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SUPERCONGRUENCES FOR TRUNCATED ${}_{n+1}F_n$ HYPERGEOMETRIC SERIES WITH APPLICATIONS TO CERTAIN WEIGHT THREE NEWFORMS

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ABSTRACT. We prove general results on supercongruences between values of truncated ${}_{n+1}F_n$ hypergeometric functions and their character analogs. As a consequence of the main results of this paper, we prove Beukers-type supercongruences for certain weight three newforms.

1. INTRODUCTION

In [RV1], Fernando Rodriguez-Villegas discovered numerically a number of Beukers-type supercongruences for hypergeometric Calabi-Yau manifolds of dimension $d \leq 3$. Specifically, he observed supercongruences between the truncated fundamental period of the Picard-Fuchs differential equation of the manifold and an expression derived from the number of its $\mathbb{F}_p$-points. This had been motivated by his joint work with Candelas and de la Ossa [COV]. Here we prove general results on supercongruences between values of truncated ${}_{n+1}F_n$ hypergeometric functions and their character analogs. As a consequence of these results, we prove some of the observed supercongruences for manifolds of dimension $d = 2$. Supercongruences of this type were first observed by Beukers [B] in connection with the Apéry numbers used in the proof of the irrationality of $\zeta(3)$. Ahlgren and Ono [AO] proved Beukers' supercongruence conjecture relating Apéry numbers to the coefficients of a certain weight four newform.

In [RV1] and [RV2], Rodriguez-Villegas identified four modular K3 surfaces with potential supercongruences. We define Dedekind's eta function by the infinite product:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \ q := e^{2\pi i z}. \tag{1.1}$$

We then define the integers $a(n), b(n)$, and $c(n)$ by

$$\sum_{n=1}^{\infty} a(n)q^n := \eta^6(4z) \in S_3(\Gamma_0(16), (\tfrac{-4}{d})), \tag{1.2}$$

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$$\sum_{n=1}^{\infty} b(n)q^n := \eta^3(6z)\eta^3(2z) \in S_3(\Gamma_0(12), (\tfrac{-3}{d})), \tag{1.3}$$

$$\sum_{n=1}^{\infty} c(n)q^n := \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z) \in S_3(\Gamma_0(8), (\tfrac{-2}{d})). \tag{1.4}$$

These weight three newforms are related to modular K3 surfaces. They are extensively studied in [SB], where, among other results, the authors prove several modulo p congruences. From [RV1] and [RV2] we are able to formulate the following:

Conjecture. *If $p \geq 5$ is a prime, then*

$$\sum_{n=0}^{p-1} \frac{(2n)!^3}{n!^6} 64^{-n} \equiv a(p) \pmod{p^2}, \tag{1.5}$$

$$\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n} \equiv b(p) \pmod{p^2}, \tag{1.6}$$

$$\sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n} \equiv c(p) \pmod{p^2}, \tag{1.7}$$

$$\sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n} \equiv \gamma(p)a(p) \pmod{p^2}, \tag{1.8}$$

where $\gamma(p) := -1$ if $p \equiv 5 \pmod{12}$ and $\gamma(p) := 1$ otherwise.

It should be noted that (1.5) has already been proved by several individuals including Ishikawa [I], Van Hamme [vH], and Ahlgren [A]. The numbers 64, 108, 256, 1728 are called the conifold points (see [RV1]). Here we prove several cases of these conjectures.

To state our results, we recall basic facts about characters and Jacobi sums and introduce some notation. We denote by $\mathbb{F}_q$ the finite field with $q = p^r$ elements, where p is a prime. We extend all multiplicative characters $\chi : \mathbb{F}_q^{\times} \to \mathbb{C}_p$, including the trivial character ϵ_q, to $\mathbb{F}_q$ by setting $\chi(0) := 0$. If A and B are two characters on $\mathbb{F}_q$, then we define $\binom{A}{B}$ in terms of the Jacobi sum by

$$\binom{A}{B} := \frac{B(-1)}{q} J_r(A, \bar{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\bar{B}(1-x), \tag{1.9}$$

where $J_r(\cdot,\cdot)$ is a Jacobi sum over $\mathbb{F}_{p^r}$. We recall some useful properties of binomial coefficients ([G], (2.6)-(2.7)):

$$\binom{A}{B} = \binom{A}{A\overline{B}} \text{ and } \binom{A}{B} = \binom{B\overline{A}}{B} B(-1). \tag{1.10}$$

