

## THE OSC DOES NOT IMPLY THE SOSC FOR INFINITE ITERATED FUNCTION SYSTEMS

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ABSTRACT. It is shown that every class of contracting similitudes  $\{f_1, \dots, f_N\}$  on  $R^s$  satisfying the OSC and such that  $\dim_H \mathcal{K}_0 < s$ , where  $\mathcal{K}_0$  denotes the corresponding fractal, can be extended to an infinite family of contracting similitudes which still satisfies the OSC but the SOSC does not hold.

### 1. NOTATION AND DEFINITIONS

We consider  $R^s$  ( $s \geq 1$ ) with the Euclidean norm  $|\cdot|$ . By  $\text{dist}(A, B)$  we denote the Hausdorff distance of two compact subsets  $A, B$  of  $R^s$ . Recall that a map  $f : R^s \rightarrow R^s$  is called a *similitude* with Lipschitz constant  $c \in (0, 1)$  if  $|f(x) - f(y)| = c|x - y|$  for  $x, y \in R^s$ . Let  $I$  be a countable index set with at least two elements, and let  $\{f_i : R^s \rightarrow R^s : i \in I\}$  be a collection of contracting similitudes – a so-called *ifs*. The class of maps  $\{f_i : i \in I\}$  is said to satisfy the *Open Set Condition (OSC)* if there is a nonempty and bounded open set  $U \subset R^s$  such that  $f_i(U) \subset U$  for all  $i$ , and  $f_i(U) \cap f_j(U) = \emptyset$  for  $i \neq j$ .

Put  $I^* = \bigcup_{n \geq 1} I^n$  and for  $\mathbf{i} \in I^n$ ,  $\mathbf{i} = (i_1, \dots, i_n)$ ,  $n \geq 1$ , set

$$f_{\mathbf{i}} = f_{i_1} \circ \dots \circ f_{i_n}.$$

If  $\mathbf{i} \in I^* \cup I^\infty$  and  $n \geq 1$  does not exceed the length of  $\mathbf{i}$ , we denote by  $\mathbf{i}_n$  the word  $(i_1, \dots, i_n)$ . Let  $\{f_i : i \in I\}$  satisfy the OSC with a set  $U$ . Set  $X = \text{cl}U$ . Observe now that given  $\mathbf{i} \in I^\infty$ , the compact sets  $f_{\mathbf{i}_n}(X)$ ,  $n \geq 1$ , are decreasing and their diameters converge to zero. This implies that the set

$$\pi(\mathbf{i}) = \bigcap_{n=0}^{\infty} f_{\mathbf{i}_n}(X)$$

is a singleton and therefore this formula defines a map  $\pi : I^\infty \rightarrow X$ . The main object of our interest will be the set

$$\mathcal{K} = \pi(I^\infty) = \bigcup_{\mathbf{i} \in I^\infty} \bigcap_{n=0}^{\infty} f_{\mathbf{i}_n}(X),$$

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called the *attractor* or *fractal* for  $\{f_i : i \in I\}$ . We easily check that

$$\mathcal{K} = \bigcup_{i \in I} f_i(\mathcal{K}).$$

The *Strong Open Set Condition (SOSC)* holds if the set  $U$  in the definition of the OSC can be chosen in such a way that  $\mathcal{K} \cap U \neq \emptyset$ .

For every finite word  $\mathbf{i} = (i_1, \dots, i_n)$  in the alphabet  $I$  we will use the following abbreviation:

$$c_{\mathbf{i}} = c_{i_1} \cdots c_{i_n}.$$

## 2. THE THEOREM

Let  $U \subset R^s$  be an open set, and let  $f_1, \dots, f_N$  be given. Assume that  $\{f_1, \dots, f_N\}$  satisfies the OSC with the set  $U$ . A. Schief proved that if  $f_i$ ,  $i \in \{1, \dots, N\}$ , are contracting similitudes, then the OSC implies the SOSC (see [6]). Y. Peres *et al.* showed that the above implication is valid for conformal contractions (see [5]). Recently, infinite families of contracting maps have been extensively studied (see for example [3]). However, the question (see Problem 7.5 in [3]) of whether the above implication holds in this case seems to be still open. The main aim of our note is to show that for infinite iterated function systems the OSC does not imply the SOSC.

