

NEW PROPERTIES OF MROWKA'S SPACE $\nu\mu_0$

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(Communicated by Alan Dow)

ABSTRACT. We extend the technique of Mrowka to show that his space $\nu\mu_0$ has the property that $\dim \nu\mu_0^n = n$ while $\text{ind } \nu\mu_0^n = 0$, assuming his extra set-theoretic hypothesis. We also show that $\nu\mu_0$ is N -compact, so assuming the extra axiom, there is an N -compact metric space with no N -compact completion.

Recently, Mrowka has written two very interesting papers ([M2], [M3]) concerning an example of his, which he calls $\nu\mu_0$. This space is a zero-dimensional metric space, and he has shown, assuming a special set-theoretic axiom $S(\mathfrak{c})$, that every completion of $\nu\mu_0$ contains an interval and that every completion of its square contains a square. Since $\dim \nu\mu_0^2 \leq 2$, this implies that, at least consistently, there is a metric space for which $\text{ind} = 0$ while $\dim = 2$, and also for which the ind of any completion is greater than 1.

In this note we show two further properties of $\nu\mu_0$. In particular we show, also assuming $S(\mathfrak{c})$, that every completion of $\nu\mu_0^n$ contains an n -cube. It follows that $\dim \nu\mu_0^n = n$ while $\text{ind } \nu\mu_0^n = 0$. This, at least consistently, answers the old question of what the possible values for $\dim - \text{ind}$ are. This result should be viewed as merely the generalization and organization of ideas found in Mrowka's work on the two-dimensional case. We also show that $\nu\mu_0$ is N -compact; hence, there is an N -compact space for which (consistently) there is no N -compact completion. Previously, there were three known examples of complete, not strongly zero-dimensional N -compact metric spaces ([K1], [K3]), but no examples for which completion preserving N -compactness was known to be impossible.

The author would like to gratefully thank the referee, who offered many improvements to the presentation of these results.

1. BASICS AND BACKGROUND

1.1. Notation. All spaces are assumed metrizable unless otherwise noted. We use \mathfrak{c} to denote the cardinality of the continuum, and N to denote the positive integers. Let $\{r_i : i \in N\}$ enumerate \mathcal{Q} , the rationals in $[0, 1]$, let \mathcal{P} denote the irrationals in $[0, 1]$ and let C denote the Cantor set (2^N with the product topology).

1.2. The statement $S(\mathfrak{c})$. Let A be a set. $W \subseteq A^N$ is *countable on all lines parallel to the j th axis*, where $j \in N$, if whenever $\sigma \in W$, $\{\tau(j) : \tau \in W, \sigma(i) = \tau(i)$

Received by the editors May 2, 1998 and, in revised form, September 9, 2000.

2000 *Mathematics Subject Classification.* Primary 54F45.

Key words and phrases. Dimension, metric space, completion, N -compact, product space.

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if $i \neq j$ is a countable set. The following axiom was formulated by Mrowka [M1], and was recently proved consistent by Dougherty ([D]).

$S(\mathbf{c})$: The space A^N , where A is a discrete space of cardinality \mathbf{c} , cannot be written as a countable union of closed sets each of which is countable on all lines parallel to some axis.

The statement $S(\mathbf{c})$ is a large cardinal assumption; in particular, it implies that in L there are weakly compact cardinals below \mathbf{c}^V .

1.3. n -Bernstein sets. A subset B of a complete metric space M is *Bernstein* provided that $B \cap K \neq \emptyset$ and $(M \setminus B) \cap K \neq \emptyset$ whenever K is a perfect subset of M . A subset S of $\prod_{i \in I} X_i$ is *oblique* if whenever σ and τ are in S , $\sigma(i) \neq \tau(i)$ for all $i \in I$. The Bernstein set $B \subset C$ is *n -Bernstein* provided B^n intersects each oblique perfect subset of C^n . It is not difficult to construct, by transfinite induction, a subset of C that is n -Bernstein for all n , but not all Bernstein sets are n -Bernstein.

1.4. Tree products. If T and U are trees, then the *tree product* of T and U , denoted $T \otimes U$, is the collection $\{(t, u) : t \in T, u \in U, \text{ht}_T(t) = \text{ht}_U(u)\}$. Using the partial order $(t, u) \leq (t^*, u^*)$ iff $t \leq_T t^*$ and $u \leq_U u^*$, $T \otimes U$ is itself a tree. Let $T^{(n)} = T \otimes T \otimes \cdots \otimes T$ (n times). Tree products will be used to define a special base in section 3.

