

EXAMPLES CONCERNING HEREDITY PROBLEMS OF WCG BANACH SPACES

SPIROS A. ARGYROS AND SOPHOCLES MERCOURAKIS

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. We present two examples of WCG spaces that are not hereditarily WCG. The first is a space with an unconditional basis, and the second is a space X such that X^{**} is WCG and X^{**} does not contain ℓ^1 . The non-WCG subspace Y of X has the additional property that Y^{**} is not WCG and X/Y is reflexive.

1. INTRODUCTION

Thirty years ago H. P. Rosenthal ([R]) showed that there exists a non-WCG subspace of $L^1(\mu)$, with μ an appropriate probability measure, settling in the negative the heredity problem for WCG spaces. The aim of the present paper is to provide some more examples related to some heredity questions stated in the literature. More precisely our examples concern the following.

Q₁ : Is every WCG space with an unconditional basis hereditary WCG?

This is a well-known problem recently stated in [Z].

Q₂ : If X^{**} is WCG, does X share the same property?

This problem was posed by W. Johnson and J. Lindenstrauss [JL].

Q₃ : Is every WCG space not containing ℓ^1 hereditarily WCG?

K. John and V. Zizler have shown that WCG with a Fréchet smooth norm are hereditary WCG [JZ]. M. Fabián has extended this result [F1]. Thus **Q₃** concerns the weaker assumption that ℓ^1 does not embed into X . Let us mention that in [MS] it has been shown that under Martin's axiom every WCG space with $\dim X < 2^\omega$ is also hereditarily WCG.

In the first section, answering **Q₁**, we present an example of a WCG space X with an unconditional basis that has a non-WCG subspace Y . Additionally Y is generated by a block basis of X and thus it also has an unconditional basis. This example is unexpectedly easy and is built on the well-known Talagrand's compactum. However it shares some remarkable properties. Thus the unconditional basis of X is decomposed into countably many pieces, each one generating a Hilbert space. On the other hand, the unconditional basis of the subspace Y is extended to an M -basis of X failing Johnson's property (i.e. there is no countable partition of it with each piece being a relatively weakly compact set) and which also is decomposed into countably many pieces, each one generating a non-WCG subspace of X .

Received by the editors July 16, 2003 and, in revised form, October 23, 2003.

2000 *Mathematics Subject Classification*. Primary 46B20, 46B26, 03E05.

Key words and phrases. WCG Banach space, unconditional basis, tree.

The second section is combinatorial in nature. It contains one of the basic ingredients of the second example. In particular we show the following.

Theorem. *There exists a sequence $(T_n)_n$ of trees satisfying the following:*

- (i) *Each T_n is of height ω and for $n_1 \neq n_2$, if S_i is a segment of T_{n_i} , $i = 1, 2$, then $|S_1 \cap S_2| \leq 1$.*
- (ii) *For every partition $(D_n)_n$ of $T = \bigcup_n T_n$ there exists $n_0 \in \mathbb{N}$ such that $D_{n_0} \cap T_{n_0}$ contains an infinite segment of T_{n_0} .*

We actually show that the remarkable Reznichenko sequence of trees satisfies the above-stated properties.

In the third section we present a Banach space X with the following properties.

- (i) Both X and X^{**} are WCG and do not contain ℓ^1 .
- (ii) There exists a subspace Y of X such that both Y and Y^{**} are not WCG.
- (iii) The quotient space X/Y is reflexive.

This example answers in the negative **Q₃** and is also related to **Q₂**, which still remains open. The space X is of the form $(\sum \oplus X_n)_2$ where each X_n is James's space $J(T_n)$ and $(T_n)_n$ is a Reznichenko sequence. The unconditional counterpart of this example provides another example of a space with an unconditional basis containing a non-WCG subspace Y with the additional property that X/Y is reflexive. We finally mention that in the first section we provide a necessary property, weaker than Johnson's property, satisfied by any M -basis of a WCG space.

2. THE HEREDITY PROBLEM FOR SPACES WITH AN UNCONDITIONAL BASIS

In the first part of this section we prove the aforementioned property of M -bases. This will be used for checking the non-WCG property in the case of spaces without unconditional basis. In the second part we present the first of our examples.

Proposition 2.1. *Let X be a WCG Banach space, and let $\{(x_a, f_a) : a \in T\}$ be any M -basis of X . Then there exist a sequence T_m , $m \geq 1$, of subsets of T with $T = \bigcup_{m=1}^{\infty} T_m$ and a partition $\{T_\delta : \delta \in \Delta\}$ of T into countable sets such that, if $(a_n) \subseteq T$ and $(\lambda_n) \subseteq \mathbb{R}$ with the properties:*

- (a) $a_n \in T_{\delta_n}$, $n \geq 1$, for some sequence (δ_n) of distinct points of Δ and
- (b) $\liminf_{n \rightarrow \infty} \left\| \sum_{k=1}^n \lambda_k f_{a_k} \right\| < +\infty$,

then we have, for every $\varepsilon > 0$ and $m \in \mathbb{N}$ that the set

$$\{n \in \mathbb{N} : a_n \in T_m \text{ and } |\lambda_n| \geq \varepsilon\} \text{ is finite.}$$

Proof. It is clear that the set $K = \left\{ \frac{f_a}{\|f_a\|} : a \in T \right\}$ is w^* discrete with $K \cup \{0\}$ w^* compact. Since X is WCG there exists a weakly discrete total subset $\{y_\gamma : \gamma \in \Gamma\}$ of X so that the set $\{y_\gamma : \gamma \in \Gamma\} \cup \{0\}$ is weakly compact (see [AL]). We define a function $F : T \times \Gamma \rightarrow \mathbb{R}$, by the rule $F(\alpha, \gamma) = f_\alpha(y_\gamma)$, for $\alpha \in T$, $\gamma \in \Gamma$ and easily conclude that for any pair $(\alpha, \gamma) \in T \times \Gamma$ both of the sets $\{\alpha' \in T : F(\alpha', \gamma) \neq 0\}$ and $\{\gamma' \in \Gamma : F(\alpha, \gamma') \neq 0\}$ are at most countable. By applying a result of the first-named author (see [A] or [ACGJM] Lemma 2.4 or Lemma 1.6.2 in [F]), we can find partitions $\{\Gamma_\delta : \delta \in \Delta\}$ and $\{T_\delta : \delta \in \Delta\}$ of Γ, T into countable sets with the following property: For every $\delta_1, \delta_2 \in \Delta$, $\delta_1 \neq \delta_2$, and for every $\alpha \in T_{\delta_1}$ and $\gamma \in \Gamma_{\delta_2}$, we have $f_\alpha(y_\gamma) = 0$.

