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CLOSED SETS WHICH ARE NOT CS^∞-CRITICAL

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Abstract. In this paper we first observe that the complement of a countable closed subset of an n-dimensional manifold M has large $(n-1)$-homology group. In the last section we use this information to prove that, under some topological conditions on the given manifold, certain families of fibers, in the total space of a fibration over M, are not critical sets for some special real or S^1-valued functions.

1. Introduction

Let M, N be differentiable manifolds, $\mathcal{F} \subseteq C^\infty(M,N)$ be a family of smooth mappings and $f : M \to N$ be a differentiable mapping. Denote by $C(f)$ its critical set and recall that $C(f)$ is closed.

A closed subset of M is called *$\mathcal{F}$-critical* if $C = C(f)$ for some differentiable mappling $f \in \mathcal{F}$. A $C^\infty(M,N)$-critical set will be called *N-critical* and an $\mathbf{R}$-critical set will be simply called *critical*. Given a closed subset C of M, the question is: *is it $\mathcal{F}$-critical?* This is a fundamental problem which has been treated in [5] and [6] for $\mathcal{F} = C^\infty(M,\mathbf{R})$ and its subfamily of smooth proper functions. For instance the *Antoine's Necklace* of $\mathbf{R}^3$ is a properly critical set [5] while the circle $S^1 \subseteq \mathbf{R}^2$ is not a critical set [6]. On the other hand the finite subsets of M having cardinality strictly smaller then $cat(M)$-the Lusternick-Schnirelmann category of M, are not critical if M is compact, because it is well known that any function $f : M \to \mathbf{R}$ has at least $cat(M)$ critical points. Finally, when M^m is immersible into $\mathbf{R}^{m+1}$, the S^m-non-criticality of certain subsets of M provides immediate information on the set of zeros of the Gauss-Kronecker curvature associated to an arbitrary immersion of M into $\mathbf{R}^{m+1}$, taking into account that the mentioned set of zeros is actually the critical set of the associated Gauss mapping of the given immersion. For instance, the product $S^k \times S^n$, $k+n \geq 3$ has, for any immersion $f : S^k \times S^n \to \mathbf{R}^{k+n+1}$, infinitely many points of zero Gauss-Kronecker curvature, simply because the finite subsets of $S^k \times S^n$ are not S^{k+n}-critical [7].

Let us consider the family $CS^\infty(M,N) := \{f \in C^\infty(M,N) \mid B(f) \cap f(R(f)) = \emptyset\}$, where $R(f)$ is the set of regular points of f while $B(f) = f(C(f))$ is the set of

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its critical values. We say that a mapping from $CS^\infty(M,N)$ *separates the critical values by the regular ones.*

Remark 1.1. Let $p:\tilde{N}\to N$ be a covering mapping and $\tilde{f}:M\to\tilde{N}$ be a differentiable mapping. If $\tilde{f}\notin CS^\infty(M,\tilde{N})$, then $p\circ\tilde{f}\notin CS^\infty(M,N)$. Therefore if $g\in CS^\infty(M,N)$ and $\tilde{g}\in C^\infty(M,\tilde{N})$ is a lifting of g, then $\tilde{g}\in CS^\infty(M,\tilde{N})$.

Indeed, since $\tilde{f}\notin CS^\infty(M,\tilde{N})$, it follows that there exist $x_0\in C(f)$, $x_1\in R(f)$ such that $\tilde{f}(x_0)=\tilde{f}(x_1)$. Because p is a local diffeomorphism, this implies that $x_0\in C(p\circ\tilde{f})$, $x_1\in R(p\circ\tilde{f})$ and obviously $(p\circ\tilde{f})(x_0)=(p\circ\tilde{f})(x_1)\in B(p\circ\tilde{f})\cap(p\circ\tilde{f})(R(p\circ\tilde{f}))$, that is, $p\circ\tilde{f}\notin CS^\infty(M,N)$.

Proposition 1.2. (*i*) *$f\in CS^\infty(M,N)$ iff $C(f)=f^{-1}(B(f))$.*

(*ii*) *If M is a connected differentiable manifold and $f\in CS^\infty(M,\mathbf{R})$ is such that $R(f)=M\backslash C(f)$ is also connected, then $f(R(f))=(m_f,M_f)$, where $m_f=\inf\limits_{x\in M}f(x), M_f=\sup\limits_{x\in M}f(x)$ and $B(f)\subseteq\{m_f,M_f\}\cap\mathbf{R}$. Moreover, if M is compact, then $m_f,M_f\in\mathbf{R}$ and $B(f)=\{m_f,M_f\}$.*

Proof. (*i*) Indeed if $f\in CS^\infty(M,N)$ we only have to show that $f^{-1}(B(f))\subseteq C(f)$, because the other inclusion is always true. Hence if $p\in f^{-1}(B(f))$, it follows that $f(p)\in B(f)=f(C(f))\subseteq N\backslash f(R(f))$, that is, $f(p)\notin f(R(f))$, meaning that $p\notin R(f)$.