If $A_0, A_1, \ldots A_n$, and $B_1, B_2, \ldots B_n$ are characters on $\mathbb{F}_q$ and if $x \in \mathbb{F}_q$, then Greene [G] defines ${}_{n+1}F_n$ *Gaussian hypergeometric series* by

(1.11)

$${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \mid x\right)_q := \frac{q}{q-1} \sum_{\chi} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x),$$

where the sum runs over all characters χ on $\mathbb{F}_q$. We note that this definition lives in some extension of $\mathbb{Q}_p$. For certain choices of characters, the right-hand side actually is in $\mathbb{Z}_p$.

If m is a positive integer, then we define the truncated hypergeometric series by

$$(1.12)\qquad {}_{n+1}F_n\left(\begin{matrix} a_0, & a_1, & \dots, & a_n \\ & b_1, & \dots, & b_n \end{matrix} \mid x\right)_{\mathrm{tr}(m)} := \sum_{k=0}^{m-1} \frac{(a_0)_k (a_1)_k \cdots (a_n)_k}{k!(b_1)_k \cdots (b_n)_k} x^k,$$

where $(a)_k := a(a+1)\cdots(a+k-1)$.

If $n \in \mathbb{N}$, we define the p-adic Γ-function on the ring $\mathbb{Z}_p$ of p-adic integers by

$$(1.13)\qquad \Gamma_p(n) := (-1)^n \prod_{j<n,\, p\nmid j} j \text{ and } \Gamma_p(x) := \lim_{n\to x} \Gamma_p(n),\ x \in \mathbb{Z}_p,$$

where in the limit we take any sequence of positive integers that approaches x in the p-adic sense. We recall three basic properties of the p-adic Γ-function. If $p \geq 5$ is a prime and $x, y \in \mathbb{Z}_p$, then the following are true. We have

$$(1.14)\qquad \Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & \text{if } x \in \mathbb{Z}_p^*, \\ -\Gamma_p(x) & \text{if } x \in p\mathbb{Z}_p. \end{cases}$$

If $n \geq 1$, then

$$(1.15)\qquad x \equiv y \pmod{p^n} \Rightarrow \Gamma_p(x) \equiv \Gamma_p(y) \pmod{p^n}.$$

If $R(x)$ denotes the reduction of x modulo p to the range $\{1, \dots, p\}$, then

$$(1.16)\qquad \Gamma_p(x)\Gamma_p(1-x) = (-1)^{R(x)}.$$

We are now able to state the results of this paper. Let ϕ_q denote the character of order 2 on $\mathbb{F}_q$, and let ϵ_q denote the trivial character on $\mathbb{F}_q$. In the sequel we shall drop the subscript q since it will be obvious from the context.

Theorem 1. *If p is a prime, $p \equiv 1 \pmod{d_i}$ with $1 \leq m_i < d_i$, ρ_i is a character of order d_i on $\mathbb{F}_p$, and $\sum_{i=1}^{n+1} \frac{m_i}{d_i} \geq n-1$, then*

$$\begin{aligned} &{}_{n+1}F_n\left(\begin{matrix} \frac{m_1}{d_1}, & \frac{m_2}{d_2}, & \dots, & \frac{m_{n+1}}{d_{n+1}} \\ & 1, & \dots, & 1 \end{matrix} \mid 1\right)_{tr(p)} \\ &\qquad \equiv (-1)^n p^n \cdot {}_{n+1}F_n\left(\begin{matrix} \rho_1^{m_1}, & \rho_2^{m_2}, & \dots, & \rho_{n+1}^{m_{n+1}} \\ & \epsilon_p, & \dots, & \epsilon_p \end{matrix} \mid 1\right)_p - \delta \cdot p \pmod{p^2}, \end{aligned}$$

where $\delta := \begin{cases} 0 & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} > n-1, \\ \prod_{i=1}^{n+1} \Gamma_p(1-\frac{m_i}{d_i}) & \text{if } \sum_{i=1}^{n+1} \frac{m_i}{d_i} = n-1. \end{cases}$