Now set $T_m = \{\alpha \in T : \text{there exists } \delta \in \Delta \text{ with } \alpha \in T_\delta \text{ and } \gamma \in \Gamma_\delta : |f_\alpha(y_\gamma)| \geq \frac{1}{m}\}$, for $m \geq 1$; then clearly $T = \bigcup_{m=1}^\infty T_m$. We shall show that the families $\{T_m : m \in \mathbb{N}\}$ and $\{T_\delta : \delta \in \Delta\}$ satisfy the desired properties. So let $(\alpha_n) \subseteq T$ and $(\lambda_n) \subseteq \mathbb{R}$ having properties (a) and (b) of the theorem. Assume that there exist $\varepsilon > 0$, $m_0 \in \mathbb{N}$ and a subsequence (α_{k_n}) of (α_n) so that $\{\alpha_{k_n} : n \geq 1\} \subseteq T_{m_0}$ and $|\lambda_{k_n}| \geq \varepsilon$, for all $n \geq 1$. Let (γ_n) be a sequence of (necessarily distinct) points of Γ such that $|f_{\alpha_{k_n}}(y_{\gamma_n})| \geq \frac{1}{m_0}$ for all $n \geq 1$. Property (b) and the fact that any closed ball of X^* is an Eberlein compact set in the w^* topology yield that there exists a subsequence $(x_{\mu_n}^*)$ of the sequence $x_n^* = \sum_{k=1}^n \lambda_k f_{\alpha_k}$, $n \geq 1$, with $x_{\mu_n}^* \xrightarrow{w^*} x^* \in X^*$. Fix any $\nu \in \mathbb{N}$. Then we have (by also using property (a)) that, $|x^*(y_{\gamma_\nu})| = \lim_{n \rightarrow \infty} |(\sum_{k=1}^{\mu_n} \lambda_k f_{\alpha_k})(y_{\gamma_\nu})| = |\lambda_{k_\nu}| \cdot |f_{\alpha_{k_\nu}}(y_{\gamma_\nu})| \geq \frac{|\lambda_{k_\nu}|}{m_0}$, for large n . Therefore, $|x^*(y_{\gamma_\nu})| \geq \frac{|\lambda_{k_\nu}|}{m_0} \geq \frac{\varepsilon}{m_0}$, for all $\nu \in \mathbb{N}$, which contradicts the fact that $y_{\gamma_\nu} \xrightarrow{w} 0$. The proof of the proposition is complete. \square

Theorem 2.2. *Let X be a WCG Banach space, and let $\{(x_\alpha, f_\alpha) : \alpha \in T\}$ be any M -basis of X . Then there exists a sequence D_m , $m \geq 1$, of subsets of T with $T = \bigcup_{m=1}^\infty D_m$ such that, if $m \in \mathbb{N}$, (α_n) is any sequence of distinct points of D_m and (λ_n) is any sequence in \mathbb{R} with $\liminf_{n \rightarrow \infty} \|\sum_{k=1}^n \lambda_k f_{\alpha_k}\| < +\infty$, then $\lambda_n \rightarrow 0$.*

Proof. Let $\{T_m : m \in \mathbb{N}\}$ and $\{T_\delta : \delta \in \Delta\}$ be families as in the statement of the above theorem. Let, for $\delta \in \Delta$, $T_\delta = \{\alpha_1^\delta, \dots, \alpha_n^\delta, \dots\}$ be a one-to-one enumeration of T_δ and for $k \in \mathbb{N}$, $E_k = \{\alpha \in T : \alpha = \alpha_k^\delta, \text{ for some } \delta \in \Delta\}$. We also set $T_{n,k} = T_n \cap E_k$, for $n, k \in \mathbb{N}$, hence $T = \bigcup_{n \geq 1, k \geq 1} T_{n,k}$. It is clear that the countable family $\{T_{n,k} : n, k \in \mathbb{N}\}$, which we enumerate as $\{D_m : m \in \mathbb{N}\}$, satisfies the conclusion of the theorem. \square

An easy consequence of the above theorem is the well-known characterization of WCG Banach spaces having an unconditional basis due to Johnson (see [R] and [F]).

Theorem 2.3 (Johnson). *Let X be a WCG Banach space with an unconditional basis $\{x_\gamma : \gamma \in \Gamma\}$. Then the following are equivalent:*

- (i) X is WCG.
- (ii) *There exists a sequence Γ_n , $n \geq 1$, of subsets of Γ with $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$ so that each set $\{x_\gamma : \gamma \in \Gamma_n\} \cup \{0\}$ is weakly compact.*

Proof. It is clear that only the implication (i) \Rightarrow (ii) requires proof. For this we consider the partition Γ_n , $n \geq 1$, of the set Γ resulting from Theorem 2.2. It is easy to see that the family $\{x_\gamma : \gamma \in \Gamma_n\}$ does not contain an ℓ^1 -sequence and hence it is a weakly relatively compact set. \square

Most of the remaining part of this section is devoted to the construction of a WCG Banach space X with an unconditional basis that contains a non-WCG subspace also having an unconditional basis. The space X will be a direct sum, in the ℓ^2 -norm of a sequence X_n , $n \geq 1$, of WCG Banach spaces each with an unconditional basis. The definition of the spaces X_n , $n \geq 1$, is based on an example of a compact space given by Talagrand ([T], Th. 4.3) and on a rather standard method of construction of Banach spaces having an unconditional basis, which can be found for instance in [AM] (Def. 3.13 and 2.1 (iii)). We briefly describe the method.

We recall that a family \mathcal{A} of subsets of a nonempty set Γ is said to be adequate if it satisfies the following properties:

- (i) $\{\gamma\} \in \mathcal{A}$, for all $\gamma \in \Gamma$,
- (ii) $A \subseteq B \in \mathcal{A} \Rightarrow A \in \mathcal{A}$ and
- (iii) if $A \subseteq \Gamma$ such that every finite subset of A belongs to \mathcal{A} , then $A \in \mathcal{A}$.

Note that if \mathcal{A} is an adequate family of subsets of Γ , then the space $K \equiv K(\mathcal{A}) = \{\chi_A : A \in \mathcal{A}\}$ is a compact subspace of the compact space $\{0, 1\}^\Gamma$.

Let \mathcal{A} be an adequate family of subsets of the set Γ ; then we can define, by using \mathcal{A} , a Banach space $X \equiv X_{1,2}(\mathcal{A})$ in the following way: X will be the completion of the space $c_{00}(\Gamma)$ of finitely supported real functions on Γ , under the norm

$$\|f\| = \sup\left\{\left(\sum_{i=1}^n \left(\sum_{\gamma \in A_i} |f(\gamma)|\right)^2\right)^{1/2} : A_i \cap A_j = \emptyset, i \neq j \text{ and } A_i \in \mathcal{A}\right\}.$$

It is easy to verify that the usual basis of $c_{00}(\Gamma)$, i.e. the family $\{e_\gamma : \gamma \in \Gamma\}$ is an unconditional normalized (boundedly complete) basis for the Banach space X (see [AM], Def. 3.13).

We define now the example of the compact space of Talagrand mentioned above. In order to do this we need some terminology and notation. We denote by Σ the set $\mathbb{N}^{\mathbb{N}}$ of sequences of positive integers, endowed with the product topology (where \mathbb{N} has the discrete topology), and by S the set of finite sequences of positive integers.

If $\sigma \in \Sigma$ and $n \in \mathbb{N}$, then $\sigma|n$ is the finite sequence $(\sigma(1), \dots, \sigma(n))$; for $s \in S$, $s = (k_1, \dots, k_n)$ and $\sigma \in \Sigma$, the notation $s < \sigma$ means that $\sigma|n = s$. It is clear that the countable family $\{V_s, s \in S\}$, where $V_s = \{\sigma \in \Sigma : s < \sigma\}$ for $s \in S$, is a basis for the topology of Σ consisting of open and closed sets.

