

LAGRANGE MULTIPLIERS FOR FUNCTIONS DERIVABLE ALONG DIRECTIONS IN A LINEAR SUBSPACE

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ABSTRACT. We prove a Lagrange multipliers theorem for a class of functions that are derivable along directions in a linear subspace of a Banach space where they are defined. Our result is available for topological linear vector spaces and is stronger than the classical one even for two-dimensional spaces, because we only require the differentiability of functions at critical points. Applying these results we generalize the Lax-Milgram theorem. Some applications in variational inequalities and quasilinear elliptic equations are given.

1. INTRODUCTION

The concept of Gâteaux-differentiability is compatible to weak solutions of partial differential equations but not to generalized solutions of partial differential equations. In some cases the considered functionals are only derivable along directions in a linear subspace of a Banach space where they are defined. In [2, 3, 6, 9, 12] these cases have been studied under some generalized Palais-Smale conditions and mountain-pass lemma techniques. In the present paper we prove a Lagrange multipliers theorem for those functionals. Our result is available for topological linear vector spaces and is stronger than the classical one even for two-dimensional spaces (see [7], p.293), because we only require the differentiability of functions at critical points.

The essential point of the Lagrange multipliers theorem is to provide the collinearity of two vectors (usually denoted by $Df(x)$ and $Dg(x)$). We shall reduce this problem to the two-dimensional case by the following idea : Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors in \mathbb{R}^n with $y_1 \neq 0$. In order to prove the collinearity of x and y it is sufficient to show that (x_1, x_i) and (y_1, y_i) are collinear for any i in $\{2, \dots, n\}$. Indeed $x = \frac{x_1}{y_1}y$ in this case. Using this reduction we neither need the compactness conditions on functionals nor the completeness of infinite-dimensional spaces.

In the second section we give simple proofs of Lagrange multiplier theorems without any assumption on completeness and local convexity of considered spaces. We only need these results for the Banach spaces in the last two sections, but we do

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not get any extra complication for the proofs in general cases of topological vector spaces.

In the third section we point out that the Lax-Milgram theorem can be obtained by using the Lagrange multiplier theorem, and we generalize its classical version (see [4], p.84), then apply it to solve an elliptic problem.

In the last two sections we apply our results to variational inequalities and quasi-linear elliptic eigenvalue problems.

2. LAGRANGE MULTIPLIERS THEOREMS FOR G -DIFFERENTIABLE FUNCTIONS IN TOPOLOGICAL LINEAR SPACES

Let U be an open subset of a topological vector space E , G be a linear subspace of E , and f be a function from U into \mathbb{R} . For any $(x, h) \in U \times G$ we define $U(x, h) = \{ t \in \mathbb{R} : x + th \in U \}$ and $f_{(x, h)}(t) = f(x + th) \forall t \in U(x, h)$.

Since U is open in E , it is clear that $U(x, h)$ is an open subset of \mathbb{R} .

We say : (i) f is G -continuous on U if and only if $f_{(x, h)}$ is continuous at 0 for any (x, h) in $U \times G$.

(ii) f is G -differentiable at x if and only if there exists a linear mapping $Df(x)$ from G into \mathbb{R} such that

$$\lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t} = Df(x)(h) \quad \forall h \in G.$$

(iii) f is G -differentiable on U if f is G -differentiable at every x in U .

(iv) f is strongly G -differentiable at x if and only if there exists a linear mapping $Df(x)$ from G into \mathbb{R} such that

$$\lim_{(s, t) \rightarrow 0} \frac{f(x + sth + sk) - f(x)}{s} = Df(x)(k) \quad \forall h, k \in G.$$

Example 2.1. Let G be a linear subspace of a topological vector space E and f be a linear mapping from E into \mathbb{R} , which may be discontinuous on E . It is clear that f is G -continuous on E . Fix three vectors x, h and k in E . We see that

$$\frac{f(x + sth + sk) - f(x)}{s} = f(k) + tf(h) \quad \forall s, t \in \mathbb{R} \setminus \{0\}.$$

Therefore, f is strongly G -differentiable at any x in E for any linear subspace G of E .

Example 2.2. Let U be an open subset of a normed linear space E , x be in U and f be a mapping from U into \mathbb{R} such that f is Fréchet differentiable at x . It is clear that f is strongly G -differentiable at x for any linear subspace G of E .