Conversely, if $C(f)=f^{-1}(B(f))$, assume that $B(f)\cap f(R(f))\neq\emptyset$ and consider $q\in f(C(f))\cap f(R(f))$. This means that there exists $p_1\in C(f),p_2\in R(f)$ such that $f(p_1)=f(p_2)=q$, that is $p_2\in f^{-1}(q)\subseteq f^{-1}(B(f))=C(f)$, which is of course a contradiction with the obvious fact that $R(f)\cap C(f)=\emptyset$.

(*ii*) If f is non-constant, then obviously $f(R(f))\subseteq(m_f,M_f)$. On the other hand, we can choose two regular values q_1,q_2 in $Imf\subseteq[m_f,M_f]$ arbitrarily close to m_f and M_f respectively. We can also take two regular points $p_1\in f^{-1}(q_1),p_2\in f^{-1}(q_2)$ and connect p_1,p_2 by a differential path in $R(f)$. Because f is separating critical points by the regular ones, this path is applied by f on a segment of regular values in Imf connecting q_1 with q_2. Hence the interval $[q_1,q_2]$ is completely contained in $Imf\backslash B(f)$. Consequently we have shown in this way that $(m_f,M_f)\subseteq Imf\backslash B(f)=f(R(f))$ and that $B(f)\subseteq\{m_f,M_f\}\cap\mathbf{R}$. □

In this paper we are going to prove that the collection of fibers in the total space of certain fibrations $p:E\to M$, over closed countable subsets of the base space, is neither $CS^\infty(E,\mathbf{R})$-critical nor $CS^\infty(E,S^1)$-critical under certain topological conditions on the total and base spaces and on the fiber of the considered fibrations.

Let us observe that any closed countable subset A of a manifold has countably many isolated points. Indeed otherwise the subset $I\subseteq A$ of isolated points would be finite, possible empty, and $A\backslash I$ would be a countable perfect subset of the given manifold. But it is folklore that perfect subsets of complete metric spaces are not countable. Therefore such a subset can be represented as $A=I\cup A'=\{a_1,a_2,\ldots\}\cup A'$, where A' is the derived set of A, that is, the set of accumulation points.

We close this section by recalling a previously proved theorem, which involves closed countable sets. It has been proved in [7] for some particular closed countable sets, the arguments for arbitrary ones being given in [8].

Theorem 1.3. *Let M be an n-dimensional differentiable manifold ($\partial M = \emptyset$) and A be a closed countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $f : P \to M$ is a continuous map such that $f(\partial P) \subseteq M\backslash A$, then there exists a continuous map $g : P \to M$ such that $g(P) \subseteq M\backslash A$, $g|_{\partial P} = f|_{\partial P}$ and $f \simeq g(rel\, \partial P)$. If M is connected, then one particularly gets, using the particular case $P = [0,1]$, that $M\backslash A$ is also connected.*

2. Basic results

We start this section by proving that the complement of a closed countable subset of a given n-dimensional manifold has large $n-1$ homology group. The manifold M will be with empty boundary all along the paper.

Proposition 2.1. *Let M be an n-dimensional differential manifold, $n \geq 2$, and $A = I \cup A'$ be a closed countable subset of M, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of A and A' is its derived set. If $H_{n-1}(M) \simeq 0$, then for each $k \geq 1$ there exists a surjective group homomorphism*

$$\delta_k : H_{n-1}(M\backslash A) \to \mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k),$$

where $A_k = \{a_{k+1}, a_{k+2} \ldots\} \cup A'$. Moreover, if M is either not compact or compact but not orientable, then $H_{n-1}(M\backslash A) \simeq \mathbf{Z}^k \oplus H_{n-1}(M\backslash A_k)$, for each $k \geq 1$, that is, $H_{n-1}(M\backslash A)$ has free abelian subgroups of arbitrarily large rank.