For $m \in \mathbb{N}$, let

$$\mathcal{A}_m = \{A \subseteq \Sigma : \sigma, \tau \in A, \sigma \neq \tau \text{ implies } \sigma|m = \tau|m \text{ and } \sigma(m+1) \neq \tau(m+1)\}.$$

Thus every $A \in \mathcal{A}_m$ is at most countable. It is easily verified that:

- (a) \mathcal{A}_m is adequate for all $m \in \mathbb{N}$,
- (b) $\mathcal{A}_m \cap \mathcal{A}_n = \{\{\sigma\} : \sigma \in \Sigma\}$, for $m \neq n$, and
- (c) $\mathcal{A} = \bigcup_{m=1}^{\infty} \mathcal{A}_m$ is also adequate.

Note. The compact space

$$K \equiv K(\mathcal{A}) = \{\chi_A : A \in \mathcal{A}\}$$

is the first example of a compact space so that $C(K)$ is \mathcal{K} -analytic in its weak topology (that is, K is a Talagrand compact) without being a WCG Banach space (that is, K is not an Eberlein compact). The space K was constructed by Talagrand in [T] (Th. 4.3).

The following lemma is due to Talagrand. Its proof is an easy application of Baire's category theorem (see [T] and [F], Lemma 1.6.1).

Lemma 2.4. *Let (Γ_n) , $n \geq 1$, be a sequence of subsets of Σ such that $\Sigma = \bigcup_{n=1}^{\infty} \Gamma_n$. Then there exist $n_0, m_0 \in \mathbb{N}$ and an infinite member A of \mathcal{A}_{m_0} such that $A \subseteq \Gamma_{n_0}$.*

Now we are coming to the construction of the desired WCG Banach space X . We set for every $m \in \mathbb{N}$, $X_m = X_{1,2}(\mathcal{A}_m)$.

Proposition 2.5. *For every $m \in \mathbb{N}$, X_m is a WCG Banach space with an unconditional basis.*

Proof. It is clear from the previous remarks that the family $\{e_\sigma : \sigma \in \Sigma\}$ is an unconditional basis for X_m , for every $m \in \mathbb{N}$. Fix any $m \in \mathbb{N}$. It is clear that

$$\{e_\sigma : \sigma \in \Sigma\} = \bigcup_{|s|=m+1} \{e_\sigma \in V_s\}$$

(where, for $s \in S$, $|s|$ denotes the length of s).

Let $A \in \mathcal{A}_m$ and $s \in S$ with $|s| = m+1$. Then obviously $|A \cap V_s| \leq 1$ and therefore we get, from the definition of the norm of the space X_m that the complemented subspace $Y_s = \overline{\text{span}}\{e_\sigma : \sigma \in V_s\}$ of X_m is isometric with a Hilbert space, and the family $\{e_\sigma : \sigma \in V_s\}$ is the canonical basis for Y_s . It follows in particular that each member of the countable family of sets $\{e_\sigma : \sigma \in V_s\} \cup \{0\}$, $s \in S, |s| = m+1$, is weakly compact in X_m , so we get from (1) that X is WCG.

The proof of the proposition is complete. \square

Now we set $X = (\sum_{m=1}^{\infty} \oplus X_{1,2}(\Sigma \times \{m\}, \mathcal{A}_m)_2)$, i.e. X is equal to the direct sum, in the ℓ^2 -norm, of the Banach spaces X_m , $m \geq 1$ defined above.

Theorem 2.6. *The space X is WCG with an unconditional basis containing a non-WCG closed linear subspace Y that also has an unconditional basis.*

Proof. The Banach space X is WCG as a direct sum of WCG Banach spaces in the ℓ^2 -norm. It is also clear that the family $\{e_{(\sigma,m)} : \sigma \in \Sigma, m \in \mathbb{N}\}$ is an unconditional normalized basis for X . Now we are going to define the non-WCG subspace Y of X .

So let $x_\sigma = \sum_{m=1}^{\infty} \frac{1}{2^{m/2}} \cdot e_{(\sigma,m)}$, for $\sigma \in \Sigma$; then clearly $\|x_\sigma\| = 1$ for every $\sigma \in \Sigma$. We also set $Y = \overline{\text{span}}\{x_\sigma : \sigma \in \Sigma\}$ (= the closed linear span of the vectors $x_\sigma, \sigma \in \Sigma$ in X). We shall show that Y is the desired subspace of X . The family $\{x_\sigma : \sigma \in \Sigma\}$ is an unconditional basis of Y as a block basis of the unconditional basis $\{e_{(\sigma,m)} : \sigma \in \Sigma, m \in \mathbb{N}\}$ of X . Assume for the purpose of contradiction that Y is WCG. Then Theorem 2.3 yields that there exists a sequence $\Gamma_n, n \geq 1$, of subsets of Σ with $\Sigma = \bigcup_{n=1}^{\infty} \Gamma_n$ so that each set $\{x_\sigma : \sigma \in \Gamma_n\} \cup \{0\}$, $n \in \mathbb{N}$, is weakly compact in Y . Also by Lemma 2.4 there exist $m_0, n_0 \in \mathbb{N}$ and an infinite member A of \mathcal{A}_{m_0} with $A \subseteq \Gamma_{n_0}$. Let $A = \{\sigma_k : k \in \mathbb{N}\}$ be an enumeration of A . Then clearly we have that

$$(1) \quad x_{\sigma_k} \xrightarrow{w} 0.$$

We consider the series $\sum_{k=1}^{\infty} e_{(\sigma_k, m_0)}^*$. Since the subspace $\overline{\text{span}}\{e_{(\sigma_k, m_0)} : k \in \mathbb{N}\}$ of X is isometric with the space $\ell^1(\mathbb{N})$, this series defines an element $x^* \in X^*$. Let λ be any positive integer. Then we have $x^*(x_{\sigma_\lambda}) = (\sum_{k=1}^{\infty} e_{(\sigma_k, m_0)}^*)(\sum_{m=1}^{\infty} \frac{1}{2^{m/2}} e_{(\sigma_\lambda, m)}) = e_{(\sigma_\lambda, m_0)}^*(\frac{1}{2^{m_0/2}} \cdot e_{(\sigma_\lambda, m_0)}) = \frac{1}{2^{m_0/2}}$, which contradicts (1). The proof of the theorem is complete. \square

It is well-known that any WCG Banach space X admits an M -basis $\{(x_\gamma, f_\gamma) : \gamma \in \Gamma\}$ so that the set $\{x_\gamma : \gamma \in \Gamma\} \cup \{0\}$ is weakly compact (see [HHZ]). It is natural to ask whether any M -basis of a WCG Banach space satisfies Johnson's property (i.e. there is a partition $(\Gamma_n)_n$ of Γ such that $\{x_\gamma : \gamma \in \Gamma_n\}$ is weakly relatively compact for all $n \in \mathbb{N}$). With the next result we shall provide an example of an M -basis that fails Johnson's property. Additionally this example shows that the conclusion of Theorem 2.2 is in a sense the best possible.