First we have an “implicit mapping theorem” for G -differentiable mappings.

Lemma 2.3. Let U be an open subset of a topological vector space X , G be a linear subspace of X , g be a real G -continuous function on U . Let u be in U , and v and w be in G . Assume that g is G -differentiable at u and $Dg(u)(w) \neq 0$. Put $\beta = -\frac{Dg(u)(v)}{Dg(u)(w)}$. Then there exist two sequences of real numbers $\{\alpha_n\}$ and $\{\delta_n\}$ converging to 0 such that α_n is positive and $g(u_n) = g(u)$ for any integer n , where $u_n = u + \alpha_n(\delta_n + \beta)w + \alpha_nv$.

Proof. Replacing g with $g - g(u)$ if necessary, we can assume without loss of generality that $g(u) = 0$. By definition we see that $Dg(u)(\beta w + v) = 0$. For sufficiently small real numbers α and δ we define

$$\begin{aligned}\phi(\alpha, \beta, \delta) &= \alpha^{-1}[g(u + \alpha(\delta w + \beta w + v)) - g(u)] - Dg(u)(\delta w + \beta w + v), \\ G_\alpha(\delta) &= g(u + \alpha(\delta w + \beta w + v)).\end{aligned}$$

Since $g(u) = 0$ and g is G -differentiable at u , we have $\lim_{\alpha \rightarrow 0} \phi(\alpha, \beta, \delta) = 0$ and

$$\begin{aligned}G_\alpha(\delta) &= g(u + \alpha(\delta w + \beta w + v)) - g(u) \\ &= \alpha[Dg(u)(\delta w + \beta w + v) + \phi(\alpha, \beta, \delta)] = \alpha[\delta Dg(u)(w) + \phi(\alpha, \beta, \delta)].\end{aligned}$$

Hence there is a sequence of positive real numbers $\{\alpha_n\}$ converging to 0 such that

$$G_{\alpha_n}(-\frac{1}{n}).G_{\alpha_n}(\frac{1}{n}) < 0.$$

By (i) there is an n_0 such that the map

$$x \longmapsto G_{\alpha_n}(x) = g(u + \alpha_n(\beta w + v) + x\alpha_n w)$$

is continuous on $[-\frac{1}{n}, \frac{1}{n}]$ for any integer $n \geq n_0$. Thus there exists a δ_n in $[-\frac{1}{n}, \frac{1}{n}]$ such that $G_{\alpha_n}(\delta_n) = g(u_n) = 0$, where $u_n = u + \alpha_n(\delta_n w + \beta w + v) = u + \alpha_n(\delta_n + \beta)w + \alpha_n v$. \square

Remark 2.4. If v and w are linearly independent, then u_n and u are different for any integer n .

Let C be a subset of a topological vector space X . Then C is called a cone if αu is in C for any (α, u) in $[0, \infty) \times C$. We have the following Lagrange multiplier results for G -differentiable functions.

Theorem 2.5. *Let C be a convex cone of a topological vector space X , U be an open subset of X , G be a linear subspace of X , u be in $U \cap C \cap G$, r be a real number, and f and g be two functions on U . Put $M = \{x \in C \cap U : g(x) = r\}$.*

Assume:

- (i) $f(u)$ is the minimum (or maximum respectively) of $f(M)$,
- (ii) f is strongly G -differentiable at u and g is G -differentiable at u ,
- (iii) g is G -continuous on U and $Dg(u)(u) \neq 0$.

Then there exists a real number λ such that $\lambda Dg(u)(v - u) \leq Df(u)(v - u)$ (or $\lambda Dg(u)(v - u) \geq Df(u)(v - u)$ respectively) for any v in $C \cap G$.

Proof. Applying Lemma 2.3 for $w = u$, we can find two sequences of real numbers $\{\alpha_n\}$ and $\{\delta_n\}$ converging to 0 such that α_n is positive and $g(u_n) = r$ for any integer n , where $u_n = [1 + \alpha_n(\delta_n + \beta)]u + \alpha_n v$ and $\beta = -\frac{Dg(u)(v)}{Dg(u)(u)}$. Since C is a convex cone and $[1 + \alpha_n(\delta_n + \beta)] > 0$ for every sufficiently large n , we can suppose that u_n is in M for any integer n , and we have

$$0 \leq \alpha_n^{-1}[f(u + \alpha_n(\delta_n u + \beta u + v)) - f(u)] = Df(u)(\beta u + v) + \varphi(\alpha_n, \delta_n),$$

where $\varphi(\alpha_n, \delta_n) = \alpha_n^{-1}[f(u + \alpha_n(\delta_n u + \beta u + v)) - f(u)] - Df(u)(\beta u + v)$.