Let  $\mathcal{K}_0$  denote the attractor for  $\{f_1, \dots, f_N\}$ . Assume that  $\dim_H \mathcal{K}_0 < s$ . Hence  $\sum_{i=1}^N c_i^s < 1$ , where  $c_i$  denotes the Lipschitz constant of  $f_i$ ,  $i = 1, \dots, N$  (see [2], [4]). Without loss of generality we may assume that  $c_i \leq c_1$  for all  $i \in \{1, \dots, N\}$ .

Let  $(a_n)_{n \geq 0}$  be a strictly decreasing sequence of reals such that  $1 < a_n \leq 2$  for all  $n \in \mathbb{N}$ .

Since  $\mathcal{L}^s(\mathcal{K}_0) = 0$ , where  $\mathcal{L}^s$  is the Lebesgue measure in  $R^s$ , we have  $U \setminus \mathcal{K}_0 \neq \emptyset$ . Choose  $x \in U \setminus \mathcal{K}_0$ . Since  $x \notin \mathcal{K}_0$ , there exists  $r > 0$  such that  $f_{j_1} \circ \dots \circ f_{j_m}(B(x, 2r)) \cap B(x, 2r) = \emptyset$  for  $j_1, \dots, j_m \in \{1, \dots, N\}$ ,  $m \in \mathbb{N}$ , and  $B(x, 2r) \subset U$ . We may also assume that  $\text{dist}(B(x, 2r), \mathcal{K}_0) > r$ .

The following lemma is crucial for our considerations.

**Lemma.** *Let  $((x_n, r_n))_{n \geq 1}$  be a dense sequence in  $\text{cl}U \times (0, r/2]$ , and let  $B_n = B(x_n, r_n)$  for  $n \in \mathbb{N}$ . Then there are contracting similitudes  $f_{N+1}, f_{N+2}, \dots$  such that there exist sequences  $(i_n)_{n \geq 0}$  of integers,  $(U_n)_{n \geq 0}$  of open subsets of  $U$  and  $(F_n)_{n \geq 0}$  of compact subsets of  $R^s$  satisfying:*

- (1)  $(i_n)_{n \geq 0}$  is nondecreasing,  $i_n \rightarrow \infty$  as  $n \rightarrow \infty$ ;
- (2)  $\{f_1, \dots, f_{i_n}\}$  satisfies the OSC with the set  $U_n$ ,  $\text{dist}(B(x, 2r), \mathcal{K}_n) > r$  and  $\dim_H \mathcal{K}_n < s$ , where  $\mathcal{K}_n$  denotes the attractor for  $\{f_1, \dots, f_{i_n}\}$ ,  $n \in \mathbb{N} \cup \{0\}$ ;
- (3)  $f_{\mathbf{j}}(B(x, 2r)) \cap B(x, 2r) = \emptyset$ , where  $\mathbf{j}$  is a finite word in the alphabet  $\{1, \dots, i_n\}$ ,  $n \in \mathbb{N} \cup \{0\}$ ;
- (4)  $B(x, a_n r) \subset U_n$  for  $n \in \mathbb{N} \cup \{0\}$ ;
- (5)  $U_n = U_{n-1} \setminus F_n$ ,  $f_{\mathbf{j}}^{-1}(F_n) \cap U_{n-1} \subset F_n$ , where  $\mathbf{j}$  is a finite word in the alphabet  $\{1, \dots, i_n\}$ ,  $n \in \mathbb{N}$ ;
- (6)  $f_{i_n}(\mathcal{K}_{n-1}) \subset B_n$  if  $B_n \cap \mathcal{K}_{n-1} \neq \emptyset$  for  $n \in \mathbb{N}$ .