Proposition 2.7. *The WCG Banach space X of Theorem 2.6 admits an M -basis that cannot be written as a countable union of weakly relatively compact sets.*

Proof. For every $\sigma \in \Sigma$ we consider the subspace, $X_\sigma = \overline{\text{span}}\{e_{(\sigma,m)} : m \geq 1\}$ of X ; it is clear that each X_σ is isometric to a separable Hilbert space having the set $\{e_{(\sigma,m)} : m \in \mathbb{N}\}$ as an unconditional basis. It is also clear that $X = \sum_{\sigma \in \Sigma} \oplus X_\sigma$. By making a Gram-Schmidt orthonormalization, we can find for every $\sigma \in \Sigma$ an orthonormal basis $\{x_\sigma^n : n \geq 1\}$ of X_σ so that $x_\sigma^1 = x_\sigma (= \sum_{m=1}^{\infty} \frac{1}{2^{m/2}} e_{(\sigma,m)})$. Now we set

$$f_\sigma^n = \begin{cases} x_\sigma^n, & n \geq 2, \\ \sum_{m=1}^{\infty} \frac{1}{2^{m/2}} e_{(\sigma,m)}^*, & n = 1, \end{cases}$$

for every $\sigma \in \Sigma$ and $n \in \mathbb{N}$ and claim that the family $\{(x_\sigma^n, f_\sigma^n) : \sigma \in \Sigma, n \geq 1\}$ is an M -basis for X that satisfies the desired property. Clearly the second part of our claim requires justification. So let us assume that the set $\{x_\sigma^n : \sigma \in \Sigma, n \geq 1\}$ is equal to a countable union of weakly relatively compact sets. Then since (obviously) the unconditional basis $\{x_\sigma : \sigma \in \Sigma\}$ of the space Y is contained in the set $\{x_\sigma^n : \sigma \in \Sigma, n \geq 1\}$ we would have that the space Y is WCG, a contradiction. \square

Remark 2.8. Observe that for every $\sigma \in \Sigma$ the sequence $\{e_{(\sigma,m)} : m \in \mathbb{N}\}$ is isometric to the usual basis of $\ell^2(\mathbb{N})$. Also for a given σ the set $\{x_\sigma^n : n \geq 1\}$ is also an orthonormal basis for the space $X_\sigma = \overline{\text{span}}\{e_{(\sigma,m)} : m \in \mathbb{N}\}$. Therefore the following peculiar phenomenon occurs. In a space with an unconditional basis $(e_{(\sigma,m)})_{\sigma \in \Sigma, m \in \mathbb{N}}$ we decompose the basis into a family $\{F_\sigma = \{e_{(\sigma,m)}\}_{m \in \mathbb{N}}\}_{\sigma \in \Sigma}$ of pairwise disjoint sets, and in the space X_σ we select a new unconditional basis $(x_\sigma^n)_n$ such that the new biorthogonal system $\{x_\sigma^n\}_{\sigma \in \Sigma, n \in \mathbb{N}}$ is not an unconditional basis. However we can split this system into countably many pieces (i.e., $Q_n = \{x_\sigma^n\}_{\sigma \in \Sigma}$, each one being an unconditional basis.

Summing up all the previous results we have the following.

Theorem 2.9. *There exists a WCG space X with an unconditional basis $(e_a)_{a \in \mathcal{A}}$ such that*

- (i) *There exists a countable partition of \mathcal{A} into $(\mathcal{A}_n)_{n \in \mathbb{N}}$ such that $(e_a)_{a \in \mathcal{A}_n}$ is isometric to the usual basis of $\ell^2(\mathcal{A}_n)$.*
- (ii) *There exists an M -basis $(x_\gamma)_{\gamma \in \Gamma}$ of X not admitting a countable partition with each member being a weakly relatively compact set.*
- (iii) *For this M -basis there exists a partition $(x_\gamma)_{\gamma \in \Gamma_n}$, $n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ the space $X_n = \overline{\text{span}}\{(x_\gamma)_{\gamma \in \Gamma_n}\}$ is not WCG and $(x_\gamma)_{\gamma \in \Gamma_n}$ is a 1-unconditional basis for the space X_n .*

Proof. (i) The desired space X is the space defined above, namely, $X = (\sum_m \oplus X_m)_2$. The unconditional basis is the family $(e_{(\sigma,n)})_{\sigma \in \Sigma, n \in \mathbb{N}}$. As in the proof of Proposition 2.5 we decompose the set Σ into the families $\{V_s : |s| = n+1\}$. Each family $\{e_{(\sigma,n)} : \sigma \in V_s\}$ is isometric to the usual basis of $\ell^2(V_s)$, and the families $\{e_{(\sigma,n)} : \sigma \in V_s\}$, $|s| = n+1$, $n \in \mathbb{N}$, define a countable partition of the basis of X .

(ii) It follows from Proposition 2.7.

(iii) The proof of Proposition 2.7 yields that $x_\sigma^1 = \sum_m \frac{1}{2^{m/2}} e_{(\sigma,m)}$ and also for $n \geq 2$,

$$x_\sigma^n = \lambda_n (\mu_1 e_{(\sigma,1)} + \cdots + \mu_{n-1} e_{(\sigma,n-1)}) + \sum_{m \geq n} \frac{1}{2^{m/2}} e_{(\sigma,m)}.$$

Hence using similar arguments to the proof of the fact that $(x_\sigma \equiv x_\sigma^1)_{\sigma \in \Sigma}$ generates a non-WCG subspace of X , we obtain the same property for every family $(x_\sigma^n)_{\sigma \in \Sigma}$. The proof of the theorem is complete. \square

3. SEQUENCES OF TREES

A tree is a partially ordered set (T, \leq) such that $\{s \in T : s \leq t\}$ is a well-ordered subset of T for all $t \in T$.

Let (T, \leq) be any tree. A chain in (T, \leq) is any totally ordered (and thus well-ordered) subset of (T, \leq) . Any maximal chain is called a branch of (T, \leq) . A segment or interval I of (T, \leq) is any chain so that, $s < t < u$ and $s, u \in I$ imply that $t \in I$. An element t of T is called an immediate successor of $s \in T$ if $s < t$ and there is no $x \in T$ so that $s < x < t$.

For every $t \in T$, we set $S_t = \{u \in T : u \text{ is an immediate successor of } t\}$, $T_t = \{u \in T : t \leq u\}$ and $I_t = \{s \in T : s \leq t\}$ (I_t is called an initial segment of (T, \leq)).

We denote by $o(t)$ for $t \in T$ the order type of $I_t \setminus \{t\} = \{s \in T : s < t\}$. If α is any ordinal, then the α -th level of T , or $Lev_\alpha(T)$ is the set $\{t \in T : o(t) = \alpha\}$. The height of T , or $ht(T)$ is the least α such that $Lev_\alpha(T) = \emptyset$; clearly $Lev_0(T)$ is the set of minimal elements of T . If $Lev_0(T) = \{t\}$, then we say that t is the root of T .