Since $\lim_{n \rightarrow \infty} \varphi(\alpha_n, \delta_n) = 0$, letting n tend to ∞ we obtain

$$\begin{aligned}0 \leq Df(u)(\beta u + v) &= -\frac{Dg(u)(v)}{Dg(u)(u)}Df(u)(u) + Df(u)(v) \\ &= Df(u)(v) - \lambda Dg(u)(v)\end{aligned}$$

where $\lambda = \frac{Df(u)(u)}{Dg(u)(u)}$. Thus for any v in $C \cap G$,

$$\lambda Dg(u)(v - u) = \lambda Dg(u)(v) - Df(u)(u) \leq Df(u)(v) - Df(u)(u).$$

□

Theorem 2.6. *Let U be an open subset of a topological vector space E , G be a linear subspace of E , x_0 be in U , r be a real number, and f and g be two functions from U into \mathbb{R} . Put $M = \{x \in U : g(x) = r\}$. Assume that:*

- (i) $f(x_0)$ is the minimum (or maximum) of $f(M)$,
- (ii) f is strongly G -differentiable at x_0 and g is G -differentiable at x_0 ,
- (iii) g is G -continuous on U and $Dg(x_0) \neq 0$.

Then there exists a real number λ such that $Df(x_0)(h) = \lambda Dg(x_0)(h) \forall h \in G$.

Proof. Choose w in G such that $Dg(x_0)(w) > 0$, and fix a vector v in G . Applying Lemma 2.3, we can find two sequences of real numbers $\{\alpha_n\}$ and $\{\delta_n\}$ converging to 0 such that α_n is positive and $g(u_n) = r$ for any integer n , where $u_n = u + \alpha_n(\delta_n + \beta)w + \alpha_n v$ and $\beta = -\frac{Dg(u)(v)}{Dg(u)(w)}$. Since U is open, we can suppose that u_n is in M for any integer n . Arguing as in the proof of Theorem 2.5, we obtain

$$0 \leq [Df(x_0) - \lambda Dg(x_0)](v).$$

Replacing v in the above inequality by $-v$ we get the theorem. □

Remark 2.7. Without using the classical implicit function theorem (see [7], p.362) in the proof of Theorem 2.6, we do not need the completeness of E , which is crucial in the classical Lagrange multiplier theorems.

We obtain a result on the signs of Lagrange multipliers in the following theorem.

Theorem 2.8. *Let U be an open subset of a topological vector space E , G be a linear subspace of E , x_0 be in U , r be a real number, f and g be two functions from U into \mathbb{R} and $M = \{x \in U : g(x) \geq r\}$. Assume that:*

- (i) $f(x_0)$ is the minimum of $f(M)$,
- (ii) f is strongly G -differentiable at x_0 and g is G -differentiable at x_0 ,
- (iii) g is G -continuous on U and $Dg(x_0) \neq 0$.

Then there exists a non-negative real number λ such that $Df(x_0)(h) = \lambda Dg(x_0)(h) \forall h \in G$.

Proof. Put $\alpha = g(x_0)$, $S = \{x \in U : g(x) = \alpha\}$ and $N = \{x \in U : g(x) \geq \alpha\}$. We have $\alpha \geq r$ and

$$f(x_0) = \min f(S) = \min f(N).$$

According to Theorem 2.6 there exists a real number λ such that

$$(2.1) \quad Df(x_0)(h) = \lambda Dg(x_0)(h) \quad \forall h \in G.$$

Now we prove that $\lambda \geq 0$. Let k be in G such that $Dg(x_0)(k) > 0$. Since g is G -differentiable at x_0 , we have

$$\lim_{t \rightarrow 0} \frac{g(x_0 + tk) - g(x_0)}{t} = Dg(x_0)(k) > 0.$$

Thus, there exists a $\theta > 0$ such that for any $t \in (0, \theta)$ we have

$$g(x_0 + tk) - g(x_0) > 0$$

or $(x_0 + tk) \in N$. Consequently, since f is G -differentiable at x_0 and $f(x_0) = \min f(N)$, we have

$$(2.2) \quad 0 \leq \lim_{t \rightarrow 0^+} \frac{f(x_0 + tk) - f(x_0)}{t} = Df(x_0)(k).$$

Since $Dg(x_0)k > 0$, combining (2.1) and (2.2) we see that $\lambda \geq 0$. \square

Denote by M' the set $\{x \in U : g(x) \leq r\}$. We note that if $f(x_0) = \max f(M')$, we obtain the same result. We have a similar result for $\lambda \leq 0$, if $f(x_0) = \max f(M)$, or $f(x_0) = \min f(M')$.