*Proof.* We set  $i_0 = N$ ,  $U_0 = U$  and  $F_0 = \emptyset$ . Now assume that we are given  $i_p, f_1, \dots, f_{i_p}$ ,  $F_p$  and  $U_p$  for some  $p \in \mathbb{N} \cup \{0\}$ . We will define  $i_{p+1}$ ,  $f_{i_{p+1}}$ ,  $F_{p+1}$  and  $U_{p+1}$  in such a way that conditions (1)–(6) will be satisfied. If  $B_{p+1} \cap \mathcal{K}_p = \emptyset$ , then we set  $i_{p+1} = i_p$ ,  $F_{p+1} = F_p$  and  $U_{p+1} = U_p$ . If  $B_{p+1} \cap \mathcal{K}_p \neq \emptyset$ , then  $i_{p+1} = i_p + 1$ . Set  $r_0 = \text{dist}(B(x, 2r), \mathcal{K}_p)$ . Obviously  $r_0 > r$ . Choose  $\eta \in (0, r_0 - r)$

and set  $\varepsilon = (1 - c_1)\eta$ .  $y \in \text{int}(B(x, a_p r) \setminus B(x, a_{p+1} r))$ . Choose  $\mathbf{i}$  to be a finite word in the alphabet  $\{1, \dots, i_p\}$  such that  $f_{\mathbf{i}}(y) \in B(x_0, \varepsilon)$ , and let  $\kappa > 0$  be such that  $B(y, \kappa) \subset \text{int}(B(x, a_p r) \setminus B(x, a_{p+1} r))$  and  $f_{\mathbf{i}}(B(y, \kappa)) \subset B(x_0, \varepsilon)$ . Since  $\mathcal{L}^s(f_{\mathbf{i}}(B(y, \kappa))) > 0$ ,  $\mathcal{L}^s(\mathcal{K}_p) = 0$  and  $\mathcal{K}_p$  is compact, there exist  $y_0 \in B(y, \kappa)$  and  $\kappa_0 < \kappa$  such that  $\overline{B}(y_0, \kappa_0) \subset B(y, \kappa)$  and  $f_{\mathbf{i}}(\overline{B}(y_0, \kappa_0)) \cap \mathcal{K}_p = \emptyset$ , where  $\overline{B}(y_0, \kappa_0)$  denotes the closed  $\kappa_0$ -ball centered at  $y_0$ . Set  $W = f_{\mathbf{i}}(\overline{B}(y_0, \kappa_0))$ . We see at once that

$$\#\{\mathbf{j} : \mathbf{j} \text{ is a finite word in the alphabet } \{1, \dots, i_p\} \text{ and } f_{\mathbf{j}}^{-1}(W) \cap \text{cl}U \neq \emptyset\} < \infty.$$

Moreover since  $f_{\mathbf{i}}(y_0)$  does not belong to  $\mathcal{K}_p$ , it is not a fixed point for  $f_{\mathbf{j}}$ , where  $\mathbf{j}$  is an arbitrary finite word in the alphabet  $\{1, \dots, i_p\}$ . Then, if we choose  $\kappa_0$  small enough, we can also assume that  $W \cap f_{\mathbf{j}}^{-1}(W) = \emptyset$  for all finite words  $\mathbf{j}$ . Define

$$F_{p+1} = \bigcup \left( f_{\mathbf{j}}^{-1}(W) \cap \text{cl}U \right),$$

where the above union is over all finite words in the alphabet  $\{1, \dots, i_p\}$ , and observe that by the above  $F_{p+1}$  is compact. Further, since  $\mathcal{K}_p = \text{cl} \bigcup_{\mathbf{j}} \text{Fix}(f_{\mathbf{j}})$ , where the union is taken over all finite words in the alphabet  $\{1, \dots, i_p\}$  (see [1]), there exists  $\mathbf{j}_0$  such that  $\text{Fix}(f_{\mathbf{j}_0}) \in B(x_0, \varepsilon)$ . Without loss of generality we may assume that  $\text{diam} f_{\mathbf{j}_0}(U) < c_1 \kappa_0$  and  $\sum_{i=1}^p c_i^s + c_{\mathbf{j}_0}^s < 1$ . Define  $f_{i_{p+1}} = f_{\mathbf{j}_0} + (f_{\mathbf{i}}(y_0) - \text{Fix}(f_{\mathbf{j}_0}))$  and  $U_{p+1} = U_p \setminus F_{p+1}$ .

