PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 3, Pages 693–697
S 0002-9939(04)07562-8
Article electronically published on September 29, 2004

A NOTE ON LOCALIZATIONS OF PERFECT GROUPS

BERNARD BADZIOCH AND MARK FESHBACH

(Communicated by Paul Goerss)

ABSTRACT. We describe a perfect group whose localization is not perfect.

1. Introduction

A localization is a type of a functor $L\colon \mathbf{Groups} \to \mathbf{Groups}$ that is idempotent (that is, $LLG \cong LG$) and admits a coaugmentation $\eta\colon G \to LG$ [1]. Localizations are ubiquitous in group theory: abelianization, killing of the p–torsion and inversion of a prime in a group are all examples of such functors.

The natural question – which classes of groups are preserved by all localizations – has been a focus of a lot of study recently. This work yields both classes that are preserved (e.g. abelian groups, nilpotent groups of class 2 [2]), and those that do not have this property, such as finite [3] or solvable groups [5]. The goal of this note is to prove the following.

Theorem 1.1. *The class of perfect groups is not closed with respect to taking localizations. That is, there exists a perfect group P and a localization $\eta\colon P \to LP$ such that LP is not perfect.*

This answers a question posed by Casacuberta in [1].

The groups P and LP we construct in the proof of Theorem 1.1 are infinite. It would be interesting to know if one can find a finite perfect group with a nonperfect localization. The following shows however that the localized group would have to be infinite.

Proposition 1.2. *If $\eta\colon P \to LP$ is a localization of a perfect group P and LP is finite, then LP is a perfect group.*

The main results of this paper were obtained independently by J. Rodríguez, J. Scherer and A. Viruel in [7].

2. Proofs

Our main tool will be the following fact, which characterizes all possible localizations of a group. It is a direct ramification of the definition of localization functors (see e.g. [1, Lemma 2.1]).

Received by the editors January 10, 2003 and, in revised form, November 9, 2003.
2000 *Mathematics Subject Classification.* Primary 20J15; Secondary 55P60, 20F38.

Lemma 2.1. *A homomorphism $\eta\colon G \to H$ is a localization of G with respect to some localization functor iff η induces a bijection of sets*

$$\mathrm{Hom}(H, H) \xrightarrow{\eta^*} \mathrm{Hom}(G, H).$$

As an application we obtain

Lemma 2.2. *If $\eta\colon G \to H$ is a localization and G is a perfect group, then there are no nontrivial homomorphisms $H/[H, H] \to H$.*

Proof. Let $g\colon H/[H, H] \to H$ be any homomorphism, and let f denote the composition $H \to H/[H, H] \xrightarrow{g} H$. Since G is perfect the composition $f \circ \eta$ is the trivial map. Lemma 2.1 implies then that f is also trivial, and thus so is g. □

Since every finite group H admits a nontrivial map $H/[H, H] \to H$ unless H is perfect, Proposition 1.2 is a consequence of Lemma 2.1.

Next, we turn to the proof of Theorem 1.1. We start with

Construction of the group P. For $n \geq 0$, let $\tilde{P}_n$ be a free group on 2^n generators $x_1^{(n)}$, $x_2^{(n)}$,…,$x_{2^n}^{(n)}$, and let $\tilde{\phi}_n\colon \tilde{P}_n \to \tilde{P}_{n+1}$ be a group homomorphism defined by

$$\tilde{\phi}_n(x_i^{(n)}) = [x_{2i-1}^{(n+1)}, x_{2i}^{(n+1)}]$$

where $[a, b] = a^{-1}b^{-1}ab$ is the commutator of a and b. Define $\tilde{P} := \varinjlim_n \tilde{P}_n$. Notice that since $\tilde{P}$ is generated by the elements $x_i^{(n)} \in [\tilde{P}, \tilde{P}]$, the group $\tilde{P}$ is perfect. Let K be the smallest normal subgroup of $\tilde{P}$ containing the elements $[x_1^{(0)}, x_i^{(n)}]$ for all $n \geq 0$, $1 \leq i \leq 2^n$. Define $P := \tilde{P}/K$.