1.5. The spaces $\nu\mu$ and $\nu\mu_0$. Fix a subset B of C that is n -Bernstein for all $n \in N$. Let $A = C \setminus B$; then A and B both have cardinality \mathbf{c} . The space $\nu\mu$ is a subset of $C^N \times [0, 1]$ (with a finer topology), and $\nu\mu_0$ is a subspace of $\nu\mu$. An element x of $\nu\mu$ will be denoted by (\mathbf{x}, t) where $\mathbf{x} \in C^N$ and $t \in [0, 1]$.

Our description of Mrowka's spaces differs slightly from [M2] and [M3], but it might help to relate them to the spaces in [R], [K2], and [K3].

The space $\nu\mu = P_1 \cup P_2$ where:

(i) $P_1 = \{(\mathbf{x}, t) : t \in \mathcal{P}, \text{ and } \mathbf{x} \in A^N\}$ and

(ii) $P_2 = \bigcup_{i \in N} P_{2,i}$ where

$P_{2,i} = \{(\mathbf{x}, r_i) : \mathbf{x}(j) \in A \text{ if } j \neq i \text{ and } \mathbf{x}(i) \in C\}$.

Also, $\nu\mu_0$ is the subset $P_1 \cup P_2^0$ of $\nu\mu$ with $P_2^0 = \bigcup_{i \in N} P_{2,i}^0$ where

$P_{2,i}^0 = \{(\mathbf{x}, r_i) : \mathbf{x}(j) \in A \text{ if } j \neq i \text{ and } \mathbf{x}(i) \in B\}$.

For $t \in [0, 1]$ and $n \in N$, let $d_n(t) = \min(\{1/n\} \cup \{|t - r_i| : i \leq n, t \neq r_i\})$.

For $x = (\mathbf{x}, t) \in P_1$ and $n \in N$, let $U_n(x) = \{y = (\mathbf{y}, s) : |t - s| < d_n(t) \text{ and } \mathbf{y}(i) = \mathbf{x}(i), \text{ for } 1 \leq i \leq n\}$.

For $x = (\mathbf{x}, r_i)$ in $P_{2,i}$, and $n \in N$, let $V_n(x) = \{y = (\mathbf{y}, s) : |r_i - s| < d_n(r_i), \mathbf{y}(j) = \mathbf{x}(j) \text{ if } j \neq i \text{ and } j \leq n, \text{ and if } i \leq n, \text{ then } \mathbf{y}(i)(k) = \mathbf{x}(i)(k) \text{ for } k \leq n\}$.

The $\{U_n(x) : n \in N, x \in P_1\} \cup \{V_n(x) : n \in N, x \in P_2\}$ form a base for a metrizable topology on $\nu\mu$ (for example, the metrization theorem of Frink (see [K2]) can be used to show this); the restriction of this topology to $\nu\mu_0$ is zero-dimensional.

Note that $\nu\mu$ can be viewed as a subset of $C^N \times [0, 1]$, and P_1 can be viewed as $A^N \times \mathcal{P}$ where each factor of A^N is discrete. By rearranging coordinates, $P_{2,i}$ can be viewed as $A^N \times C \times \{r_i\}$, and $P_{2,i}^0$ can be viewed as $A^N \times B \times \{r_i\}$ (B has the subspace topology inherited from C). These facts will be used in section 3.

1.6. Properties of $\nu\mu_0$. Mrowka discovered $\nu\mu$ and $\nu\mu_0$ and proved the following two theorems, the first in [M2] and the second in [M3].

Theorem 1.1. *$S(c)$ implies that every completion of $\nu\mu_0$ contains an interval.*

Theorem 1.2. *$S(c)$ implies that every completion of $\nu\mu_0^2$ contains a square.*

In section 3 we prove the general analogue of these:

Theorem 1.3. *$S(c)$ implies that, for all $n \geq 1$, every completion of $\nu\mu_0^n$ contains an n -cube.*

In section 2 we prove:

Theorem 1.4. *The space $\nu\mu_0$ is N -compact.*

This theorem assumes no extra set-theoretic conditions, but the conclusion that $\nu\mu_0$ has no N -compact completion requires Theorem 1.1.

2. PROOF OF THEOREM 1.4

We make use of the following well-known characterization of N -compactness.