We see at once that conditions (1), (3), (5) and (6) hold. Now we check condition (2). From the construction it follows that  $f_i(U_{p+1}) \cap f_j(U_{p+1}) = \emptyset$  for  $i, j \in \{1, \dots, i_{p+1}\}, i \neq j$ . To finish the proof of the OSC we have to show that  $f_i(U_{p+1}) \subset U_{p+1}$  for  $i \in \{1, \dots, i_{p+1}\}$ . We see at once that  $f_{i_{p+1}}(U_{p+1}) \subset U_{p+1}$ . Let  $f_i(z) \notin U_{p+1}$  for some  $z \in U_n$  and  $i \in \{1, \dots, i_{p+1} - 1\}$ . Then  $f_i(z) \in F_{p+1}$  and hence  $z \in f_i^{-1}(F_{p+1})$ . Further, since  $z \in f_i^{-1}(F_{p+1}) \cap U_p \subset F_{p+1}$ , we obtain  $z \notin U_{p+1} = U_p \setminus F_{p+1}$ , which is a contradiction. From the construction it follows that

$$\text{dist}(\mathcal{K}_{p+1}, \mathcal{K}_p) \leq \varepsilon(1 - c_1)^{-1}$$

and consequently

$$\begin{aligned} \text{dist}(\mathcal{K}_{p+1}, B(x, 2r)) &\geq \text{dist}(\mathcal{K}_{p+1}, B(x, 2r)) - \text{dist}(\mathcal{K}_{p+1}, \mathcal{K}_p) \\ &\geq \text{dist}(\mathcal{K}_{p+1}, B(x, 2r)) - \varepsilon(1 - c_1)^{-1} \\ &= \text{dist}(\mathcal{K}_{p+1}, B(x, 2r)) - \eta > r, \end{aligned}$$

which finishes the proof of (2). To prove (4) assume that  $B(x, a_{p+1} r) \setminus U_{p+1} \neq \emptyset$ . We have  $B(x, a_{p+1} r) \subset B(x, a_p r) \subset U_p$ . Since  $U_{p+1} = U_p \setminus F_{p+1}$ , we have  $B(x, a_{p+1} r) \cap F_{p+1} \neq \emptyset$ . By (3) we see that if  $f_{\mathbf{i}}(B(x, a_p r)) \cap f_{\mathbf{j}}(B(x, a_p r)) \neq \emptyset$ , then  $\mathbf{i} = \mathbf{j}$ . Therefore if  $f_{\mathbf{i}}^{-1}(W) \cap B(x, a_p r) \neq \emptyset$  and  $f_{\mathbf{j}}^{-1}(W) \cap B(x, a_p r) \neq \emptyset$ , we have  $\mathbf{i} = \mathbf{j}$ . From this it follows that there is a unique finite word  $\mathbf{i}$  in the alphabet  $\{1, \dots, i_{p+1}\}$  such that  $f_{\mathbf{i}}^{-1}(W) \subset B(x, a_p r)$ . Since  $f_{\mathbf{i}}^{-1}(W) \subset B(x, a_p r) \setminus B(x, a_{p+1} r)$ , we have  $B(x, a_{p+1} r) \cap F_{p+1} = \emptyset$ , which, in turn, is a contradiction. The proof of (4) is finished.  $\square$

**Theorem.** *Every class of contracting similitudes  $\{f_1, \dots, f_N\}$  on  $R^s$  satisfying the OSC and such that  $\dim_H \mathcal{K}_0 < s$ , where  $\mathcal{K}_0$  denotes the corresponding fractal, can be extended to an infinite family of contracting similitudes which still satisfies the OSC but the SOSOC does not hold.*