Definition 3.1. A sequence T_n , $n \geq 1$, of trees will be called a Reznichenko sequence of trees if:

- (i) $|Lev_0(T_n)| = 1$ for all $n \geq 1$ and $Lev_0(T_n) \cap Lev_0(T_m) = \emptyset$, for $m \neq n$.
- (ii) $ht(T_n) = \omega$ for all $n \geq 1$ and for every $t \in T_n$ the set S_t is uncountable.
- (iii) For every I_1, I_2 segments of T_{n_1}, T_{n_2} respectively, $n_1 \neq n_2$, we have that $|I_1 \cap I_2| \leq 1$.
- (iv) For every non-empty subset M of \mathbb{N} , I_n initial segments of T_n , $n \in M$, such that $I_n \cap I_m = \emptyset$ for $m \neq n$, there exist uncountably many $t \in T = \bigcup_{n=1}^{\infty} T_n$ so that $t \in S_{\max I_n}$ for all $n \in M$.

We refer to [A] and [F] for proofs of the existence of a sequence of Reznichenko trees. But for the convenience of the reader we give below a sketch of the proof of the existence of such a sequence of trees.

Proposition 3.2. *There exists a sequence of Reznichenko trees.*

An outline of the proof. Every tree T_n , $n \in \mathbb{N}$, will be realized as $\bigcup_{\xi < \omega_1} T_n^\xi$ with T_n^ξ , $\xi < \omega_1$, $n \in \mathbb{N}$, satisfying the following inductive properties:

- (i) T_n^ξ is a tree, $ht(T_n^\xi) \leq \omega$, $T_n^\xi \subseteq [0, \xi] \times 2^\omega$ and $(0, n)$ is the root of T_n^ξ (where 2^ω is considered as an ordinal number).
- (ii) For $\zeta < \xi < \omega_1$, T_n^ζ is a subtree of T_n^ξ (especially here by the term subtree we mean that, if $t \in T_n^\zeta$, then $\{s \in T_n^\xi : s < t\} \subseteq T_n^\zeta$).
- (iii) If $\xi = \zeta + 1$, M is a non-empty subset of \mathbb{N} and, for $n \in M$, I_n is an initial segment of T_n^ζ such that $I_n \cap I_m = \emptyset$ for $m \neq n$, then the set $\{t \in 2^\omega : (\xi, t) \text{ is an immediate successor of } \max I_n \text{ for all } n \in M\}$ is uncountable.
- (iv) If I_1, I_2 are initial segments of $T_{n_1}^\xi, T_{n_2}^\xi$, $n_1 \neq n_2$, then $|I_1 \cap I_2| \leq 1$.

We proceed inductively and set $T_n^0 = \{(0, n)\}$ for $n \geq 1$. Let $\xi < \omega_1$ and assume that T_n^ζ have been constructed for all $\zeta < \xi$ and $n \in \mathbb{N}$.

If ξ is a limit ordinal, we set $T_n^\xi = \bigcup_{\zeta < \xi} T_n^\zeta$, for $n \in \mathbb{N}$. If $\xi = \zeta + 1$, we consider all sets of the form $K = \{I_n : n \in M\}$ where M is any non-empty subset of \mathbb{N} , I_n

is an initial segment of T_n^ζ for $n \in M$ and $I_n \cap I_m = \emptyset$ for $m \neq n$. The cardinality of all such K is 2^ω , and we order them as $\{K_t^\xi : t < 2^\omega\}$ so that each K appears uncountably many times in this enumeration. We define T_n^ξ , for a given $n \in \mathbb{N}$ in the following way. We set for every $(\eta, t) \in T_n^\zeta$, $(\eta, t)_\xi^+ = \{(\xi, t') : \text{there exists } I \in K_{t'}^\xi, I \subseteq T_n^\zeta \text{ and } \max I = (\eta, t)\}$. Then we set

$$T_n^\xi = T_n^\zeta \cup \{(\eta, t)_\xi^+ : (\eta, t) \in T_n^\zeta\}$$

and order T_n^ξ in the obvious way.

The inductive construction is complete. It is clear that the inductive properties (i) to (iv) stated above imply that T_n , $n \geq 1$, is a sequence of Reznichenko trees. \square

Note. It can be shown that the compact space $K = \{\chi_I : I \text{ is a segment of } T_n, \text{ for some } n \in \mathbb{N}\} \cup \{0\}$ is still another example of a Talagrand compact space that is not an Eberlein compact space (cf. [A], [F]).

In the sequel we prove a Baire category type property of sequences of Reznichenko trees.

Proposition 3.3. *Let T_n , $n \geq 1$, be a sequence of Reznichenko trees, and let $T = \bigcup_{n=1}^\infty T_n$. If $T = \bigcup_{k=1}^\infty D_k$, then there exists $k_0 \in \mathbb{N}$ such that the set $D_{k_0} \cap T_{k_0}$ contains an infinite segment of the tree T_{k_0} .*

Proof. We first make the following

Claim. There exists $k_0 \in \mathbb{N}$ and $a_0 \in T_{k_0}$ such that for every $\beta \in T_{k_0}^{a_0} = \{\beta \in T_{k_0} : a_0 \leq \beta\}$ we have $|S_\beta \cap D_{k_0}| \geq \omega_1$.

Proof of the Claim. Assume for the purpose of contradiction that the claim does not hold. Therefore for every $k \in \mathbb{N}$ and every $a \in T_k$ there exists $\beta \in T_k^a$ such that $|S_\beta \cap D_k| \leq \omega$. We observe that for each $k \in \mathbb{N}$ and every finite subset F of T_k , not containing the root of T_k , there exists $a \in T_k$ such that $T_k^a \cap F = \emptyset$ and also $\{\beta : \beta \leq a\} \cap F = \emptyset$. Therefore proceeding by induction we construct a sequence of pairwise disjoint initial segments I_{β_k} , $k \in \mathbb{N}$, respectively, such that $|S_{\beta_k} \cap D_k| \leq \omega$ for all $k \in \mathbb{N}$.

Since by property (iv) of the definition of Reznichenko trees (Definition 3.1) the set $\bigcap_{k=1}^\infty S_{\beta_k}$ is uncountable, there exists $n \in \mathbb{N}$ such that $|\bigcap_{k=1}^\infty S_{\beta_k} \cap D_n| \geq \omega_1$, a contradiction.

Now let k_0 be the natural number given by the claim. Then it is easy to construct by repeated use of the claim the desired infinite segment (beginning from a_0) in the set $D_{k_0} \cap T_{k_0}$. \square

4. THE HEREDITY PROBLEM FOR SPACES NOT CONTAINING ℓ^1

Given any tree (T, \leq) , a James space denoted by $J(T)$ is defined as follows: $J(T)$ is the completion of the linear space $c_{00}(T)$ under the norm (“James-norm”)

$$\|f\| = \sup\left\{\left(\sum_{i=1}^n \left(\sum_{t \in S_i} f(t)\right)^2\right)^{1/2} : S_1, \dots, S_n \text{ are pairwise disjoint segments of } (T, \leq)\right\}$$

(see [HO]).