Remark 2.9. The signs of Lagrange multipliers are important to solve some problems. In [8] Le and Schmitt applied the regularity of solutions of elliptic equations to get the positivity of Lagrange multipliers. Their method may not be available if the equations are strongly singular. Our main idea in the proof of the above theorem is as follows : choose a convenient value level of function g when we apply the Lagrange multiplier theorem.

3. A LAX-MILGRAM THEOREM

In this section we point out that the Lax-Milgram theorem can be obtained by using the Lagrange multipliers theorem. Furthermore, applying techniques in section 2 we get the following version of Lax-Milgram's theorem.

Theorem 3.1. *Let E be a reflexive Banach space, $\{E_n\}$ be an increasing sequence of closed subspaces of E , and V be $\bigcup_{n=1}^{\infty} E_n$. Let f be a function from $E \times V$ into \mathbb{R} and T be a bounded linear mapping from V into \mathbb{R} , where the norm on V is the one induced from E . Suppose the following conditions are satisfied.*

(i) *For fixed n in \mathbb{N} , $f_n \equiv f|_{E_n \times E_n}$ is a bilinear bounded function, and for any fixed v in V , $f(\cdot, v)$ is a linear bounded functional on E .*

(ii) *f is coercive on V , i.e., there exists $C > 0$ such that*

$$f(u, u) \geq C\|u\|^2 \quad \forall u \in V.$$

Then there exists a vector z in E such that

$$f(z, v) = T(v) \quad \forall v \in V.$$

Proof. Denote by T_n the restriction of T on E_n . If $T \equiv 0$, we choose $z = 0$. If $T \neq 0$, we can suppose that $T_1^{-1}(\{1\})$ is not empty. Put $\phi_n(u) = \frac{1}{2}f_n(u, u)$ for any u in E_n . It is clear that ϕ_n and T_n are strongly E_n -differentiable. Now putting $S_n = T_n^{-1}(\{1\})$, we see that $S_n \subset S_{n+1}$ for all n in \mathbb{N} . Since T_n is a linear bounded functional on E_n , it is weakly continuous on E_n . By Theorem 1.2 in ([13], p.4), there exists u_n in E_n such that $\phi_n(u_n) = \min \phi_n(S_n) \forall n \in \mathbb{N}$. By Theorem 2.6, there exists a number λ_n such that $D\phi_n(u_n)(v) = \lambda_n DT_n(u_n)(v) \forall v \in E_n$ or

$$(3.1) \quad f(u_n, v) = \lambda_n T(v) \quad \forall v \in E_n, n \in \mathbb{N},$$

It follows that $\lambda_n = f(u_n, v)$ for any v in S_n and

$$(3.2) \quad 0 < C\|u_n\|^2 \leq f_n(u_n, u_n) = \lambda_n.$$

Since T is a bounded linear mapping on V , there is a positive real number β such that

$$(3.3) \quad \beta < \|x\|^2 \quad \forall x \in T^{-1}(1).$$

Combining (3.2) and (3.3) we get a positive real number α such that $\lambda_n > \alpha$ for any integer n . Put $z_n = \lambda_n^{-1}u_n$ for any integer n . Because $S_n \subset S_{n+1}$ and $\phi(u_n) = \min \phi(S_n)$, we see that $\{\phi(u_n)\}_n$ is non-increasing. Therefore,

$$C\|u_n\|^2 \leq 2\phi(u_1).$$

This implies that $\{\|u_n\|_E\}$ is bounded in \mathbb{R} . Since E is a reflexive Banach space, we can assume that $\{u_n\}$ weakly converges to a vector u in E . Fix a w in S_1 . Since $f(\cdot, w)$ belongs to the dual space of E , we have $\lim_{n \rightarrow \infty} f(u_n, w) = f(u, w)$. Thus $\lim_{n \rightarrow \infty} \lambda_n = \lambda \equiv f(u, w)$. Therefore, $\lambda \geq \alpha > 0$ and $\{z_n\}$ weakly converges to $z \equiv \lambda^{-1}u$ in E .