*Proof.* We show that an infinite ifs given by the above lemma will satisfy the OSC but will not satisfy the SOSC. Set  $U_0 = U \setminus \text{cl} \bigcup_{i=1}^{\infty} F_i$ . From conditions (4) and (5) it follows that  $U_0 \neq \emptyset$ . Observe that  $f_k(U_0) \subset U_0$  for  $k \in \mathbb{N}$ . Indeed, for every  $k$  there exists  $n_0 \in \mathbb{N}$  such that  $i_{n_0} \geq k$ . Then  $f_k(U_{n_0}) \subset U_{n_0}$ . Let  $f_k(z) \notin U_0$  for some  $z \in U_0$ . Therefore there exists a sequence  $(y_n)_{n \geq 1}$ ,  $y_n \in \bigcup_{i=1}^{\infty} F_i$ ,  $n \in \mathbb{N}$ , such that  $y_n \rightarrow f_k(z) \in U_{n_0} = U \setminus \bigcup_{i=1}^{n_0} F_i$ . Hence we may assume that  $y_n \notin F_j$  for  $j = 1, \dots, n_0$ . On the other hand,  $f_k^{-1}(F_j) \cap U_0 \subset F_j$  for  $j > n_0$  and

$$|f_k^{-1}(y_n) - z| = c_k^{-1}|y_n - f_k(z)| \rightarrow 0$$

as  $n \rightarrow \infty$ . Since  $f_k^{-1}(y_n) \in \bigcup_{i=n_0+1}^{\infty} F_i$  for all large  $n$ , we have  $z \in \text{cl} \bigcup_{i=n_0+1}^{\infty} F_i \subset \text{cl} \bigcup_{i=1}^{\infty} F_i$ , which is a contradiction.

Now we show that  $f_p(U_0) \cap f_q(U_0) = \emptyset$  for  $p \neq q$ . Namely, choose  $m$  such that  $i_m > \max\{p, q\}$ . Then we have  $U_0 \subset U_m$  and  $\{f_1, \dots, f_{i_m}\}$  satisfies the OSC with  $U_m$ .

Finally, we show that  $\{f_1, f_2, \dots\}$  does not satisfy the SOSC. Let  $\mathcal{K}$  be its attractor. It is obvious that  $\bigcup_{i=1}^{\infty} \mathcal{K}_i \subset \mathcal{K} \subset \text{cl} \bigcup_{i=1}^{\infty} \mathcal{K}_i$ . Suppose to the contrary that the SOSC holds. Then there exist  $z \in \mathcal{K}$  and an open set  $V$  such that  $z \in V$  and  $f_i(V) \cap f_j(V) = \emptyset$  for  $i \neq j$ . However, since  $V$  is open and  $z \in \mathcal{K}$ , there exists  $m \in \mathbb{N}$  such that  $\mathcal{K}_m \cap V \neq \emptyset$ . On the other hand, since  $\mathcal{K}_m = \text{cl} \bigcup \text{Fix } f_{\mathbf{i}}$ , where the union is over all finite words  $\mathbf{i}$  in the alphabet  $\{1, \dots, i_m\}$  (see [1]), without loss of generality we can assume that  $z$  is a fixed point of  $f_{\mathbf{i}}$  for some finite word  $\mathbf{i}$  in the alphabet  $\{1, \dots, i_m\}$ . Let  $\tau > 0$  be such that  $B(z, \tau) \subset V$ . Then  $f_{\mathbf{i}}(B(z, \tau)) = B(z, c_{\mathbf{i}}\tau)$ . Further let  $n \in \mathbb{N}$  be such that  $i_{n+1} \geq i_m$  and  $B_{n+1} = B(x_{n+1}, r_{n+1})$  satisfies  $B_{n+1} \cap \mathcal{K}_m \neq \emptyset$  and  $B_{n+1} \subset B(z, c_{\mathbf{i}}\tau)$ . Then  $B(z, c_{\mathbf{i}}\tau) \subset f_{i_{n+1}}(V)$  and  $f_{i_{n+1}}(\mathcal{K}_m) \subset B(z, c_{\mathbf{i}}\tau)$ . Hence  $f_{\mathbf{i}}(V) \cap f_{i_{n+1}}(V) \neq \emptyset$ , which is a contradiction.  $\square$

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