Corollary 1. *If p is a prime, $p \equiv 1 \pmod{d_i}$, $1 \leq m_i < d_i$, and ρ_i is a character of order d_i on $\mathbb{F}_p$, then*

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} \frac{m_1}{d_1}, 1-\frac{m_1}{d_1}, & \frac{m_2}{d_2}, & 1-\frac{m_2}{d_2} \\ & 1, & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \\ &\qquad \equiv -p^3 \cdot {}_4F_3\left(\begin{matrix} {\rho_1}^{m_1}, & \overline{\rho_1}^{m_1}, & {\rho_2}^{m_2}, & \overline{\rho_2}^{m_2} \\ & \epsilon_p, & \epsilon_p, & \epsilon_p \end{matrix} \mid 1\right)_p \\ &\qquad\quad - (-1)^{\frac{m_1}{d_1}(p-1)+\frac{m_2}{d_2}(p-1)} p \pmod{p^2}. \end{aligned}$$

Corollary 2. *If p is a prime, $p \equiv 1 \pmod{d}$, $1 \leq m < d$, and ρ is a character of order d on $\mathbb{F}_p$, then*

$${}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{m}{d}, & 1-\frac{m}{d} \\ & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \equiv p^2 \cdot {}_3F_2\left(\begin{matrix} \phi_q, & \rho^m, & \overline{\rho}^m \\ & \epsilon_p, & \epsilon_p \end{matrix} \mid 1\right)_p \pmod{p^2}.$$

Theorem 2. *If p is a prime, $p \equiv -1 \pmod{d_i}$, $1 \le m_i < d_i$, and ρ_i is a character of order d_i on $\mathbb{F}_{p^2}$, then*

$$
{}_4F_3\left(\begin{matrix} \frac{m_1}{d_1}, & 1-\frac{m_1}{d_1}, & \frac{m_2}{d_2}, & 1-\frac{m_2}{d_2} \\ & 1, & 1, & 1 \end{matrix} \,\Big|\, 1\right)^2_{tr(p)}
\equiv -p^6 \cdot {}_4F_3\left(\begin{matrix} {\rho_1}^{m_1}, & \overline{\rho_1}^{m_1}, & {\rho_2}^{m_2}, & \overline{\rho_2}^{m_2} \\ & \epsilon_{p^2}, & \epsilon_{p^2}, & \epsilon_{p^2} \end{matrix} \,\Big|\, 1\right)_{p^2} \pmod{p^2}.
$$

Theorem 3. *If p is a prime, $p \equiv -1 \pmod{d}$, $1 \le m < d$, and ρ is a character of order d on $\mathbb{F}_{p^2}$, then*

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{m}{d}, & 1-\frac{m}{d} \\ & 1, & 1 \end{matrix} \,\Big|\, 1\right)^2_{tr(p)} \equiv p^4 \cdot {}_3F_2\left(\begin{matrix} \phi_q, & \rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}, & \epsilon_{p^2} \end{matrix} \,\Big|\, 1\right)_{p^2} \pmod{p^2}.
$$

For (1.5) – (1.8), we are able to prove the following.

Theorem 4. *Let $p \ge 5$ be a prime.*

(1) *We have* $\sum_{n=0}^{p-1} \frac{(2n)!^3}{n!^6} 64^{-n} \equiv a(p) \pmod{p^2}$.

(2) *If* $p \equiv 1 \pmod 3$, *then* $\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n} \equiv b(p) \pmod{p^2}$.
If $p \equiv 2 \pmod 3$, *then* $\left(\sum_{n=0}^{p-1} \frac{(3n)!(2n)!}{n!^5} 108^{-n}\right)^2 \equiv b(p)^2 \pmod{p^2}$.