We note the following easily verified facts:

- (i) If S is a nonempty subset of (T, \leq) , then S is also a tree with the induced order from (T, \leq) and the Banach space $J(S)$ is isometrically embedded in the space $J(T)$. In particular, if (S, \leq) has the property $\{s \in T : s < t\} \subseteq S$ for every $t \in S$, then the space $J(S)$ is complemented in $J(T)$.
- (ii) Given any $\xi \leq ht(T)$ let us denote, for convenience, by $T(\xi)$ the ξ -th level of (T, \leq) (i.e., the set $Lev_\xi(T)$) and by T_ξ the union $\bigcup_{\zeta < \xi} T(\zeta + 1)$. Then the natural map $P_\xi : J(T) \rightarrow J(T_\xi) : f \rightarrow f|_{T_\xi}$ is a norm one projection (see (i) above).
- (iii) If we denote by (D, \leq) the dyadic tree (i.e., that tree with a unique root, $ht(D) = \omega$ and so that each element of (D, \leq) has exactly two immediate successors), then every countable tree (S, \leq) with $ht(S) \leq \omega$ is order isomorphic to a subtree of (D, \leq) .

Note. The James tree space is the James space $J(D)$, where (D, \leq) is the dyadic tree. It is well-known that $J(D)$ is ℓ^2 -saturated, i.e. every infinite-dimensional subspace of $J(D)$ contains a further subspace isomorphic to the Hilbert space $\ell^2(\mathbb{N})$ (see [J] and [HO]). Using this fact, Hagler and Odell proved in [HO] that this is a general phenomenon for the class of James-type Banach spaces, $J(T)$, where (T, \leq) is any tree. For the sake of completeness we give below a more extensive description of the involving part of their argument.

Theorem 4.1 (Hagler - Odell). *Let (T, \leq) be a tree. Then every infinite-dimensional subspace Y of $J(T)$ contains an infinite-dimensional subspace Z isomorphic to a subspace of $J(D)$ (and hence by the above remarks $J(T)$ is ℓ^2 -saturated).*

Proof. Let ξ be the least ordinal $\zeta \leq ht(T)$ such that the operator $P_\zeta : Y \rightarrow J(T_\zeta)$ is not strictly singular (i.e., there exists an infinite-dimensional subspace Y_1 of Y so that $P_\zeta|_{Y_1}$ is an isomorphic embedding). Let Z_0 be an infinite-dimensional subspace of Y so that $P_\xi : Z_0 \rightarrow J(T_\xi)$ is an isomorphic embedding. Hence there exists $\theta > 0$ such that $\|P_\xi(z)\| \geq \theta$ for every $z \in Z_0$ with $\|z\| = 1$. We distinguish two cases:

(I) ξ is a successor ordinal, say $\xi = \zeta + 1$. Since the operator $P_\zeta : Y \rightarrow J(T_\zeta)$ is strictly singular, for every infinite-dimensional subspace Z of Y and each $\varepsilon > 0$ there exists $z_\varepsilon \in Z$, $\|z_\varepsilon\| = 1$ with $\|P_\zeta(z_\varepsilon)\| < \varepsilon$. Let $\varepsilon_1 \in (0, \theta)$, pick $z_1 \in Z_0$, $\|z_1\| = 1$ so that $\|P_\zeta(z_1)\| < \varepsilon_1$. Thus $\|P_{\zeta+1}(z_1)\| \geq \theta$ and $\|(P_{\zeta+1} - P_\zeta)(z_1)\| \geq \theta - \varepsilon_1$. We choose a finite subset F_1 of the level $T(\zeta + 1)$ of (T, \leq) and a vector $w_1 \in \langle e_a : a \in F_1 \rangle$ such that $\|P_{\zeta+1}(z_1) - w_1\| < \varepsilon_1$. Note that $\|w_1\| \geq \theta - \varepsilon_1$. We set $Z_1 = \{z \in Z_0 : e_a^*(z) = 0, \text{ for } a \in F_1\}$ and note that Z_1 is infinite-dimensional. We proceed by induction and for an appropriate decreasing sequence $(\varepsilon_n) \subseteq (0, \theta)$, choose a sequence $(z_n) \subseteq Z_0$, a sequence $(w_n) \subseteq J(T)$ supported on a sequence of finite pairwise disjoint sets (F_n) with $\bigcup_{n=1}^\infty F_n \subseteq T(\zeta + 1)$ so that $\|z_n\| = 1$, $\|P_{\zeta+1}(z_n) - w_n\| \leq \varepsilon_n$ and $\|w_n\| \geq \theta - \varepsilon_n$ for $n \geq 1$. The sequence (w_n) is clearly equivalent to the canonical basis of $\ell^2(\mathbb{N})$, and if the sum $\sum_{n=1}^\infty \varepsilon_n$ is sufficiently small, then the sequence $(P_{\zeta+1}(z_n))$ is equivalent to (w_n) . So we are done in the case when ξ is a successor ordinal.

(II). Assume that ξ is a limit ordinal. Let $z_1 \in Z_0$, $\|z_1\| = 1$, thus $\|P_\xi(z_1)\| \geq \theta$. Let $\varepsilon_1 \in (0, \theta)$, pick a finite subset $F_1 \subseteq T_\xi$ and $w_1 \in \langle e_a : a \in F_1 \rangle$ such that $\|P_\xi(z_1) - w_1\| < \varepsilon_1$, hence $\|w_1\| \geq \theta - \varepsilon_1$. Set $\zeta_1 = \max\{o(a) : a \in F_1\}$, clearly $\zeta_1 < \xi$. Let again (ε_n) be an appropriate sequence in the interval $(0, \theta)$. We

construct by induction sequences $(z_n) \subseteq Z_0, (w_n) \subseteq J(T)$ and a sequence of ordinals $\zeta_1 < \zeta_2 < \dots < \zeta_n < \dots < \xi$ such that

- (a) each w_n is supported on a finite set F_n such $F_1 \subseteq T_{\zeta_1}$ and $F_n \subseteq T_\xi \setminus T_{\zeta_{n-1}}$ for $n \geq 2$,
- (b) $\|z_n\| = 1$ and $\|P_\xi(z_n) - w_n\| < \varepsilon_n$, for $n \geq 1$.

The tree $F = \bigcup_{n=1}^\infty F_n$ (with the induced order from (T, \leq)) is clearly countable and $ht(F) = \omega$. Therefore, by facts (i) and (ii) the James space $J(F)$ is embedded into the space $J(D)$. Since the James tree space $J(D)$ is ℓ^2 -saturated its subspace X , generated by the basic sequence $(w_n)_{n \geq 2}$, contains isomorphically the space $\ell^2(\mathbb{N})$. On the other hand (b) implies that (if the sum $\sum_{n=1}^\infty \varepsilon_n$ is quite small) X is isomorphic to the space $\overline{\text{span}}\{P_\xi(z_n) : n \geq 1\}$, which is embedded into Z_0 and thus into Y . □

In the following theorem some properties of James spaces $J(T)$ on trees (T, \leq) of height ω are stated. We give an outline of the proof of this theorem.