Now fix a v in V . There is an integer number n_0 such that $v \in E_n$ for any $n \geq n_0$. By (3.1),

$$f(z_n, v) = T(v) \quad \forall n \geq n_0.$$

Letting n tend to infinity, by (i), we get the theorem. □

Remark 3.2. The vector z in the above theorem may not be in V (see Theorem 3.3 in this paper). If $V = E$ and f is well-defined and continuous on $E \times E$, Theorem 3.1 has been proved in [10, 11].

Let Ω be a bounded domain in \mathbb{R}^m , and let $A = (A_{ij}^{\alpha\beta})$ be a measurable matrix function from Ω into $\mathbb{R}^{N^2 \times m^2}$ such that $A_{ij}^{\alpha\beta} = A_{ij}^{\beta\alpha}$ for any $i, j = 1, \dots, N$, and any $\alpha, \beta = 1, \dots, m$. We assume that

(i) There is a function ρ in $L^\infty_{loc}(\Omega)$ such that

$$\|A(x)\| \leq \rho(x) \quad \forall x \in \Omega.$$

(ii) There is a positive real number ν such that

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i \xi_\beta^j \geq \nu|\xi|^2 \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^{N \times m}.$$

By convention, repeated Greek indices are summed from 1 to m , Latin indices from 1 to N .

Applying Theorem 3.1 we get the following result.

Theorem 3.3. *Let h be in $L^s(\Omega, \mathbb{R}^N)$, where $s > 1$ if $m = 2$ and $s = \frac{2m}{m+2}$ if $m \geq 3$. Then there exists a generalized solution u_0 in $W_0^{1,2}(\Omega, \mathbb{R}^N)$ to the following boundary problem:*

$$\begin{cases} -\partial_\beta(A_{ij}^{\alpha\beta}\partial_\alpha u^i) = h^j & \text{on } \Omega, \forall j = 1, \dots, N, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Proof. Let $\{\Omega_n\}$ be an increasing sequence of open subsets in Ω such that $\overline{\Omega}_n$ is contained in Ω_{n+1} and $\Omega = \bigcup_{n=1}^\infty \Omega_n$. Let E be the usual Sobolev space $W_0^{1,2}(\Omega, \mathbb{R}^N)$, $E_n = W_0^{1,2}(\Omega_n, \mathbb{R}^N)$ for any integer n and

$$\begin{aligned} V &= \bigcup_{n=1}^\infty E_n, \\ f(u, v) &= \int_\Omega A_{ij}^{\alpha\beta} \partial_\alpha u^i \partial_\beta v^j dx \quad \forall (u, v) \in E \times V. \end{aligned}$$

By the Sobolev embedding theorem we can define

$$T(v) = \int_{\Omega} h^j v^j dx \quad \forall v \in E,$$

and see that T is a bounded linear mapping from E into \mathbb{R} . Applying the foregoing theorem we can find a function u_0 in E such that

$$(3.4) \quad f(u_0, v) = T(v) \quad \forall v \in V.$$

We get the result by using $v = (\delta_j^1, \dots, \delta_j^N)\varphi$ in (3.4), where φ is in $C_c^\infty(\Omega)$, j is in $\{1, \dots, N\}$, and δ_j^k is the Kronecker number. \square

4. AN APPLICATION TO VARIATIONAL INEQUALITIES

Denote by X the space of all u in the Sobolev space $W^{2,2}((0, l))$ such that u and u' are in $W_0^{1,2}((0, l))$. Then X is a normed space with the $W^{2,2}((0, l))$ -norm. Fixing an integrable function P on $(0, l)$, for any u in X we define

$$f(u) = \frac{1}{2} \int_0^l |u''(x)|^2 dx \quad \text{and} \quad g(u) = \frac{1}{2} \int_0^l P(x)|u'(x)|^2 dx.$$

Let C be a closed and convex cone of X such that there exists u_0 in C with $g(u_0) = r \neq 0$. We are seeking solutions (u, λ) in $C \times \mathbb{R} \setminus \{0\}$ such that

$$(Q) \quad Df(u)(v - u) \geq \lambda Dg(u)(v - u) \quad \forall v \in C.$$

Put $M = \{u \in C : g(u) = r\} = C \cap g^{-1}(\{r\})$.