(3) *If* $p \equiv 1 \pmod 4$, *then* $\sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n} \equiv c(p) \pmod{p^2}$.
If $p \equiv 3 \pmod 4$, *then* $\left(\sum_{n=0}^{p-1} \frac{(4n)!}{n!^4} 256^{-n}\right)^2 \equiv c(p)^2 \pmod{p^2}$.

(4) *If* $p \equiv 1 \pmod 6$, *then* $\sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n} \equiv a(p) \pmod{p^2}$.
If $p \equiv 5 \pmod 6$, *then* $\left(\sum_{n=0}^{p-1} \frac{(6n)!}{(3n)!n!^3} 1728^{-n}\right)^2 \equiv a(p)^2 \pmod{p^2}$.

In sections 2 and 3, we prove Theorems 1-3 using the method of proof in [M2] (i.e. we use basic character theory, the Gross-Koblitz formula [GK], and properties of the p-adic Γ-function). For these proofs, the arguments are similiar enough to those in [M2] that we only point out the changes made in the strategy. The key change is in dealing with the strange combinatorial expressions involving harmonic numbers that we encounter. In [M2], using Wilf-Zeilberger theory, the author evaluated two families of expressions explicitly in terms of p (see (5.28), (6.21)). Here, by writing the expressions in a different way and by using new techniques, we avoid WZ-theory, Corollary 2 is immediate and Corollary 1 uses (1.16).

In section 4, we prove Theorem 4 using Corollary 2 and Theorem 3. In addition, we need to evaluate the Gaussian hypergeometric series in terms of the trace of Frobenius. To accomplish this we borrow an idea from Ono [O] and use a character analog of Whipple's theorem for classical ${}_3F_2$ hypergeometric series. This analog was found by Greene [G], and it yields an expression in terms of Jacobi sums. Using several theorems of Berndt, Evans, and Williams ([BE], [BEW]), and a theorem of Beukers and Stienstra [SB], we evaluate these Jacobi sums in terms of the coefficients of the respective weight three modular forms.

2. Proof of Theorem 1

We begin this section with a lemma and a proposition. The proof of the lemma is trivial.

Lemma 2.1. *If $p \geq 5$ is a prime and $n \geq 1$, then*

$$\sum_{k=1}^{p-1} k^n \equiv \begin{cases} 0 & \pmod p \text{ if } p-1 \not| n, \\ -1 & \pmod p \text{ if } p-1 | n. \end{cases}$$

Proposition 2.2. *Let m and d be integers such that $1 \leq m < d$. If $p \equiv 1 \pmod d$ is a prime, then define r such that $p = dr+1$.*

(1) *If $0 \leq j \leq mr$, then $(\frac{m}{d})_j \equiv \Gamma_p(1-\frac{m}{d})((d-m)r+j)! \pmod p$.*
(2) *If $mr+1 \leq j \leq p-2$, then $(\frac{m}{d})_j(\frac{d}{mp}) \equiv \Gamma_p(1-\frac{m}{d})\frac{((d-m)r+j)!}{p} \pmod p$.*

Proof of Proposition 2.2. We first prove (1). From Proposition (1.14), we have

$$\Gamma_p(\tfrac{m}{d}+j) = (-1)^j(\tfrac{m}{d})_j\Gamma_p(\tfrac{m}{d}). \tag{2.1}$$

Using (1.15) and (1.13), we obtain

$$\Gamma_p(\tfrac{m}{d}+j) \equiv \Gamma_p((d-m)r+1+j) \equiv (-1)^{(d-m)r+1+j}((d-m)r+j)! \pmod p. \tag{2.2}$$

We then equate the two expressions and use Proposition (1.16).