Theorem 2.1. *A zero-dimensional space X is N -compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed.*

Suppose that \mathcal{U} is a clopen ultrafilter on $\nu\mu_0$ with the countable intersection property. For each $k, n \in N$ and $b \in 2^n$, let $H(k, b)$ denote $\{x \in \nu\mu_0 : b \subset \mathbf{x}(k)\}$. Then $H(k, b)$ is a clopen set in $\nu\mu_0$. Fixing k and n , $\{H(k, b) : b \in 2^n\}$ is a finite clopen partition of $\nu\mu_0$, and thus there is $b_n \in 2^n$ with $H(k, b_n) \in \mathcal{U}$. Clearly $b_m \subset b_n$ when $m > n$; let $s_k = \bigcup_n b_n$.

Letting $W = \bigcap_{k \in N} \bigcap_{n \in N} H(k, b_n)$, W is nonempty by the countable intersection property of \mathcal{U} , and if $(\mathbf{x}, t) \in W$, then $\mathbf{x}(k) = b_k$ for all $k \in N$. Now either $\mathbf{x}(i) \in A$ for all i or, for exactly one i , $\mathbf{x}(i) \in B$. In the latter case, $t = r_i$ and W is a singleton; hence \mathcal{U} is fixed. In the other case, $W \subseteq \{\mathbf{x}\} \times \mathcal{P}$, so W is a zero-dimensional separable and metrizable space and is therefore N -compact. It follows that the trace of \mathcal{U} on W has a nonempty intersection, and \mathcal{U} is fixed.

3. PROOF OF THEOREM 1.3

From the description of $\nu\mu$ in section 1, by a rearrangement of coordinates, $\nu\mu^n$ can be viewed as a subset of $(C^N)^n \times [0, 1]^n$, with $(A^N)^n \times [0, 1]^n \subset \nu\mu^n$.

Suppose $F = \bigcup_{i \in N} F_i$ is an F_σ set in $\nu\mu^n \setminus \nu\mu_0^n$. The goal is to find $x \in (A^N)^n$ such that the set $\{x\} \times [0, 1]^n$ is disjoint from F . From this it follows that the G_δ set $\nu\mu^n \setminus F$ of $\nu\mu^n$ contains an n -cube. An application of Lavrentieff's Theorem (see [E]) then yields Theorem 1.3.

Partition $[0, 1]^n$ and along with it $\nu\mu^n$ into countably many sets as follows. For each $u \subseteq \{1, 2, \dots, n\}$ and $f : u \rightarrow \mathcal{Q}$, let $T(u, f) = \{y \in [0, 1]^n : \text{if } i \in u, \text{ then } y(i) = f(i), \text{ and } y(i) \in \mathcal{P} \text{ otherwise}\}$. Correspondingly, let $S(u, f) = \{(x_1, x_2, \dots, x_n) \in \nu\mu^n : (t_1, t_2, \dots, t_n) \in T(u, f)\}$. The $x \in (A^N)^n$ that we want must satisfy, for all u and f , $\{x\} \times T(u, f) \cap F = \emptyset$.

Notice that $S(u, f) = R_1 \times R_2 \times \dots \times R_n$ where, if $i \in u$, and $f(i) = r_{j(i)}$, then R_i is $P_{2, j(i)}$, and if i is not in u , then $R_i = P_1$. It follows that $S(\emptyset, \emptyset) = (A^N)^n \times \mathcal{P}^n$, which is a subset of $\nu\mu_0^n$ and so does not intersect F ; hence we focus on nonempty u .

Fix a nonempty u and an f ; if $x \in (A^N)^n$ is such that $\{x\} \times T(u, f)$ intersects F , then it intersects some F_i . This means that x is in the projection along $[0, 1]^n$

of $F_i \cap S(u, f)$. By rearranging coordinates and letting $k = |u|$ we can view $S(u, f)$ as $A^N \times C^k \times T(u, f)$. In this view, $F_i \cap S(u, f)$ is disjoint from $A^N \times B^k \times T(u, f)$, which is a subset of $\nu\mu_0$.

We now need:

Theorem 3.1. *Suppose $0 \leq k \leq n$, $u \subseteq \{1, 2, \dots, n\}$ with $|u| = k$, $f : u \rightarrow \mathcal{Q}$ and F is a closed subset of $A^N \times C^k \times T(u, f)$ that does not intersect $A^N \times B^k \times T(u, f)$. Then there are closed sets $\{K_i : i \in N\}$ of $A^N \times C^k$ each of which is countable on all lines parallel to some axis, and such that the projection of F to $A^N \times C^k$ is contained in $\bigcup\{K_i\}$.*

Observe that the intersection of a K_i from Theorem 3.1 with $A^N \times A^k$ is closed in $A^N \times A^k$ if each factor is assumed discrete.