Lemma 4.1. *Under the above conditions we have:*

(i) f is weakly lower semi-continuous, coercive and strongly X -differentiable on X and

$$Df(u)(h) = \int_0^l u'' h'' dx \quad \forall u, h \in X.$$

(ii) g is weakly continuous and X -differentiable on X and

$$Dg(u)(h) = \int_0^l P u' h' dx \quad \forall u, h \in X.$$

Proof. By the Poincaré inequality $|f(u)|^{1/2}$ is equivalent to the norm on X . Therefore f is weakly lower semi-continuous and coercive on X . For any u, h, k in X and any s, t in \mathbb{R} we have

$$f(u + sth + sk) - f(u) = \int_0^l s u'' (th'' + k'') dx + \frac{1}{2} \int_0^l s^2 |(th'' + k'')|^2 dx,$$

$$\lim_{(s, t) \rightarrow 0} \frac{f(u + sth + sk) - f(u)}{s} = \int_0^l u'' k'' dx.$$

Let $\{u_n\}$ be an arbitrary sequence converging weakly to u in X . Then $\{\|u_n\|_X\}$ is bounded. Because of the compactness of the embedding from $W^{2,2}((0, l))$ into $C^1([0, l])$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging to u in $C^1([0, l])$,

and thus $\{u'_{n_k}\}$ converges to u' in $C([0, l])$. Moreover $\{u'_{n_k}\}$ is bounded in $C([0, l])$. Since $P \in L^1((0, l))$, we have

$$\begin{aligned} |g(u_{n_k}) - g(u)| &= \frac{1}{2} \left| \int_0^l P(|u'_{n_k}|^2 - |u'|^2) dx \right| \leq \frac{1}{2} \int_0^l |P| (|u'_{n_k}| + |u'|) (|u'_{n_k} - u'|) dx \\ &\leq C \int_0^l |P| \sup_{x \in [0, l]} |(u'_{n_k} - u')(x)| dx \leq C \|P\|_{L^1((0, l))} \|u'_{n_k} - u'\|_{C([0, l])}. \end{aligned}$$

Therefore for any sequence $\{u_n\}$ weakly converging to u we can find a subsequence $\{u_{n_k}\}$ such that $\{g(u_{n_k})\}$ converges to $g(u)$. This implies that $\{g(u_n)\}$ converges to $g(u)$ and g is weakly continuous.

For any u, h in X and any t in \mathbb{R} we have

$$g(u + th) - g(u) = t \int_0^l P u' h' dx + \frac{1}{2} t^2 \int_0^l P |h'|^2 dx.$$

Thus

$$\lim_{t \rightarrow 0} \frac{g(u + th) - g(u)}{t} = \int_0^l P u' h' dx.$$

□

Theorem 4.2. *Under the above conditions there exists a solution (u, λ) of (Q).*

Proof. Since g is weakly continuous, $g^{-1}(\{r\})$ is a weakly closed subset of X . From Theorem III.7 in [4], since C is closed and convex, we have that C is weakly closed. Consequently, $M = C \cap g^{-1}(\{r\})$ is weakly closed. Therefore, by Lemma 4.1 (i) and Theorem 1.2 in [13] there exists a function $u \in M$ such that

$$f(u) = \min_{v \in M} f(v).$$

Moreover, since $g(u) = r \neq 0$, we have $Dg(u)(u) = 2g(u) \neq 0$. By Lemma 4.1 and Theorem 2.5 we obtain the variational inequality (Q) with $\lambda = \frac{Df(u)(u)}{Dg(u)(u)}$, which is not equal to 0. □

Remark 4.3. If $P > 0$, the real number λ is also positive. If $P \equiv 1$ the problem (Q) is solved in Section 64.6 in [14].