Proposition 2.3. *The group P is perfect and $x_1^{(0)}$ is a central element of P. Moreover, $x_1^{(0)}$ is an element of infinite order, and as a consequence P is a nontrivial group.*

Proof. The first two claims are obvious. To see that $x_1^{(0)} \in P$ has infinite order notice that P can be viewed as a limit

$$P = \varinjlim_n P_n$$

where P_n is a group with the presentation

$$P_n := \langle x_1^{(n)}, \ldots, x_{2^n}^{(n)} \mid [x_1^{(0)}, x_i^{(n)}] = 1, \quad i = 1, \ldots, 2^n \rangle$$

(by abuse of notation we denote here by $x_1^{(0)}$ the image of the element $x_1^{(0)}$ under the map $P_0 \to P_n$). It is then enough to show that $x_1^{(0)}$ has infinite order in P_n for all $n \geq 0$. To see this consider $GL(\mathbb{Z}, 2^n + 1)$ – the group of invertible matrices of dimension $2^n + 1$ with integer coefficients. For $n > 0$ there is a homomorphism

$$\psi_n\colon P_n \to GL(\mathbb{Z}, 2^n + 1)$$

defined by $\psi_n(x_i^{(n)}) = e^1_{i,i+1}$, where $e^a_{i,j}$ denotes the matrix with 1's on the diagonal, a as the (i, j)-th entry, and 0's elsewhere. One can check that $\psi_n(x_1^{(0)}) = e^{\pm 1}_{1,2^n+1}$. Since $(e^{\pm 1}_{1,2^n+1})^k = e^{\pm k}_{1,2^n+1}$, this is a non-torsion element of $GL(\mathbb{Z}, 2^n + 1)$, and as a consequence $x_1^{(0)}$ has infinite order in P_n as claimed. □

Construction of the map $\boldsymbol{\eta}\colon \mathbf{P} \to \mathbf{LP}$. Let $\mathbb{Q}$ be the group of rational numbers. Define

$$LP := P \oplus \mathbb{Q}/\langle (x_1^{(0)}, -1)\rangle,$$

and let the map $\eta\colon P \to LP$ be given by the composition of the inclusion $P \hookrightarrow P\oplus\mathbb{Q}$ and the projection $P\oplus\mathbb{Q} \to LP$. Since η is a monomorphism we will identify P with its image $\eta(P)$. Notice that P is a normal subgroup of LP and that $LP/P \cong \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is not a perfect group, neither is LP.

It remains to prove that η is a localization of P. By Lemma 2.1 this amounts to showing that any homomorphism $f\colon P \to LP$ admits a unique factorization

$$\begin{array}{ccc} P & \xrightarrow{\ \eta\ } & LP \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & LP \end{array}$$

Uniqueness of $\bar{\mathbf{f}}$. Assume that $\bar{f}_1, \bar{f}_2\colon LP \to LP$ are homomorphisms such that $\eta\bar{f}_1 = \eta\bar{f}_2$, and consider the homomorphism

$$g := (\bar{f}_1|_{\mathbb{Q}} - \bar{f}_2|_{\mathbb{Q}})\colon \mathbb{Q} \to LP.$$

We have $\bar{f}_1(1) = \bar{f}_1(x_1^{(0)}) = \bar{f}_2(x_1^{(0)}) = \bar{f}_2(1)$, and thus $\mathbb{Z} \subseteq \ker g$. Therefore we get a factorization

$$g\colon \mathbb{Q}/\mathbb{Z} \to LP$$

and $g \equiv 1$ iff $\bar{f}_1 = \bar{f}_2$. Thus, our claim is a consequence of the following.

Lemma 2.4. *The group LP is torsion free.*

Proof. Let $(w, \frac{p}{q})$ represent a torsion element of LP. Then $q \cdot (w, \frac{p}{q}) = (w^q, p) = w^q(x_1^{(0)})^p$ is a torsion element in P. Consider the group $R := P/\langle x_1^{(0)}\rangle$. The element $w^q(x_1^{(0)})^p = w^q$ is torsion in R, and thus so is w. Notice that $R = \varinjlim_n R_n$ where

$$R_n = \langle x_1^{(n)}, \dots, x_{2^n}^{(n)} | x_1^{(0)} = 1\rangle.$$

It follows that w is a torsion element in R_n for n large enough. On the other hand, R_n is a group with one relator given by a word that is not a proper power of any element in the free group. By [4, Thm. 4.12, p. 266], R_n must be torsion free. Therefore $w = 1$ in R, and so $(w, \frac{p}{q}) = ((x_1^{(0)})^l, \frac{p}{q}) = l + \frac{p}{q} \in \mathbb{Q} \subseteq LP$ for some $l \in \mathbb{Z}$. Since by assumption $(w, \frac{p}{q})$ is a torsion element, it must be trivial. □