For (2), the argument is similar. We use Proposition (1.14) to obtain

$$\Gamma_p(\tfrac{m}{d}+j) = (-1)^j \cdot \tfrac{d}{mp} \cdot (\tfrac{m}{d})_j\Gamma_p(\tfrac{m}{d}), \tag{2.3}$$

and we use Proposition (1.15) and (1.13) to obtain

$$\Gamma_p(\tfrac{m}{d}+j) \equiv \Gamma_p((d-m)r+1+j) \equiv (-1)^{(d-m)r+1+j} \cdot \tfrac{1}{p} \cdot ((d-m)r+j)! \pmod p. \tag{2.4}$$

We note that the expressions in (2.3) and (2.4) are p-integral. The terms with the p's in their denominator are only present to cancel out their reciprocals. □

Proof of Theorem 1. Recalling the notation of Theorem 1, we define $r_i := \frac{p-1}{d_i}$. We also define the harmonic number $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Without loss of generality, we can assume $m_1r_1 \leq m_2r_2 \leq \cdots \leq m_{n+1}r_{n+1}$. Using basic character theory, the Gross-Koblitz formula, and p-adic Γ-function properties, we follow the method of proof in [M2, section 5] to obtain

(2.5)

$$(-1)^n p^n \cdot {}_{n+1}F_n\left(\begin{matrix}\rho_1^{m_1}, & \rho_2^{m_2}, & \ldots, & \rho_{n+1}^{m_{n+1}} \\ & \epsilon_p, & \ldots, & \epsilon_p\end{matrix} \;\Big|\; 1\right)_p \equiv p\Big\{\sum_{m_1r_1+1}^{m_2r_2}\left(\prod_{i=1}^{n+1}\frac{(\frac{m_i}{d_i})_j}{j!}\right)\left(\frac{jd_1}{m_1p}\right)$$
$$+ \sum_{j=0}^{m_1r_1}\Big[\prod_{i=1}^{n+1}\frac{(\frac{m_i}{d_i})_j}{j!}\Big]\Big[1 + j \cdot \Big[\sum_{i=1}^{n+1}\big(H_{(d_i-m_i)r_i+j} - H_j\big)\Big]\Big]\Big\}$$
$$+ \sum_{j=0}^{m_2r_2}\prod_{i=1}^{n+1}\frac{\left(\frac{m_i}{d_i}\right)_j}{j!} \pmod{p^2}.$$

This is analogous to (5.25) in [M2]. If $m_1r_1 = m_2r_2$, then only the second sum in the braces is present. Using Proposition 2.2 and arguing as we did for (5.27) in

[M2] yields

$$(-1)^n p^n \cdot {}_{n+1}F_n\left(\begin{matrix}\rho_1^{m_1}, & \rho_2^{m_2}, & \dots, & \rho_{n+1}^{m_{n+1}} \\ & \epsilon_p, & \dots, & \epsilon_p\end{matrix} \;\Big|\; 1\right)_p$$

$$\equiv \sum_{j=0}^{m_2r_2}\prod_{i=1}^{n+1}\frac{\left(\frac{m_i}{d_i}\right)_j}{j!} + p\cdot\left(\prod_{i=1}^{n+1}\Gamma_p(1-\tfrac{m_i}{d_i})\right)\cdot A \pmod{p^2}, \tag{2.6}$$

where

$$A := \sum_{j=0}^{m_2r_2}\left[\prod_{i=1}^{n+1}\frac{((d_i-m_i)r_i+j)!}{j!}\right]\cdot\left[1+j\cdot\sum_{i=1}^{n+1}\left(H_{(d_i-m_i)r_i+j}-H_j\right)\right]. \tag{2.7}$$

We determine when $A\equiv 0 \pmod p$ and when $A\equiv 1 \pmod p$. We can extend the sum in A to $p-1$ to obtain

$$A \equiv \sum_{j=0}^{p-1}\left[\prod_{i=1}^{n+1}\frac{((d_i-m_i)r_i+j)!}{j!}\right]\cdot\left[1+j\cdot\sum_{i=1}^{n+1}\left(H_{(d_i-m_i)r_i+j}-H_j\right)\right] \pmod p. \tag{2.8}$$

In other words, for $j\geq m_2r_2+1$ the factorials for $i=1$ and $i=2$ each contain a factor of p; moreover, at most one p is cancelled by a term of the harmonic number. Noting that

$$((d_i-m_i)r_i+j)!/j! = (j+1)_{(d_i-m_i)r_i}, \tag{2.9}$$

we can rewrite the right-hand side:

$$A \equiv \sum_{j=0}^{p-1}\frac{d}{dj}\left[j\prod_{i=1}^{n+1}(j+1)_{(d_i-m_i)r_i}\right] \pmod p. \tag{2.10}$$

