theorem from [1] cited directly following the definition of property A, we obtain the following generalization of Tu's theorem.

Theorem 4. *Let* $G(\mathcal{Y})$ *be a developable complex of groups over a finite scwol* $\mathcal{Y}$ *with corresponding development* $\mathcal{X}$ *and fundamental group* π*. Suppose that* $\operatorname{asdim}|\mathcal{X}|$ *is finite and that the stabilizers of the action have property A. Then,* π *has property A.*

5. Example

The following example illustrates that one must consider the large-scale structure of the development.

Consider a finitely presented group Γ that does not have property A; see [10], [7]. Since the group is finitely presented, there is a finite complex K so that $\pi_1(K) = \Gamma$. Thus, by taking the complex of groups with each vertex trivial and the scwol whose geometric realization is equal to the complex K, one obtains the fundamental group of the complex of groups equal to the group Γ. The vertex groups have finite asdim and the group acts on the universal cover; so the complex of groups is developable. The complex K is finite, yet the group Γ does not have finite asdim and does not have property A.

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