Proof. We first recall that $M\backslash A$ is connected, because of Theorem 1.3. To prove the stated isomorphism, we will use the Mayer-Vietoris sequence for the following two spaces:

$$X_k = M\backslash\{a_1, a_2, \ldots, a_k\},\ Y_k = M\backslash A_k.$$

Taking into account the fact that $X \cap Y = M\backslash A$ and $X_k \cup Y_k = M$ for each $k \geq 1$, we get the following exact sequence:

$$\to H_n(M) \to H_{n-1}(M\backslash A) \xrightarrow{\Delta_k} H_{n-1}(X_k) \oplus H_{n-1}(Y_k) \to H_{n-1}(M) \to, \tag{1}$$

where $\Delta_k(\bar{z}) = \big(H_k(i_k)(\bar{z}), H_k(j_k)(\bar{z})\big)$ and $i_k : M\backslash A \hookrightarrow X_k$, $j_k : M\backslash A \hookrightarrow Y_k$ are the inclusions. Because of the exactness of the sequence (1) we can conclude that for each $k \geq 1$ the group homomorphism Δ_k is surjective because $H_{n-1}(M) \simeq 0$. Using the homology sequence of the pair (M, X_k) we have

$$\to H_n(X_k) \to H_n(M) \to H_n(M, X_k) \to H_{n-1}(X_k) \to H_{n-1}(M) \to . \tag{2}$$

On the other hand, excising a suitable open subset of X_k, one can get an isomorphism $H_n(M, X_k) \simeq \mathbf{Z}^k$. Therefore, when M is either not compact or compact and not orientable, we get that $H_{n-1}(X_k) \simeq \mathbf{Z}^k$, $H_n(M)$ being trivial in both cases [4, p. 166, 167].

If M is compact orientable, the exact sequence (2) provide us, up to some group isomorphisms, the following short exact sequence:

$$0 \to \mathbf{Z} \to \mathbf{Z}^k \to H_{n-1}(X_k) \to 0,$$

because $H_n(X_k) \simeq 0$, taking into account that X_k is a connected non-compact n-dimensional manifold. It ensures us that, when M is compact orientable, there exists $d \geq 1$ such that $H_{n-1}(X_k) \simeq \mathbf{Z}/d\mathbf{Z} \oplus \mathbf{Z}^{k-1}$.

In any case there exists a surjective group homomorphism

$$\psi_k : H_{n-1}(X_k) \to \mathbf{Z}^{k-1},$$

which is defined by composing the isomorphism $H_{n-1}(X_k) \simeq \mathbf{Z}/d\mathbf{Z} \oplus \mathbf{Z}^{k-1}$ or $H_{n-1}(X_k) \simeq \mathbf{Z}^k$ with a suitable projection that simply forgets the first term of the direct sum. Consequently the group homomorphism

$$\delta_k : H_{n-1}(M\backslash A) \to \mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k),\ \delta_k := (\psi_k \times id_{H_{n-1}(Y_k)}) \circ \Delta_k$$

is obviously surjective, and the proof is now complete. □

Corollary 2.2. *Let M be an n-dimensional differential manifold, $n \geq 2$, and $A = I \cup A'$ be a closed countable subset of M, where $I = \{a_1, a_2, \ldots\}$ is the set of isolated points of A and A' is its derived set. If G is an abelian group and $\varphi : G \to H_{n-1}(M\backslash A)$ is a surjective group homomorphism, then G is not finitely generated. In particular, $H_{n-1}(M\backslash A)$ is not finitely generated.*

Proof. Assume that G is a finitely generated group and $\varphi : G \to H_{n-1}(M\backslash A)$ is a surjective group homomorphism. It follows that

$$\delta_k \circ \varphi : G \to \mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k)$$

is also surjective for each $k \geq 1$. But since $\delta_k \circ \varphi$ maps the torsion part $t(G)$ of G into the torsion part $t\big(\mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k)\big) = \mathbf{Z}^{k-1} \oplus t\big(H_{n-1}(M\backslash A_k)\big)$ of $\mathbf{Z}^k \oplus H_{n-1}(M\backslash A_k)$, it follows that

$$\frac{G}{t(G)} \to \mathbf{Z}^{k-1} \oplus \frac{H_{n-1}(M\backslash A_k)}{t\big(\mathbf{Z}^{k-1} \oplus H_{n-1}(M\backslash A_k)\big)},\ g + t(G) \mapsto (\delta_k \circ \varphi)(g) + t(G)$$