Define the polynomial $p(j)\in\mathbb{Z}[j]$ by

$$p(j) := \frac{d}{dj}\left[j\prod_{i=1}^{n+1}(j+1)_{(d_i-m_i)r_i}\right] = \sum_{k=0}^{D}a_kj^k, \text{ where } D := \sum_{i=1}^{n+1}(d_i-m_i)r_i, \tag{2.11}$$

to obtain

$$A \equiv \sum_{j=0}^{p-1}\left(a_0+\sum_{k=1}^{D}a_kj^k\right) \equiv \sum_{k=1}^{D}a_k\sum_{j=1}^{p-1}j^k \pmod p. \tag{2.12}$$

We consider the case $D<2(p-1)$. By Lemma 2.1, the only k we need to be concerned with in the j summation is $k=p-1$. Since $p(j)$ is a derivative, the coefficient a_{p-1} will contain a factor of p. Hence in this case, $A\equiv 0 \pmod p$. We consider the case $D=2(p-1)$. Using the above information, the only k we need to concern ourselves with is $k=2(p-1)$. Since $p(j)$ is a derivative of a monic polynomial, it follows that $a_{2(p-1)}=(2p-1)$. Using Lemma 2.1, we find that $A\equiv 1 \pmod p$. Since we can extend the first sum in (2.6) from m_2r_2 to $p-1$, the theorem follows. $\square$

3. Proofs of Theorems 2 and 3

Proof of Theorem 3. We recall the notation of Theorem 3. We define n such that $p = d(n+1)-1$ and define $N_1 := m(n+1)-1$, $N_2 := (d-m)(n+1)-1$. Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2, we obtain

(3.1)

$$p^4 \cdot {}_3F_2\left(\begin{matrix}\phi, & \rho^m, & \overline{\rho}^m \\ & \epsilon_{p^2}, & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} \equiv \left(\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{m}{d})_k(1-\frac{m}{d})_k}{k!^3}\right)^2$$
$$+ 2\cdot p \cdot \left(\sum_{k=0}^{N_1} k \cdot \frac{(\frac{1}{2})_k(\frac{m}{d})_k(1-\frac{m}{d})_k}{k!^3}\right) \cdot \left(\Gamma_p(\tfrac{1}{2})\Gamma_p(\tfrac{m}{d})\Gamma_p(1-\tfrac{m}{d})\right) \cdot B \pmod{p^2},$$

where

$$(3.2)\quad B := \left(\sum_{j=0}^{\frac{p-1}{2}} \frac{(\frac{p-1}{2}+j)!}{j!}\frac{(N_1+j)!}{j!}\frac{(N_2+j)!}{j!}\left[H_{\frac{p-1}{2}+j} + H_{N_1+j} + H_{N_2+j} - 3H_j\right]\right).$$

We point out that line (3.1) is similar to (6.21) of [M2]. We note that the case where $m = 1$, $d = p+1$ is handled like it is in (6.15) of [M2]. Arguing as in section 2, we obtain

$$(3.3)\qquad B \equiv \sum_{j=0}^{p-1} \frac{d}{dj}\left[(j+1)_{\frac{p-1}{2}}(j+1)_{N_1}(j+1)_{N_2}\right] \pmod{p}.$$

If we let d be the degree of the polynomial in terms of j, we see that $d < 2(p-1)$. Arguing as in section 2, we have that $B \equiv 0 \pmod{p}$. □