Theorem 4.2. *Let (T, \leq) be a tree of height ω ($ht(T) = \omega$) and set $X = J(T)$. Then we have*

- (i) *the Banach space X is WCG;*
- (ii) *$X^{**} \cong X \oplus \ell^2(\mathcal{B}_T)$, where \mathcal{B}_T is the set of (infinite) branches of (T, \leq) (if (T, \leq) has no infinite branch, then X is reflexive), and hence in particular X^{**} is WCG and does not contain ℓ^1 .*

Proof. (i) It is easy to verify that if $T(n)$ is the n -th level ($n \in \mathbb{N}$) of (T, \leq) , then the set $\{e_t : t \in T(n)\}$ is isometrically equivalent to the canonical basis of the space $\ell^2(T(n))$. Thus each set $\{e_t : t \in T(n)\} \cup \{0\}$ is weakly compact in X . Hence X is WCG.

(ii) For any nonempty segment S the functional S^* is defined as

$$S^*(x) = \sum_{t \in S} x(t),$$

and it is easy to verify that $\|S^*\| = 1$. Consider the set

$$D = \left\{ \sum_{i=1}^n \lambda_i S_i^* : \lambda_i \in \mathbb{R}, \{S_i\}_{i=1}^n \text{ pairwise disjoint segments of } (T, \leq) \right.$$

$$\left. \text{and } \sum_{i=1}^n \lambda_i^2 \leq 1 \right\}.$$

It is easy to verify that D is a 1-norming set for X and also that it is norm dense in its w^* -closure. Since ℓ^1 does not embed in X , Haydon's classical theorem [H] yields that D generates in norm the space X^* . Thus we conclude that the set $\{S^* : S \text{ is a segment of } T\}$ generates in norm the space X^* (e.g. [A], [F]).

Let Z be the closed linear span of the set $\{e_t^* : t \in T\}$ in X^* . Then it can be shown as in the case of the dyadic tree (D, \leq) (e.g. [LS]) that Z is a predual of X and $X^*/Z \cong \ell^2(\mathcal{B}_T)$ (if there is no infinite branch of (T, \leq) , i.e., if $\mathcal{B}_T = \emptyset$, then clearly $Z = X^*$ and the space X is reflexive). Assume that $\mathcal{B}_T \neq \emptyset$. Then we get that $X^{**} \cong X \oplus \ell^2(\mathcal{B}_T)$. Thus the space X^{**} is WCG as a direct sum of two WCG spaces; it is also not difficult to see that (for the same reason) X^{**} is ℓ^2 -saturated and hence does not contain the space ℓ^1 . □

Theorem 4.3. *There exists a WCG Banach space X such that X^{**} is also WCG not containing ℓ^1 . Moreover there exists a closed subspace Y of X with the properties*

- (i) *The spaces Y and Y^{**} are not WCG.*
- (ii) *The quotient X/Y is a reflexive space.*

Proof. We start with the definition of the space X . Let T_m , $m \geq 1$, be a sequence of Reznichenko trees. Set $X_m = J(T_m \times \{m\})$ and $X = (\sum_{m=1}^{\infty} \oplus X_m)_2$. Since each tree T_m is of height ω , we have that each X_m is a WCG space and also that $X_m^{**} \cong X_m \oplus \ell^2(\mathcal{B}_m)$ (where \mathcal{B}_m is the set of branches of T_m) is also a WCG space. Therefore the spaces X and X^{**} are WCG and do not contain ℓ^1 . We pass now to define the subspace Y . For every $t \in T = \bigcup_{m=1}^{\infty} T_m$ we set $D_t = \{m \in \mathbb{N} : t \in T_m\}$ and $x_t = \sum_{m \in D_t} \frac{1}{2^{m/2}} \cdot e_{(t,m)}$ (clearly $\|x_t\| \leq 1$). Finally we define $Y = \overline{\text{span}}\{x_t : t \in T\}$.

Claim I. There exists a family $\{f_t : t \in T\} \subseteq Y^*$ so that the family $\{(x_t, f_t) : t \in T\}$ is an M -basis for Y .

Proof of Claim I. It is clear that it is enough to find a family of functionals $\{x_t^* : t \in T\} \subseteq Y^*$ such that $x_t^*(x_t) \neq 0$ for $t \in T$ and $x_t^*(x_s) = 0$ for $t \neq s$ that separates the points of Y (then we set $f_t = \frac{x_t^*}{x_t^*(x_t)}$ for $t \in T$). We consider the natural embedding $I : Y \rightarrow X$ ($I(x) = x$, for $x \in Y$) and its dual operator $I^* : X^* \rightarrow Y^*$. Letting $t \in T$, it is easy to verify that all functionals $\{I^*(e_{(t,m)}^*) : m \in D_t\}$ of the space Y have the same kernel, which is the subspace Z of Y that is generated from the vectors x_s , $s \in T, t \neq s$. Therefore if we normalize some one of them and denote it by x_t^* , we shall have that, for every $m \in D_t$, there exists a real number $\lambda_{(t,m)} \neq 0 : I^*(e_{(t,m)}^*) = \lambda_{(t,m)} \cdot x_t^*$. It is clear that the family $\{x_t^* : t \in T\}$ satisfies the desired properties.

We note that if $t \in T$ and $m \in D_t$, then we have $x_t^*(x_t) = \frac{I^*(e_{(t,m)}^*)(x_t)}{\lambda_{(t,m)}} = \frac{1}{2^{m/2} \cdot \lambda_{(t,m)}}$ and thus

$$(2) \quad f_t = \frac{x_t^*}{x_t^*(x_t)} = \frac{I^*(e_{(t,m)}^*)}{\lambda_{(t,m)} \cdot x_t^*(x_t)} = 2^{m/2} \cdot I^*(e_{(t,m)}^*).$$

Claim II. Let $m \in \mathbb{N}$ and let $b = \{t_1 < \dots < t_n < \dots\}$ be any branch of the tree T_m . Then the series $\sum_{k=1}^{\infty} f_{t_k}$ is weak* convergent in Y^* (and thus the sequence $\sum_{k=1}^n f_{t_k}$, $n \geq 1$ is norm bounded).

Proof of Claim II. It is clear that the subspace $Y_b = \overline{\text{span}}\{e_{(t_k,m)} : k \geq 1\}$ of X is isomorphic to the James space and complemented (in X_m and hence) in X . Therefore the series $\sum_{k=1}^{\infty} e_{(t_k,m)}^*$ is weak* convergent to some $x^* \in X^*$. Therefore,

$$(3) \quad 2^{m/2} \cdot I^*(x^*) \stackrel{w^*}{=} \sum_{k=1}^{\infty} 2^{m/2} \cdot I^*(e_{(t_k,m)}^*).$$

Since $m \in D_{t_k}$, for all $k \geq 1$, we get from (2) above that $2^{m/2} \cdot I^*(e_{(t_k,m)}^*) = f_{t_k}$, for all $k \geq 1$. It then follows immediately from (3) that the series $\sum_{k=1}^{\infty} f_{t_k}$ is weak* convergent in Y^* .