is a well-defined group homomorphism denoted by γ_k. But since G is finitely generated, there exists a unique natural number q such that $G \simeq \mathbf{Z}^q \oplus t(G)$ and obviously $\frac{G}{t(G)} \simeq \mathbf{Z}^q$. Therefore, up to some group isomorphisms, γ_{q+2} acts surjectively from $\mathbf{Z}^q$ to $\mathbf{Z}^{q+1} \oplus \frac{H_{n-1}(M\backslash A_{q+2})}{t(H_{n-1}(M\backslash A))}$. Because the projection

$$pr_1 : \mathbf{Z}^{q+1} \oplus \frac{H_{n-1}(M\backslash A_{q+2})}{t(H_{n-1}(M\backslash A))} \to \mathbf{Z}^{q+1}$$

is surjective, it follows that $pr_1 \circ \gamma_{q+2} : \mathbf{Z}^q \to \mathbf{Z}^{q+1}$ is also surjective, which is impossible because such a group homomorphism does not exist. □

Let $p : E \to M$ be a fibration whose base space M is an n-dimensional manifold, and let $A \subseteq M$ be a closed countable subset. In order to prove that the natural group homomorphism $h_{n-1} : \pi_{n-1}(E\backslash p^{-1}(A)) \to H_{n-1}(E\backslash p^{-1}(A))$ is an isomorphism, we will show that each homotopy class

$$[F] \in [(P, \partial P), (E, E\backslash p^{-1}(A))]$$

contains a mapping whose image avoids the subset $p^{-1}(A)$, where P is a compact connected differentiable manifold such that $\dim P < \dim N$ and $\partial P \neq \emptyset$.

Theorem 2.3. *Let $p : E \to M$ be a fibration whose base space M is an n-dimensional differentiable manifold, and let A be a closed countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $F : P \to E$ is a continuous map such that $F(\partial P) \subseteq E\backslash p^{-1}(A)$, then there exists a continuous map $G : P \to E$ such that $G(P) \subseteq E\backslash p^{-1}(A)$, $G|_{\partial P} = F|_{\partial P}$ and $F \simeq G(rel\, \partial P)$. If E is connected, then one particularly gets, using the particular case $P = [0, 1]$, that $E\backslash p^{-1}(A)$ is also connected.*

Proof. Applying Theorem 1.3, there exists a homotopy $H : P \times [0,1] \to M$ of $p \circ F$ such that $ImH(\cdot,1) \subseteq M\backslash A$. Because $p : E \to M$ is a fibration, there exists a homotopy H' of F that covers the homotopy H, namely $H = p \circ H'$. It is easy to check that $ImH'(\cdot,1) \subseteq E\backslash p^{-1}(A)$.

Consider the homotopies $\psi : P \times [0,1] \to P$ and $\varphi : P \times [0,1] \to E$:

$$\psi(x,t) = \begin{cases} x & \text{if } x \in P\backslash Q(\partial P \times [0,2)), \\ Q((\pi_1 \circ Q^{-1})(x), \frac{2}{2-t}(\pi_2 \circ Q^{-1})(x) + \frac{2t}{t-2}) & \text{if } x \in Q(\partial P \times [t,2]), \\ (\pi_1 \circ Q^{-1})(x) & \text{if } x \in Q(\partial P \times [0,t]), \end{cases}$$

$$\varphi(x,t) = \begin{cases} H'(\psi(x,t),t) & \text{if } x \in P\backslash Q(\partial P \times [0,t)), \\ H'(Q^{-1}(x)) & \text{if } x \in Q(\partial P \times [0,t]), \end{cases}$$

where $Q : \partial P \times [0,\infty) \to U \subset P$ is a collar neighbourhood of ∂P and $\pi_1 : \partial P \times [0,\infty) \to \partial P$, $\pi_2 : \partial P \times [0,\infty) \to [0,\infty)$ are obviously the projections.