5. A QUASILINEAR ELLIPTIC EIGENVALUE PROBLEM

Denote by E and G the spaces $W_0^{1,2}(\Omega, \mathbb{R}^N)$ and $C_0^\infty(\Omega, \mathbb{R}^N)$ respectively, where Ω is a bounded domain in \mathbb{R}^m . Let g be a real functional on E , and let $(a^{\alpha\beta})_{1 \leq \alpha, \beta \leq m}$ be a symmetric, and bounded matrix function from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}^{m \times m}$. We assume that

(A1) $a^{\alpha\beta}$ is measurable in $x \in \Omega$, differentiable in $u \in \mathbb{R}^N$, and for a.e x , $\partial_u a^{\alpha\beta}(x, \cdot)$ is uniformly continuous in u , uniformly in x . Moreover, there exist a function ρ in $L^2_{loc}(\Omega)$ and a constant c such that $|f(x, u)| \leq c$ and $|\partial_u f(x, u)| \leq \rho(x)$ for any (x, u) in $\Omega \times \mathbb{R}^N$.

(A2) There is a positive real number ν such that $a^{\alpha\beta}(x, u) \xi_\alpha^i \xi_\beta^i \geq \nu |\xi|^2 \forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nm}$.

(A3) $s^2 a^{\alpha\beta}(x, su) \xi_\alpha^i \xi_\beta^i \leq a^{\alpha\beta}(x, u) \xi_\alpha^i \xi_\beta^i \forall (x, u, \xi, s) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nm} \times [0, 1]$.

(G1) g is continuous with respect to weak convergence topology in E , $g(0) = 0$ and $M = \{x \in E : g(x) \geq 1\}$ is not empty.

(G2) g is G -differentiable on E , and $\nabla g(u) \neq 0$ if $g(u) \neq 0$.

$$\text{Put } f(u) = \frac{1}{2} \int_{\Omega} a^{\alpha\beta}(x, u(x)) \partial_{\alpha} u^i(x) \partial_{\beta} u^i(x) dx \quad \forall u \in E.$$

We have the following result.

Theorem 5.1. *Under the conditions (A1), (A2), (A3), (G1) and (G2) there exist a nonnegative real number λ and u in E such that $g(u) = 1$ and $\nabla f(u) = \lambda \nabla g(u)$.*

Proof. Arguing as in the proof of Lemma 3.1 in [12] we see that f is strongly G -differentiable on E . Let $\{u_n\}$ be a sequence in M such that $\lim_{n \rightarrow \infty} f(x_n) = \inf f(M)$. Using (A2) and the Sobolev embedding theorem we can suppose that $\{u_n\}$ converges weakly to v_0 in E and $\{u_n\}$ converges to v_0 a.e on Ω . By (A1), Fatou's lemma and (G1) we see that v_0 belongs to M and $f(v_0) = \min f(M)$. By (G1) and (A3) there is s in $[0, 1]$ such that $g(sv_0) = 1$. Put $u_0 = sv_0$. We see that $u_0 \in M$ and $f(u_0) = \min f(M)$. Using (G2) we can apply Theorem 2.8 to get the theorem. \square

Remark 5.2. In [12] the following condition was considered:

(A4) There exists a positive real number $\nu^* \leq \nu$ such that

$$-u \cdot \partial_u a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i \leq 2\nu^* |\xi|^2 \quad \forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nm}.$$

Now assume that $(a^{\alpha\beta})_{1 \leq \alpha, \beta \leq m}$ satisfies (A4). We prove that it satisfies (A3). Indeed, letting x be in $\Omega \times \mathbb{R}^N \times \mathbb{R}^{Nm} \times [0, 1]$, we have

$$\begin{aligned} s^2 a^{\alpha\beta}(x, su) \xi_{\alpha}^i \xi_{\beta}^i &= s^2 a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i \\ &+ s^2 \int_0^1 \partial_u a^{\alpha\beta}(x, u + t(s-1)u) (s-1) u \xi_{\alpha}^i \xi_{\beta}^i dt \\ &= a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i + (s^2 - 1) a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i \\ &+ s^2 \int_0^1 \frac{-(1-s)}{1+t(s-1)} \partial_u a^{\alpha\beta}(x, [1+t(s-1)]u) [1+t(s-1)] u \xi_{\alpha}^i \xi_{\beta}^i dt \\ &\leq a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i + (s^2 - 1) \nu |\xi|^2 + 2s^2 \frac{1-s}{s} \nu^* |\xi|^2 \\ &\leq a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i - (s-1)^2 \nu |\xi|^2 \leq a^{\alpha\beta}(x, u) \xi_{\alpha}^i \xi_{\beta}^i. \end{aligned}$$

Therefore we can get Theorem 3 in [12] with relaxation of compactness and some other conditions.

Our results may be applied to p -Laplace equations (see [5]).

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