Existence of $\bar{\mathbf{f}}$. We need to show that every homomorphism $f\colon P \to LP$ admits an extension $\bar{f}\colon LP \to LP$. Assume for a moment that $f(x_1^{(0)}) = (x_1^{(0)})^k \in LP$ for some $k \in \mathbb{Z}$. From the definition of LP it follows then that $\bar{f}$ can be defined by setting $\bar{f}(r) = kr$ for all $r \in \mathbb{Q}$. Next, notice that since P is perfect, $f(P) \subseteq [LP, LP] = P$. Combining these observations we get that the existence of $\bar{f}$ follows from

Lemma 2.5. *If $g\colon P \to P$ is any homomorphism, then $g(x_1^{(0)}) = (x_1^{(0)})^k$ for some $k \in \mathbb{Z}$.*

Recall the group $R = P/\langle x_1^{(0)}\rangle$ defined in the proof of Lemma 2.4. Lemma 2.5 will follow if we show that for any homomorphism $g\colon P \to R$ the element $x_1^{(0)}$ is in the kernel of g. In the proof of Proposition 2.3 we also defined the group

$$P_1 = \langle x_1^{(1)}, x_2^{(1)} \mid [x_1^{(0)}, x_i^{(1)}] = 1,\ i = 1, 2\rangle.$$

Since $x_1^{(0)}$ is not in the kernel of the map $P_1 \to P$ it is enough to show that for any $g\colon P_1 \to R$ we have $x_1^{(0)} \in \ker g$. Furthermore, since $R = \varinjlim_n R_n$ (see 2.4) and P_1 is a finitely presented group, it suffices to prove that $g(x_1^{(0)}) = 1$ for all $g \in \mathrm{Hom}(P_1, R_n)$. Finally, notice that by the definition of P_1 the elements $x_1^{(1)}$, $x_2^{(1)}$ commute with their commutator $x_1^{(0)} = [x_1^{(1)}, x_2^{(1)}]$. These observations and the presentation of R_n show that Lemma 2.5 is a special case of

Lemma 2.6. *Let F_1, F_2 be two free groups, and let u_i be a word in F_i that is not a proper power. Let G be the quotient group of $F_1 * F_2$ by the normal subgroup generated by $[u_1, u_2]$. If $x, y \in G$ are elements commuting with $[x, y]$, then $[x, y] = 1$.*

Proof. Consider the map $h\colon G \to F_1 \oplus F_2$. Its kernel K is a free group whose set of generators can be described as follows. Let S_i be a set of representatives of cosets of $\langle u_i\rangle\backslash F_i$. Then the generators of K are all commutators $[v_1, v_2]$ where $v_1 \in S_1$ represents a coset other than $\langle u_1\rangle$ and v_2 is any nontrivial element of F_2, or $v_2 \in S_2$ represents a coset different from $\langle u_2\rangle$, and v_1 is a nontrivial element of F_1. To see this recall [6, Prop. 4, p. 6] that the kernel K' of the map $F_1 * F_2 \to F_1 \oplus F_2$ is a free group whose generators are all commutators $[v_1, v_2]$ where $v_i \in F_i$ and $v_i \neq 1$. The group K is obtained as the quotient of K' by its normal subgroup generated by the set $\{w^{-1}[u_1, u_2]w \mid w \in F_1 * F_2\}$. This is equivalent to imposing the following relations in K':

$$[u_1w_1, u_2w_2] = [u_1w_1, w_2][w_2, w_1][w_1, u_2w_2]$$

where w_i is an arbitrary element of F_i. The above description of K can be derived from here. Notice that, using the above relations, any commutator $[w_1, w_2]$ such that $w_i \in F_i$ can be expressed in terms of generators of K using the formula

$$[w_1, w_2] = [w_1, s_2][s_2, s_1][s_1, w_2] \tag{2.7}$$

where $s_i \in S_i$ represents the coset $\langle u_i\rangle w_i$ for $i = 1, 2$.