We are ready to prove that Y is not WCG. Assume the contrary. Then by Claim I, the family $\{(x_t, f_t) : t \in T\}$ is an M -basis for Y and by Theorem 2.2

there exists a sequence $D_n, n \geq 1$, of subsets of T with $T = \bigcup_{n=1}^\infty D_n$ such that if a sequence $(t_k)_k \subset D_m$ for some $m \in \mathbb{N}$, then $\liminf_n \|\sum_{k=1}^n f_{t_k}\| = +\infty$. On the other hand, Proposition 3.3 yields that there exists $k_0 \in \mathbb{N}$ with the property that the set $D_{k_0} \cap T_{k_0}$ contains an infinite segment of the tree T_{k_0} . Claim II derives a contradiction and thus Y is not WCG.

For the remaining part we first observe that for the $b = \{t_1 < t_2 < \dots < t_n < \dots\}$ branch of T_{m_0} , the sequence $(e_{t_k})_k$ w^* -converges to $e_b^{**} \in X_{m_0}^{**}$. Also the space X^* is norm generated by the family $\{S^* : S \text{ is a segment of } T_m, m \in \mathbb{N}\}$. We prove the following:

Claim III. For every $b = (t_k)_k$ branch of T_{m_0} the sequence $(x_{t_k})_k$ w^* -converges to $\frac{1}{2^{m_0/2}} e_b^{**}$.

Proof of Claim III. Indeed for $m \neq m_0$, property (iii) of $(T_n)_n$ yields that $\{e_{t_k} : t_k \in T_m\}$ is either a finite set or a weakly null sequence. Therefore for S a segment of $T_m, m \neq m_0, S^*(x_{t_k}) \rightarrow 0$. Also for S a segment of $T_{m_0}, S^*(x_{t_k}) \rightarrow \frac{1}{2^{m_0/2}} e_b^{**}(S^*)$. This proves the claim. \square

It follows that for every $b \in \mathcal{B} = \bigcup_{m=1}^\infty \mathcal{B}_m, e_b^{**} \in Y^{**}$. Hence $\ell^2(\mathcal{B})$ is contained in Y^{**} , and since $X^{**} = X \oplus \ell^2(\mathcal{B})$, we obtain that $Y^{**} = Y \oplus \ell^2(\mathcal{B})$. Therefore Y^{**} is not WCG.

To see that X/Y is reflexive we simply observe that

$$(X/Y)^{**} = X^{**}/Y^{**} = (X \oplus \ell^2(\mathcal{B})) / (Y \oplus \ell^2(\mathcal{B})) = X/Y,$$

and this completes the proof of the theorem. \square

Remark 4.4. The unconditional counterpart of the space X is also defined. Namely for a tree T we consider the following norm. For $x \in c_{00}(T)$ we define

$$\|x\|_{1,2} = \sup \left(\sum_{i=1}^n \left(\sum_{t \in S_i} |x(t)| \right)^2 \right)^{1/2}$$

where the supremum is taken over all finite families $\{S_i\}_{i=1}^n$ of disjoint segments of T . We set $X_{1,2}(T)$ to be the completion of $(c_{00}(T), \|\cdot\|_{1,2})$ and clearly $X_{1,2}(T)$ has an unconditional basis. For the sequence $(T_m \times \{m\})_m$ used in the previous example we define $\tilde{X} = (\sum_{m=1}^\infty \oplus \tilde{X}_m)_2$. Clearly \tilde{X} is WCG and it has an unconditional basis.

The subspace \tilde{Y} is defined as in the case of X and also it is not WCG. For every $m \in \mathbb{N}$ and $b \in \mathcal{B}_m, b = (t_k)_k$, the sequence $(e_{t_k})_k$ is isometric to the usual basis of ℓ^1 . It is known that $(\ell^1)^{**} = \ell^1 \oplus W$. For $b \in \mathcal{B} = \bigcup_m \mathcal{B}_m$, we denote by W_b the corresponding part of the second dual of $\overline{\text{span}}\{(e_{t_k})_k\}$. Then

$$(\tilde{X})^{**} = (\tilde{X}) \oplus \left(\sum_{b \in \mathcal{B}} \oplus W_b \right)_2.$$

Using similar arguments as in the proof of Claim III of Theorem 4.3, we conclude that \tilde{X}/\tilde{Y} is reflexive.

REFERENCES

- [A] S.A. Argyros, *Weakly Lindelöf Determined Banach spaces not containing $\ell^1(\mathbb{N})$* , (1993), preprint.
- [ACGJM] S.A. Argyros, J.F. Castillo, A.S. Granero, M. Jimenez and J.P. Moreno, *Complementation and embeddings of $c_0(I)$ in Banach spaces*, *Proc. London Math. Soc.* (3) **85**, (2002), 742–768. MR 2003k:46022
- [AL] D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, *Annals of Math.*, **88**, (1968), 35–46. MR 37:4562
- [AM] S.A. Argyros and S. Mercourakis, *On weakly Lindelöf Banach spaces*, *Rocky Mountain J. Math.* **23**, (1993), 395–446. MR 94i:46016
- [F] M.J. Fabián, *Gâteaux Differentiability of Convex Functions and Topology*, John Wiley & Sons, Inc. (1997). MR 98h:46009
- [F1] M.J. Fabián, *Each weakly countably determined Asplund space admits a Fréchet differentiable norm*, *Bull. Australian Math. Soc.*, **36**, (1987), 367–374. MR 89c:46018
- [H] R.G. Haydon, *Some more characterizations of Banach spaces containing ℓ^1* , *Math. Proc. Cambridge Phil. Soc.*, **80**, (1976), 269–276. MR 54:11031
- [HHZ] P. Habala, P. Hajek, and V. Zizler, *Introduction to Banach Spaces I, II* Matfyzpress, Prague (1996).
- [HO] J. Hagler and E. Odell, *A Banach space not containing ℓ^1 whose dual ball is not weak* sequentially compact*, *Illinois J. Math.* **22**, (1978), 290–294. MR 58:2174
- [J] R.C. James, *A separable somewhat reflexive Banach space with nonseparable dual*, *Bull. Amer. Math. Soc.* **80**, (1974), 738–743. MR 54:5811
- [JL] W.B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, *Israel J. Math.* **17**, (1974), 219–230. MR 54:5808
- [JZ] K. John and V. Zizler, *Smoothness and its equivalents in weakly compactly generated Banach spaces*, *J. Funct. Anal.* **15**, (1974), 161–166. MR 54:5807
- [LS] J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain ℓ^1 and whose duals are nonseparable*, *Studia Math.* **54**, (1975), 81–105. MR 52:11543
- [MS] S. Mercourakis and E. Stamati, *A new class of weakly \mathcal{K} -analytic Banach spaces*, *Mathematika* (to appear).
- [R] H.P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, *Compos. Math.* **28**, (1974), 83–111. MR 54:5810
- [T] M. Talagrand, *Espaces de Banach faiblement \mathcal{K} -analytiques*, *Ann. of Math.* **110**, (1979), 407–438. MR 81a:46021
- [Z] V. Zizler, *Nonseparable Banach Spaces*, *Handbook of the Geometry of Banach spaces* W.B. Johnson, J. Lindenstrauss (eds.), North-Holland, Vol. 2, ch. 41, (2003), 1743–1816.

DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY OF ATHENS, ATHENS 15780, GREECE

E-mail address: sargyros@math.ntua.gr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, ATHENS 15784, GREECE

E-mail address: smercour@math.uoa.gr