To finish the argument, for each u, f and i , let $\mathcal{K}(u, f, i)$ denote the countable collection guaranteed by Theorem 3.1 when F_i is substituted for F in the statement of the theorem, and let $\mathcal{K}^*(u, f, i)$ denote the collection of intersections of elements of $\mathcal{K}(u, f, i)$ with $A^N \times A^k$. By appropriately rearranging coordinates, this means that if $x \in (A^N)^n$ is not in $\mathcal{K}^* = \bigcup_{u, f, i} \mathcal{K}^*(u, f, i)$, then $\{x\} \times T(u, f) \cap F_i = \emptyset$. Hence \mathcal{K}^* is a countable collection of closed sets in $(A^N)^n$, each of which is countable on all lines parallel to some axis, and if $x \in (A^N)^n$, $x \notin \bigcup \mathcal{K}^*$, then $\{x\} \times [0, 1]^n \cap F = \emptyset$. But $S(\mathbf{c})$ can be applied, giving that such an x exists.

To prove Theorem 3.1, we need some lemmas.

Lemma 3.2. *Let X be a space, and suppose $M \subseteq X^n$, $M \neq \emptyset$. Then one of the following holds:*

- (1) *There is an open $U \subseteq X^n$ with $U \cap M \neq \emptyset$ and k with $1 \leq k \leq n$ such that if $\sigma, \tau \in U \cap M$, then $\sigma(k) = \tau(k)$.*
- (2) *There are $\sigma, \tau \in M$ with $\sigma(i) \neq \tau(i)$ for $1 \leq i \leq n$.*

Proof. Suppose (2) does not hold and fix $\sigma \in M$. For $\delta \in M$ let $s(\delta) = \{i : \delta(i) = \sigma(i)\}$; choose τ so that $|s(\tau)|$ is minimal; $|s(\tau)| \geq 1$. Then for each $j \in \{1, 2, \dots, n\} \setminus s(\tau)$, $F_j = \{\delta \in X^n : \delta(j) = \sigma(j)\}$ is a closed set not containing τ , so there is a neighborhood U of τ missing $\bigcup_{j \notin s(\tau)} F_j$. It follows that if $\delta \in U \cap M$, then $\delta(i) = \tau(i) = \sigma(i)$ if $i \in s(\tau)$. So (1) holds.

For $\sigma \in X \times Y^n$, write $\sigma = (\sigma_x, \sigma(1), \sigma(2), \dots, \sigma(n))$ where $\sigma_x \in X$ and $(\sigma(1), \sigma(2), \dots, \sigma(n)) \in Y^n$. \square

The next lemma follows easily from Lemma 3.2.

Lemma 3.3. *Let X, Y be spaces, and suppose $M \subseteq X \times Y^n$, $M \neq \emptyset$. Then one of the following holds:*

- (1) *There is an open $U \subseteq X \times Y^n$ with $U \cap M \neq \emptyset$ and i with $1 \leq i \leq n$ such that if $\sigma, \tau \in U \cap M$, then $\sigma(i) = \tau(i)$.*
- (2) *There are $\sigma, \tau \in M$ with $\sigma(i) \neq \tau(i)$ for $1 \leq i \leq n$.*

Lemma 3.4. *If each of X and Y is a complete metric space, $n \in \mathbb{N}$, and F is a closed subset of $X \times Y^n$, then one of the following holds:*

- (1) *The projection of F to Y^n contains an oblique Cantor set, or*
- (2) *There is an open set U with $U \cap F \neq \emptyset$ and i , $1 \leq i \leq n$, such that if $\sigma, \tau \in U \cap F$, then $\sigma(i) = \tau(i)$.*

Proof. We suppose (2) does not hold and show (1) does. For $\sigma \in \{0, 1\}^k$, $k \geq 1$, let $\sigma^- = (\sigma(1), \sigma(2), \dots, \sigma(k-1)) \in \{0, 1\}^{k-1}$. Let $O_\emptyset = X \times Y^n$. By induction on k , for each $\sigma \in \{0, 1\}^k$ we find a point $x_\sigma \in F$ and an open set O_σ such that:

- (i) $x_\sigma \in O_\sigma \subseteq cl(O_\sigma) \subseteq O_{\sigma^-}$;
- (ii) the diameter of O_σ is less than $1/k$;
- (iii) if $\sigma, \tau \in \{0, 1\}^k$, $\sigma \neq \tau$ and $x \in O_\sigma, y \in O_\tau$, then for the coordinates $i \in \{1, 2, \dots, n\}$ of the projection to Y^n , $x(i) \neq y(i)$.

