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TOPOLOGICALLY KNOTTED LAGRANGIANS IN SIMPLY CONNECTED FOUR-MANIFOLDS

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(Communicated by Ronald A. Fintushel)

Abstract. Vidussi was the first to construct knotted Lagrangian tori in simply connected four-dimensional manifolds. Fintushel and Stern introduced a second way to detect such knotting. This note demonstrates that similar examples may be distinguished by the fundamental group of the exterior.

Introduction

The study of the topology of Lagrangian submanifolds took off starting in the mid 1980s. This early work is surveyed in an influential paper of Eliashberg and Polterovich [1]. The existence of smoothly knotted Lagrangian tori in some simply connected four-dimensional symplectic manifolds was recently demonstrated by S. Vidussi, [6]. The paper of Vidussi contains more history and background of the problem. Shortly thereafter, R. Fintushel and R. Stern introduced a second invariant that could distinguish smoothly knotted Lagrangian tori and applied their invariant to variations of Vidussi's construction [2]. In this paper we note that the fundamental group of the complement of a Lagrangian torus may also be used to show that it is knotted.

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Construction of knotted Lagrangian tori and the invariants of Vidussi, and Fintushel and Stern

The essence of the construction of knotted Lagrangians that was developed by Vidussi in [6] may be summarized as follows. Let M be a 3-manifold that fibers over S^1. The 4-manifold $S^1 \times M$ inherits a symplectic form expressible as $\omega = \pi_1^* d\theta \wedge p^* d\theta + \eta$, where $d\theta$ is the standard 1-form on S^1, π_1 is the natural projection $S^1 \times M \to S^1$, p is the composition of the projection onto the second factor followed by the projection of M onto S^1, and η is an area form on the fibers of M. In this manifold, any torus of the form $S^1 \times \delta$ is symplectic, i.e., ω restricts to a symplectic form whenever δ is an embedded circle transverse to all of the fibers of M. Furthermore, any torus of the form $S^1 \times \gamma$ is Lagrangian, i.e., ω restricts to

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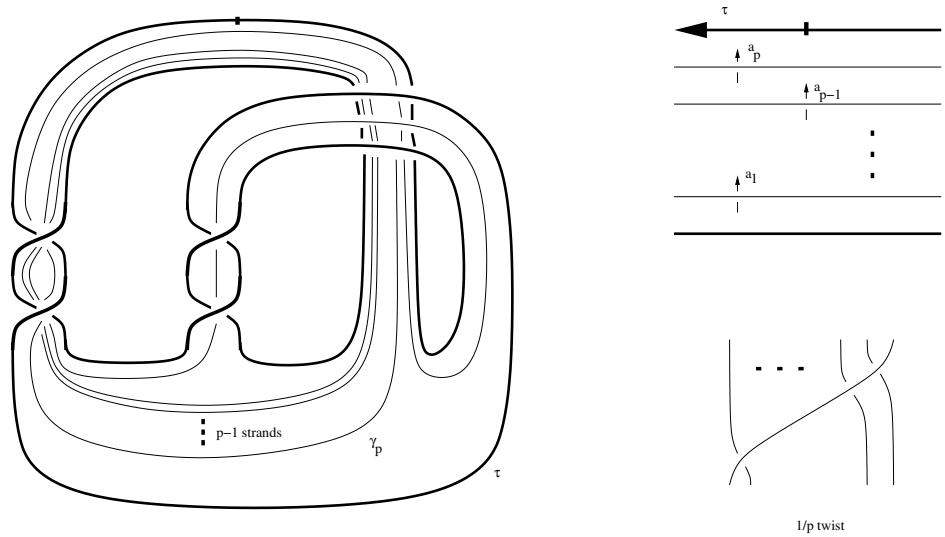

FIGURE 1. The knot γ_p

zero whenever γ is an embedded circle in one of the fibers of M. More complicated examples may be constructed by taking symplectic fiber sums with other symplectic manifolds.

In [6] Stefano Vidussi used the Seiberg-Witten invariants of the symplectic sum of an $E(1)$ with those examples summed with non-rational elliptic surfaces along the Lagrangian torus to prove that the various Lagrangian tori were not smoothly isotopic. Ron Fintushel and Ron Stern later used relative Seiberg-Witten invariants of the complement of the Lagrangian tori to prove that they are not smoothly isotopic [2]. Fintushel and Stern described a particularly simple special case of their invariant as a Lagrangian framing defect.

Fundamental groups of Lagrangian knot complements

The fundamental group of the complement of the torus is a natural invariant that could in principal detect (even topologically) non-isotopic tori. For many examples the fundamental group of the complement will not detect anything. However there are homotopic Lagrangian tori having complements with nonisomorphic fundamental groups. These examples may be chosen to be null-homologous with zero Lagrangian framing defect.