Proof of Theorem 2. We recall the notation of Theorem 2. We define n_i such that $p = d_i(n_i+1)-1$. Without lost of generality we assume $m_1/d_1 \le m_2/d_2 \le 1/2$. We define $R_i := m_i(n_i+1)-1$ and $S_i := (d_i - m_i)(n_i+1)-1$. We note $R_1 \le R_2$. Using the method of proof in [M2, section 6], and using the appropriate analog of Proposition 2.2, we obtain

$$-p^6 \cdot {}_4F_3\left(\begin{matrix}{\rho_1}^{m_1}, & \overline{\rho_1}^{m_1}, & {\rho_2}^{m_2}, & \overline{\rho_2}^{m_2} \\ & \epsilon_{p^2}, & \epsilon_{p^2}, & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2}$$
$$(3.4)\qquad \equiv \left(\sum_{k=0}^{R_2} \frac{(\frac{m_1}{d_1})_k(\frac{d_1-m_1}{d_1})_k(\frac{m_2}{d_2})_k(\frac{d_2-m_2}{d_2})_k}{k!^3}\right)^2$$
$$+ 2\cdot p \cdot \left(\sum_{k=0}^{R_1} k \cdot \frac{(\frac{m_1}{d_1})_k(\frac{d_1-m_1}{d_1})_k(\frac{m_2}{d_2})_k(\frac{d_2-m_2}{d_2})_k}{k!^3}\right)$$
$$\cdot \left(\Gamma_p(\tfrac{m_1}{d_1})\Gamma_p(\tfrac{d_1-m_1}{d_1})\Gamma_p(\tfrac{m_2}{d_2})\Gamma_p(\tfrac{d_2-m_2}{d_2})\right) \cdot C \pmod{p^2},$$

where

(3.5)

$$C := \sum_{j=0}^{R_2} \frac{(R_1+j)!}{j!}\frac{(S_1+j)!}{j!}\frac{(R_2+j)!}{j!}\frac{(S_2+j)!}{j!}\left[H_{R_1+j} + H_{S_1+j} + H_{R_2+j} + H_{S_2+j} - 4H_j\right].$$

The $m_1 = 1, d_1 = p+1$ case and the $m_1 = m_2 = 1, d_1 = d_2 = p+1$ case are handled like they are in [M2]. Arguing as we did in the proof of Theorem 3, we find that $C \equiv 0 \pmod{p}$. $\square$

4. Proof of Theorem 4

We begin with a theorem of Beukers and Stienstra that describes the coefficients of the three modular forms in question. We recall the modular forms (1.2)-(1.4).

Theorem ([SB, 14.2])**.** *If we define $\Phi_4(p) := a(p)$, $\Phi_3(p) := b(p)$, $\Phi_2(p) := c(p)$, then the p-th coefficients of the modular forms are given by*

$$\Phi_M(p) = \begin{cases} 0 & \text{if } (\frac{-M}{p}) = -1, \\ 4a^2 - 2p & \text{if } (\frac{-M}{p}) = 1,\ p = a^2 + Mb^2. \end{cases}$$

We rewrite the conjecture to motivate the use of Corollary 2 and Theorem 3.

Conjecture. *If $p \geq 5$ is a prime and $\gamma(p)$ is as before, then*

$$(4.1)\qquad {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \equiv a(p) \pmod{p^2},$$

$$(4.2)\qquad {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \equiv b(p) \pmod{p^2},$$

$$(4.3)\qquad {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \equiv c(p) \pmod{p^2},$$

$$(4.4)\qquad {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ & 1, & 1 \end{matrix} \mid 1\right)_{tr(p)} \equiv \gamma(p)a(p) \pmod{p^2}.$$

From the new formulation, we see that we need to use Corollary 2 and Theorem 3 where $m = 1$ and $d = 2, 3, 4$ or 6. For primes p where $p \equiv 1 \pmod{d}$ we use Corollary 2, for primes p where $p \equiv -1 \pmod{d}$ we use Theorem 3. The proof of Theorem 4 is thus reduced to evaluating the ${}_3F_2$ Gaussian hypergeometric series. First, we state a theorem which is a special case of Greene ([G], 4.38(ii)). The corollary follows from the two binomial coefficient properties (1.10).