Next, take elements $x, y \in G$ as in the statement of the lemma. Notice that $[x, y] \in K$ since otherwise $h(x), h(y)$ would have to commute with a nontrivial element $h([x, y]) = [h(x), h(y)]$ in $F_1 \oplus F_2$, which is impossible. Furthermore, since centralizers of all nontrivial elements in a free group are abelian and since x, y are in the centralizer of $[x, y]$, if $x, y \in K$, then we get $xy = yx$ and the statement of the lemma holds. Therefore we can assume that $x \notin K$ and $[x, y] \in K$. In this case we can uniquely represent x and $[x, y]$ in the form

$$x = x_1x_2\prod_{i=1}^{p}[a_1, b_1]^{\delta_i} \quad\text{and}\quad [x, y] = \prod_{j=1}^{q}[c_j, d_j]^{\delta_j}$$

where $x_i \in F_i$, $x_1x_2 \neq 1$, $[a_i, b_i], [c_j, d_j]$ are generators of K, and $\delta_i, \delta_j = \pm 1$. We can also assume that $[x, y]$ is represented by a cyclically reduced word in the free group K, that is, $[c_1, d_1]^{\delta_1} \neq [c_q, d_q]^{-\delta_q}$. Consider the element

$$x_2^{-n}x_1^{-n}[x, y]x_1^nx_2^n = \prod_j([x_2^n, c_jx_1^n][c_jx_1^n, d_jx_2^n][d_jx_2^n, x_1^n][x_1^n, x_2^n])^{\delta_j}. \tag{2.8}$$

Commutativity of x and $[x,y]$ implies that for any $n \in \mathbb{Z}$ this element is conjugate to $[x,y]$ in K. We will show that this is not possible unless $[x,y]=1$. One can check that the following holds.

Lemma 2.9. *Let F be a free group, and let w, v be words in F. If w is cyclically reduced and v is conjugated in F to w, then all generators of F appearing in w must appear in v.*

We apply the lemma to $w=[x,y]$ and $v = x_2^{-n}x_1^{-n}[x,y]x_1^n x_2^n$. The commutators appearing on the right-hand side of formula 2.8 are not generators of the group K. Each of them, however, can be written as a product of generators of K using formula 2.7. By Lemma 2.9 all commutators $[c_j, d_j]$ must appear among these generators. One can check however that (since u_1, u_2 are not proper powers) if $x_1 \neq 1$, $x_2 \neq 1$, and n is large enough, this can happen only if $x_1 = c^{-1}u_1^k c$, $x_2 = d^{-1}u_2^l d$ and $[x,y]=[c,d]^m$ for some $c \in F_1$, $d \in F_2$, and $k,l,m \in \mathbb{Z}$. By inspection, in this case $x_1^{-n}x_2^{-n}[x,y]x_1^n x_2^n$ is not conjugated to $[x,y]$ unless $m=0$, and $[x,y]=1$. Assume in turn that e.g. $x_2 = 1$. Then we have

$$x_1^{-n}[x,y]x_1^n = \prod_{j=1}^{q}([c_j x_1^n, d_j][d_j, x_1^n])^{\delta_j}.$$

Again, combining this with formula 2.7 we get an expression of $x_1^{-n}[x,y]x_1^n$ as a product of generators of K. In order for $[c_j, d_j]$ to appear among these generators for large n we must have $x_1 = c^{-1}u_1^k c$ and $[x,y] = \prod_j [c, d_j]^{\delta_j}$ for some $c \in F_1$, $k \in \mathbb{Z}$. As before, by inspection we obtain that also in this case $x_1^{-n}[x,y]x_1^n$ cannot be conjugate to $[x,y]$ if $[x,y] \neq 1$. □

References

1. Carles Casacuberta. On structures preserved by idempotent transformations of groups and homotopy types. In *Crystallographic groups and their generalizations (Kortrijk,1999)*, volume 262 of *Contemp. Math.*, pages 39–68. Amer. Math. Soc., Providence, RI, 2000. MR1796125 (2001i:55012)
2. Assaf Libman. Cardinality and nilpotency of localizations of groups and G-modules. *Israel J. Math.*, 117:221–237, 2000. MR1760593 (2001c:20107)
3. Assaf Libman. A note on the localization of finite groups. *J. Pure Appl. Algebra*, 148(3):271–274, 2000. MR1758733 (2001c:20108)
4. Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory: Presentations of groups in terms of generators and relations.* Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966. MR0207802 (34:7617)
5. Niamh O'Sullivan. Localizations of free soluble groups. *J. Group Theory*, 4(1):89–96, 2001. MR1808841 (2001k:20070)
6. Jean-Pierre Serre. *Trees.* Springer-Verlag, Berlin, 1980. MR0607504 (82c:20083)
7. José L. Rodríguez, Jérôme Scherer and Antonio Viruel, Preservation of perfectness and acyclicity: Berrick and Casacuberta's universal acyclic space localized at a set of primes. *Forum Mathematicum*, to appear.

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: badzioch@math.umn.edu

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
E-mail address: feshbach@math.umn.edu