The first specific example considered in [6] took the following form. Let $T(p,q)$ represent the (p,q) torus knot, and let M_τ be the result of 0-surgery on $\tau = T(2,-3)$. It is well known that M_τ fibers over S^1 [5]. Let $E(1)$ be the rational elliptic surface [4], and set $E(1)_\tau = E(1)\#_{F=S^1\times\mu}S^1 \times M_\tau$, where μ is a meridian to τ. Figure 1 shows, among other things, a knot γ_p on a fiber in the exterior of the knot τ. For the first example, we will ensure that γ_p does not link μ. By inspection, $\gamma_p = T(p,p+1)$ as a knot in S^3. As we will see the fundamental group of the complement of $S^1 \times \gamma_p$ in $E(1)_\tau$ is isomorphic to $\mathbb{Z}$. However doubling this construction in an appropriate sense will result in a fundamental group that surjects onto the fundamental group of the complement of the $(p,p+1)$ torus knot.

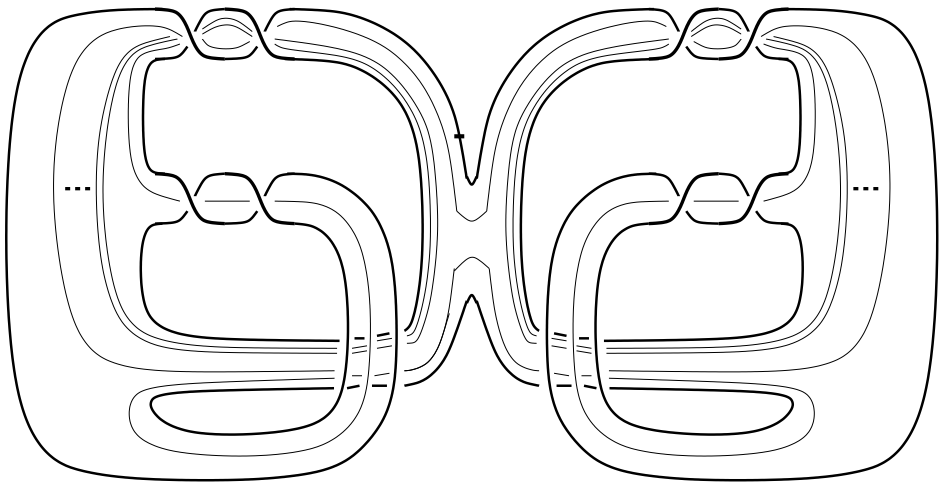

FIGURE 2. The double, $\tau\# - \overline{\tau} \perp\!\!\!\perp \gamma\# - \overline{\gamma}$

Our doubled construction begins with the knot, $\tau\# - \overline{\tau}$. Here the overline refers to taking the reflection of the knot, and the minus sign refers to switching the orientation of the knot. Our symplectic manifold is defined by $E(1)_{\tau\#-\overline{\tau}} = E(1)\#_{F=S^1\times\mu} S^1 \times M_{\tau\#-\overline{\tau}}$, where μ is a meridian to $\tau\# - \overline{\tau}$. We will consider the Lagrangian tori $S^1 \times (\gamma_p\# - \overline{\gamma}_p)$ in $E(1)_{\tau\#-\overline{\tau}}$.

Proposition 0.1. *The fundamental groups of the Lagrangian knot exteriors $S^1 \times (\gamma_p\# - \overline{\gamma}_p)$ are given by*

$$\begin{aligned}\pi_1(E(1)_{\tau\#-\overline{\tau}} - \mathring{N}(S^1 \times (\gamma_p\# - \overline{\gamma}_p))) \\ = \langle u, v, x, y | u^p v^{p+1} = 1, x^p y^{p+1} = 1, uv = xy, vu = yx\rangle.\end{aligned}$$

Proof. We will use the following alternate description of $E(1)_{\tau\#-\overline{\tau}}$. Let $X_{\tau\#-\overline{\tau}}$ denote the exterior of $\tau\# - \overline{\tau}$. We then have

$$E(1)_{\tau\#-\overline{\tau}} = S^1 \times X_{\tau\#-\overline{\tau}} \cup_{T^3} (E(1) - \mathring{N}\,(T^2)).$$