Having achieved this, letting $\mathcal{O}_k = \bigcup_{\sigma \in \{0,1\}^k} O_\sigma$, we have $\bigcap_{k \in \mathbb{N}} \mathcal{O}_k$ is a Cantor set in $X \times Y^n$ whose projection to Y^n is an oblique Cantor set, and (1) holds.

Suppose the induction is complete through stage $k-1$, and $\sigma \in \{0, 1\}^k$. We simultaneously get $x_\sigma, x_\tau, O_\sigma, O_\tau$ where τ is the sequence differing from σ in only the last position. Because (2) is not satisfied, letting $M = O_{\sigma^-} \cap F$, there is no open U with $U \cap M$ satisfying (1) of Lemma 3.3, so (2) of 3.3 holds, and there are $x_\sigma, x_\tau \in O_{\sigma^-} \cap F$ with all different Y coordinates. Then O_σ, O_τ can be chosen to be small enough neighborhoods of σ and τ to satisfy (i), (ii), and (iii). \square

Proof of Theorem 3.1. Because the projection of F into C^k is disjoint from B^k , it does not contain any oblique Cantor sets, nor does any closed subset of F . It follows from Lemma 3.4 that if G is a closed subset of F , there is a nonempty open set O with $O \cap G \neq \emptyset$ and $O \cap G$ has constant projection to one of the factors of C^k .

Since $C = 2^N, \mathcal{P} = \omega^N$ and $T(u, f)$ is identifiable with \mathcal{P}^{n-k} , a clopen base for the space $A^N \times C^k \times T(u, f)$ is naturally identified with the tree product $\mathbf{T} = A^{<\omega} \otimes (2^{<\omega})^{(k)} \otimes (\omega^{<\omega})^{(n-k)}$. Let $[t]$ denote the open set associated with $t \in \mathbf{T}$, and identify the separate groups of coordinates of t by writing $t = (t_A, t_{C^k}, t_{\mathcal{P}^{(n-k)}})$. If $t, v \in \mathbf{T}$, $t \neq v$, and $ht(t) = ht(v)$, then $[t] \cap [v] = \emptyset$.

By transfinite induction define subsets U_α of \mathbf{T} and F_α of F as follows. Let $F_0 = F$ and given F_α , let U_α be the set of $t \in \mathbf{T}$ such that $[t] \cap F_\alpha$ is nonempty, has a constant projection into one of the factors of C^k and t is minimal with respect to this property. Let $F_{\alpha+1} = F_\alpha \setminus \bigcup_{t \in U_\alpha} [t]$, and $F_\alpha = \bigcap_{\beta < \alpha} F_\beta$ if α is a limit. By the first paragraph of this proof, if $\beta < \alpha$ and $F_\beta \neq \emptyset$, then F_α is a proper subset of F_β , so the induction stops when $F_\alpha = \emptyset$.

Observe that each U_α is an antichain and that $r \neq t$ if $r \in U_\alpha, t \in U_\beta$ and $\beta < \alpha$, because $[r]$ meets F_α and $[t]$ does not. Now split $U = \bigcup_\alpha U_\alpha$ into countably many sets $\{W_{m,i,s} : m \in N, i \leq k, s \in (\omega^{<\omega})^{(n-k)}\}$ as follows: $t \in W_{m,i,s}$ iff $ht(t) = m$, if $t \in U_\alpha$, then $F_t = [t] \cap F_\alpha$ has constant projection to the i th factor of C^k and $s = t_{\mathcal{P}^{(n-k)}}$.

Finally let $K_{m,i,s}$ be the closure of the projection of $\bigcup\{F_t : t \in W_{m,i,s}\}$ into $A^N \times C^k$. Then $K_{m,i,s}$ is the union of the individual closures of the projections of the F_t , because $\{(t_A, t_{C^k}) : t \in W_{m,i,s}\}$ represents a discrete family of clopen sets in $A^N \times C^k$.

It follows that $K_{m,i,s}$ meets every line parallel to the i th coordinate of C^k in exactly one point. The family $\{K_{m,i,s} : m \in N, i \leq k, s \in (\omega^{<\omega})^{(n-k)}\}$ is as desired. \square

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