Denoting $\varphi(\cdot,1)$ by G and observing that $\varphi(\cdot,0) = F$, one can easily see that $F \simeq_\varphi G(\text{ rel } \partial P)$ and also that $G(P) \subseteq E\backslash p^{-1}(A)$, the theorem being now completely proved. □

Corollary 2.4. *If $p : E \to M$ is a fibration whose base space M is an n-dimensional differentiable manifold and A is a closed countable subset of M, then the pair $(E, E\backslash p^{-1}(A))$ is $(n-1)$-connected, that is, $\pi_q(E, E\backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1,\ldots, n-1\}$. In particular we get that $H_q(E, E\backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1,\ldots,n-1\}$ and the natural group homomorphism*

$$\chi_n : \pi_n(E, E\backslash p^{-1}(A)) \to H_n(E, E\backslash p^{-1}(A))$$

is surjective. On the other hand the inclusion $i_{E\backslash p^{-1}(A)} : E\backslash p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected, that is, the induced group homomorphism

$$\pi_q(i_{E\backslash p^{-1}(A)}) : \pi_q(E\backslash p^{-1}(A)) \to \pi_q(E)$$

is an isomorphism for $q \le n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism χ_n is an isomorphism if E is simply connected and $n \ge 3$.

Proof. The fact that $\pi_q(E, E\backslash p^{-1}(A)) = 0$ for all $q \in \{1,2,\ldots,n-1\}$ is an immediate consequence of Theorem 2.3 and of the fact that $[\alpha] \in \pi_q(E, E\backslash p^{-1}(A))$ is zero if and only if there exists $\beta \in [\alpha]$ such that $\beta(D^q) \subseteq E\backslash p^{-1}(A)$. From the Hurewicz theorem [9, pp. 394-398] one can deduce that $H_q(E, E\backslash p^{-1}(A)) \simeq 0$ for all $q \in \{1,\ldots,n-1\}$ as well as that $\chi_n : \pi_n(E, E\backslash p^{-1}(A)) \to H_n(E, E\backslash p^{-1}(A))$ is surjective. Further on, using the exact homotopy sequence

$$\cdots \to \pi_{r+1}(E, E\backslash p^{-1}(A)) \to \pi_r(E\backslash p^{-1}(A)) \to \pi_r(E) \to \pi_r(E, E\backslash p^{-1}(A)) \to \cdots,$$

and the triviality of $\pi_q(E, E\backslash p^{-1}(A))$ for $q \in \{1,2,\ldots,n-1\}$, it follows that the inclusion $i_{E\backslash p^{-1}(A)} : E\backslash p^{-1}(A) \hookrightarrow E$ is $(n-1)$-connected. Finally, since E is simply connected, it follows that $E\backslash p^{-1}(A)$ is also simply connected such that $\pi_1(E\backslash p^{-1}(A))$ acts trivially on $\pi_n(E, E\backslash p^{-1}(A))$, which means that χ_n is an isomorphism [3, p. 166]. □

Corollary 2.5. *Let $p : E \to M$ be a fibration whose base space M is an n-dimensional differentiable manifold, and let A be a closed countable subset of M.*

If the total space E is simply connected and the natural group homomorphisms $h_q^E : \pi_q(E) \to H_q(E),\ q \in \{n-1, n\}$ are isomorphisms, then the natural group homomorphism

$$h_{n-1} : \pi_{n-1}(E\backslash p^{-1}(A)) \to H_{n-1}(E\backslash p^{-1}(A))$$

is also an isomorphism.

Proof. Consider the following ladder with exact rows and commutative rectangles:

$$\begin{array}{ccccccccc} \pi_n(E) & \to & \pi_n(E, E\backslash p^{-1}(A)) & \to & \pi_{n-1}(E\backslash p^{-1}(A)) & \to & \pi_{n-1}(E) & \to & \pi_{n-1}(E, E\backslash p^{-1}(A)) \\ \downarrow h_n^E & & \downarrow \chi_n & & \downarrow h_{n-1} & & \downarrow h_{n-1}^E & & \downarrow \chi_{n-1} \\ H_n(E) & \to & H_n(E, E\backslash p^{-1}(A)) & \to & H_{n-1}(E\backslash p^{-1}(A)) & \to & H_{n-1}(E) & \to & H_{n-1}(E, E\backslash p^{-1}(A)) \end{array}$$

and conclude, using the hypothesis, Corollary 2.4 and the five lemma, that h_{n-1} is indeed an isomorphism. □

Remark 2.6. (i) If M is an n-dimensional differentiable manifold and A is a closed countable subset of M, then the pair $(M, M\backslash A)$ is $(n-1)$-connected, that is, $\pi_q(M, M\backslash A) \simeq 0$ for all $q \in \{1, \dots, n-1\}$.