Here T^2 is the fiber in $E(1)$. This allows us to apply the Seifert-Van Kampen theorem to compute $\pi_1(E(1)_{\tau\#-\overline{\tau}} - \mathring{N}\,(S^1 \times \gamma_p\# - \overline{\gamma}_p))$ as follows:

$$\begin{aligned}&\pi_1(E(1)_{\tau\#-\overline{\tau}} - \mathring{N}\,(S^1 \times \gamma_p\# - \overline{\gamma}_p)) \\ &\quad = \pi_1(S^1 \times (S^3 - \mathring{N}\,(\tau \perp\!\!\!\perp \gamma_p\# - \overline{\gamma}_p)) \cup_{T^3} (E(1) - \mathring{N}\,(T^2))) \\ &\quad = \pi_1(S^1 \times (S^3 - \mathring{N}\,(\tau\# - \overline{\tau} \perp\!\!\!\perp \gamma_p\# - \overline{\gamma}_p)))/\langle \pi_1 T^3\rangle \\ &\quad = \pi_1(S^1 \times ((S^3 - \mathring{N}\,(\tau\# - \overline{\tau} \perp\!\!\!\perp \gamma_p\# - \overline{\gamma}_p)) \cup_{T^2} S^1 \times D^2)) \cup_{\substack{S^1\times pt \\ pt\times\tau\#-\overline{\tau}}} 2D^2) \\ &\quad = \pi_1((S^3 - \mathring{N}\,(\gamma_p\# - \overline{\gamma}_p)) \cup_{\tau\#-\overline{\tau}} D^2) = \pi_1(S^3 - \mathring{N}\,(\gamma_p\# - \overline{\gamma}_p))/\langle \tau\# - \overline{\tau}\rangle.\end{aligned}$$

The third line of the above computation follows because $\pi_1(E(1) - \mathring{N}\,(T^2)) = 1$. The fourth line follows because glueing anything that kills the fundamental group

of the boundary will have the same effect. The $(S^1 \times D^2) \cup D^2 \cup D^2$ that we glue in replaces $\overset{\circ}{N}(\tau\#-\overline{\tau})$, kills the generator of the S^1 factor, and adds one more relation. To complete the proof we need to compute $\tau\# - \overline{\tau}$ in $\pi_1(S^3 - \overset{\circ}{N}(\gamma_p\# - \overline{\gamma}_p))$.

We begin by computing the element of $\pi_1(S^3 - \overset{\circ}{N}(T(p,p+1)))$ represented by τ. We will use the Wirtinger generators depicted in Figure 1. Recall that a Wirtinger generator is a loop that starts at the base point, goes over the knot, under in the direction of the arrow and then back over the knot to the base point, [5]. Using the Wirtinger relation we see that the generators on the bottom of a $1/p$ twist will be $a_1a_1a_1^{-1}$, $a_1a_pa_1^{-1}$, $\ldots$, $a_1a_2a_1^{-1}$, when the generators at the top of the twist are a_p, $\ldots$,a_1. Since $T(p,p+1)$ is just a stack of $p+1$ $1/p$-twists, this allows us to obtain the presentation $\langle z, a_1, \ldots, a_p | z = a_1 \ldots, a_pa_1, za_1z^{-1} = a_p, za_2z^{-1} = a_1, \ldots, za_pz^{-1} = a_{p-1}\rangle$, and label every strand in the diagram for $T(p,p+1)$. The element of π_1 represented by τ may now be read off of the diagram. We obtain $\tau = (y^{-1})(ya_1y^{-1})y(ya_1y^{-1})(ya_1^{-1}y^{-1})(ya_1^{-1}y^{-1}) = [a_1, y]$ where $y = a_1 \ldots a_p$. In computing this expression, we reversed the obvious Reidemeister I move on the right side of Figure 1, and grouped terms after each group of undercrossings. Since γ_p lies on a Seifert surface for τ, we know that the linking number of γ_p and τ is zero. So τ must be (as we found it) in the commutator subgroup of the knot group of γ_p. In this case τ normally generates the commutator subgroup. So the knot group of γ_p mod τ is isomorphic to the infinite cyclic group. The standard generators of the knot group of $T(p.p+1)$ are $x = a_1^{-1}y^{-1}$ and y.

We will apply the Seifert-vanKampen theorem to compute the knot group of $\gamma_p\# - \overline{\gamma}_p$, and the element $\tau\# - \overline{\tau}$ in that group. The twice punctured plane of symmetry in the complement of $\gamma_p\# - \overline{\gamma}_p$ in Figure 2 becomes a twice punctured S^2 when we add the point at infinity. Two copies of the complement of γ_p are glued together along this punctured sphere. The resulting presentation of the fundamental group is

$$\langle w, z, a_k, b_k, k = 1, \ldots, p | \\ z = a_1 \ldots a_pa_1, w = b_1 \ldots b_pb_1, za_{k+1}z^{-1} = a_k, wb_{k+1}w^{-1} = b_k, k = 1, \ldots, p\rangle,$$

and $\tau_p\# - \overline{\tau}_p$ represents $[a_1, y][b_1, v]^{-1}$ where $v = b_1 \cdots b_p$. Setting $x = z^{-1}$, $u = w^{-1}$, and rewriting the presentation of the knot group of $\gamma_p\# - \overline{\gamma}_p$ in terms of u, v, x, and y results in the stated presentation of the complement of $S^1 \times (\gamma_p\# - \overline{\gamma}_p)$. $\square$

Remark 0.2. The computation at the beginning of the previous lemma shows that the fundamental group of the complement of a Lagrangian torus of the form $S^1 \times \gamma$ in $E(1)_\kappa$ always takes the form $\pi_1(X_\gamma)/\langle\kappa\rangle$ where $\kappa \in [\pi_1(X_\gamma), \pi_1(X_\gamma)]$ is the class of κ and γ is a knot in a fiber of the fibered knot, κ.