Theorem ([G])**.** *If B is a nontrivial character on $\mathbb{F}_q$, then*

$${}_3F_2\left(\begin{matrix} \phi, & B, & \overline{B} \\ & \epsilon_q, & \epsilon_q \end{matrix} \mid 1\right)_q = B(-1)\begin{cases} 0 & \text{if } B \neq \square, \\ \binom{\chi}{\phi}\binom{\chi}{\phi\chi} + \binom{\phi\chi}{\phi}\binom{\phi\chi}{\chi} & \text{if } B = \chi^2. \end{cases}$$

Corollary 4.1. *If B is a nontrivial character on $\mathbb{F}_q$, then*

$$q^2 \cdot {}_3F_2\left(\begin{matrix} \phi, & B, & \overline{B} \\ & \epsilon_q, & \epsilon_q \end{matrix} \mid 1\right)_q = B(-1)\begin{cases} 0 & \text{if } B \neq \square, \\ J_r(\chi, \phi)^2 + J_r(\overline{\chi}, \phi)^2 & \text{if } B = \chi^2. \end{cases}$$

The following two propositions evaluate the Gaussian hypergeometric series in Corollary 2 and Theorem 3, respectively. Theorem 4 is then immediate. We recall (1.2)-(1.4) and define $\alpha_2(p) := a(p)$, $\alpha_3(p) := b(p)$, $\alpha_4(p) := c(p)$, and $\alpha_6(p) := a(p)$.

Proposition 4.2. *Fix a d, $d \in \{2,3,4,6\}$. Let p be a prime, $p \equiv 1 \pmod{d}$. If ρ_d is a character of order d on $\mathbb{F}_p$, then*

$$p^2 \cdot {}_3F_2\left(\begin{matrix}\phi, & \rho_d, & \overline{\rho_d} \\ & \epsilon_p, & \epsilon_p\end{matrix} \mid 1\right)_p = \alpha_d(p).$$

Proof of Proposition 4.2. This method comes from Ono [O], where he does the case $d = 2$. We have two cases. For the first case we consider p, $p \equiv d+1 \pmod{2d}$. Here ρ_d is not a square, so the Gaussian hypergeometric series evaluates to zero. Using the theorem of [SB] and basic Legendre symbol properties, we have that $\alpha_d(p) = 0$. For $d = 3$ this case is vacuous.

For the second case we consider p, $p \equiv 1 \pmod{2d}$. Here $\rho_d = \chi^2$ for some character χ. We consider $d = 4$. By [BEW, Theorem 3.3.1],

$$J_1(\chi,\phi)^2 + J_1(\overline{\chi},\phi)^2 = (a+ib\sqrt{2})^2 + (a-ib\sqrt{2})^2 = 4a^2 - 2p,$$

where $p = a^2 + 2b^2$. For $d = 2, 3$ and 6 we use [BEW, Theorems 3.2.1, 3.1.1, and 3.5.2], respectively. For each d, we use [SB] and see that this equals $\alpha_d(p)$. □

Proposition 4.3. *Fix a d, $d \in \{3,4,6\}$. Let p be a prime, $p \equiv -1 \pmod{d}$. If ρ_d is a character of order d on $\mathbb{F}_{p^2}$, then*

$$p^4 \cdot {}_3F_2\left(\begin{matrix}\phi, & \rho_d, & \overline{\rho_d} \\ & \epsilon_{p^2}, & \epsilon_{p^2}\end{matrix} \mid 1\right)_{p^2} = \alpha_d(p)^2 - (-1)^{\frac{p-(d-1)}{d}} 2p^2.$$

Proof of Proposition 4.3. We note that ρ_d is always a square. We have two cases to consider. For the first case, we consider p with $p \equiv -1 \pmod{2d}$. Using the theorem of [SB] we have that $\alpha_d(p) = 0$. By [BE, Theorem 2.14],

$$J_2(\chi,\phi)^2 + J_2(\overline{\chi},\phi)^2 = 2p^2.$$

For the second case, we consider p with $p \equiv d-1 \pmod{2d}$. We consider $d = 4$. By [BE, Theorem 4.6],

$$J_2(\chi,\phi)^2 + J_2(\overline{\chi},\phi)^2 = (a+ib\sqrt{2})^4 + (a-ib\sqrt{2})^4 = (4a^2-2p)^2 - 2p^2 = c(p)^2 - 2p^2,$$

where $p = a^2 + 2b^2$, and the last equality follows from [SB]. For $d = 3$ this case is vacuous. For $d = 6$ we use [BE, Theorem 4.10]. □

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