Remark 0.3. The fundamental group of the complement of $S^1 \times \gamma_p$ in $E(1)_\tau$ is always $\mathbb{Z}$. However the Fintushel-Stern Lagrangian framing defect is given by $\lambda(\gamma_p) = p+1$, so the tori $S^1 \times \gamma_p$ represent distinct isotopy classes.

Remark 0.4. For any knot of the form $\gamma\# - \overline{\gamma}$ in a fiber of a knot of the form $\kappa\# - \overline{\kappa}$, the fundamental group of the complement of $S^1 \times (\gamma\# - \overline{\gamma})$ in $E(1)_{\kappa\# - \overline{\kappa}}$ will surject onto the knot group of γ. Thus such Lagrangian tori are not isotopic to the standard nullhomologous Lagrangian tori. By symmetry the Lagrangian framing defect of Fintushel and Stern will vanish on the tori $S^1 \times (\gamma\# - \overline{\gamma})$. Vidussi's technique

of summing with an $E(1)$ in an appropriate way or Fintushel and Stern's main invariant $I(X,T)$ may still in principal detect this.

Remark 0.5. The next proposition will establish that all of the groups in the previous proposition are distinct. Thus the tori $S^1 \times (\gamma_p \# - \overline{\gamma}_p)$ form an infinite family of nonisotopic Lagrangian tori that are all homotopic.

Lemma 0.6. *The following groups are all distinct:*

$$\Gamma_p = \langle u, v, x, y | u^p v^{p+1} = 1, x^p y^{p+1} = 1, uv = xy, vu = yx \rangle.$$

Proof. We will use the order ideal of the Alexander module to distinguish these groups [5]. Introducing new variables, $t = xy$, $a = t^p v$, and $b = t^p y$ will allow the presentation to be written in the form:

$$\langle t, a, b | \prod_{k=0}^{p-1} t^{k(p+1)+1} a^{-1} t^{-(k(p+1)+1)} \prod_{k=0}^{p} t^{p^2-kp} a t^{-(k(p+1)+1)},$$
$$\prod_{k=0}^{p-1} t^{k(p+1)+1} b^{-1} t^{-(k(p+1)+1)} \prod_{k=0}^{p} t^{p^2-kp} b t^{-(k(p+1)+1)}, ata^{-1}t^{-1}tbt^{-1}b^{-1} \rangle.$$

From this presentation it is clear that the abelianization of Γ_p is infinite cyclic. Furthermore from the same presentation one can see that the commutator subgroup, $\mathcal{D}^1\Gamma_p = [\Gamma_p, \Gamma_p]$, is generated by the elements $t^k a t^{-k}$ and $t^k b t^{-k}$ for all k. The abelianization of the commutator subgroup, $(\mathcal{D}^1\Gamma_p)^{\mathrm{ab}}$, inherits the structure of a $\mathbb{Z}[t, t^{-1}]$-module with t acting by conjugation. As a $\mathbb{Z}[t, t^{-1}]$-module it has a presentation

$$(\mathcal{D}^1\Gamma_p)^{\mathrm{ab}} = \langle a, b, | \mathfrak{p}(t)a = 0, \mathfrak{p}(t)b = 0, (1-t)a - (1-t)b = 0 \rangle;$$

here

$$\mathfrak{p}(t) = \sum_{k=0}^{p} t^{p^2-kp} - \sum_{k=0}^{p-1} t^{k(p+1)+1} = \frac{(t^{p(p+1)} - 1)(t-1)}{(t^{p+1}-1)(t^p - 1)}.$$

We are making the obvious abuses of notation with t, a, and b. This module is called the Alexander module. The order ideal of this module is the ideal of $\mathbb{Z}[t, t^{-1}]$ generated by $\mathfrak{p}(t)^2$ and $(t-1)\mathfrak{p}(t)$. The number $e^{\frac{2\pi i}{p(p+1)}}$ is a zero of every polynomial in the order ideal, and the numbers $e^{\frac{2\pi i}{k(k+1)}}$ are not zeros of $\mathfrak{p}(t)^2$ for any $k > p$. This completes the proof. □

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