In particular we get that $H_q(M, M\backslash A) \simeq 0$ for all $q \in \{1, \dots, n-1\}$ and the natural group homomorphism $\chi_n : \pi_n(M, M\backslash A) \to H_n(M, M\backslash A)$ is surjective. On the other hand the inclusion $i_{M\backslash A} : M\backslash A \hookrightarrow M$ is $(n-1)$-connected, that is, the induced group homomorphism $\pi_q(i_{M\backslash A}) : \pi_q(M\backslash A) \to \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism χ_n is an isomorphism if M is simply connected and $n \geq 3$.

(ii) Let M be an n-dimensional differentiable manifold, and let A be a closed countable subset of M. If the natural group homomorphism $h_{n-1}^M : \pi_{n-1}(M) \to H_{n-1}(M)$ is surjective, then the natural group homomorphism

$$h_{n-1}^{M\backslash A} : \pi_{n-1}(M\backslash A) \to H_{n-1}(M\backslash A)$$

is also surjective. Moreover, if h_{n-1}^M, h_n^M are isomorphisms, then $h_{n-1}^{M\backslash A}$ is an isomorphism too.

(iii) If M is an n-dimensional differentiable manifold and A is a closed countable subset of M, then $\pi_{n-1}(M\backslash A)$ is not finitely generated.

(iv) Let M be an n-dimensional differential contractible manifold, $n \geq 2$, and let $A = I \cup A'$ be a closed countable subset of M, where $I = \{a_1, a_2, \dots\}$ is the set of isolated points of A and A' is its derived set. Then $M\backslash A$ is $(n-2)$-connected and $\pi_{n-1}(M\backslash A) \cong \mathbf{Z}^i \oplus \pi_{n-1}(M\backslash A_i)$, where $A_i = \{a_{i+1}, a_{i+2} \dots\} \cup A'$. In particular $\pi_{n-1}(M\backslash A)$ has free abelian subgroups of arbitrarily large rank.

Indeed the first points (i) and (ii) are immediate consequences of Corollaries 2.4 and 2.5 respectively by considering the particular fibration $id_M : M \to M$, while the third point (iii) follows easily by combining Corollary 2.2 and Remark 2.6 (ii). At the fourth point (iv) the $(n-2)$-connectedness follows easily from Remark 2.6 (i) while the isomorphism $\pi_{n-1}(M\backslash A) \cong \mathbf{Z}^i \oplus \pi_{n-1}(M\backslash A_i)$ follows from Proposition 2.1 by using the Hurewitz theorem and Remark 2.6 (ii).

3. Application

In this section we will give the topological conditions on the total and base spaces and on the fiber of a fibration in order that the collection of fibers over a closed countable subset of the base space not be CS^∞-critical.

Theorem 3.1. *Let $F \hookrightarrow E \xrightarrow{p} M^n$ be a differential fibration with compact total space, and let A be a closed countable subset of M.*

(i) If $n \geq 2$, M is a homotopy sphere and $H_1(F) \simeq 0$, then $p^{-1}(A)$ is not $CS^\infty(E, \mathbf{R})$-critical. Moreover, if E is simply connected, then $p^{-1}(A)$ is not $CS^\infty(E, S^1)$-critical too.

(ii) If $n \geq 3$, E is simply connected, $H_{n-1}(M) \simeq 0$, $\pi_{n-2}(F)$ is finitely generated and commutative when $n = 3$ and the natural group homomorphisms h^E_{n-1}, h^E_n are isomorphisms, then $p^{-1}(A)$ is neither $CS^\infty(E, \mathbf{R})$-critical nor $CS^\infty(E, S^1)$-critical.

Proof. Assume that there exists a mapping $f \in CS^\infty(E, \mathbf{R})$ such that $C(f) = p^{-1}(A)$. This means that $B(f) = \{m_f, M_f\}$ and that its restriction

$$E \setminus C(f) \to Imf \setminus B(f) = (m_f, M_f),\ p \mapsto f(p)$$

is a proper submersion, that is, via Ehresmann's theorem, a locally trivial fibration whose compact fiber we are denoting by $\mathcal{F}$. Its base space (m_f, M_f) being contractible, it follows that the inclusion $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow E \setminus C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms

$$\pi_q(i_{\mathcal{F}}) : \pi_q(\mathcal{F}) \to \pi_q(E \setminus C(f))$$

are all isomorphisms. Consequently, using the Whitehead theorem [3, p. 167] or [9, p. 399], it follows that the induced group homomorphisms

$$H_q(i_{\mathcal{F}}) : H_q(\mathcal{F}) \to H_q(E \setminus C(f)) = H_q(E \setminus p^{-1}(A))$$

are also isomorphisms.

(i) The Serre homology sequence of the fibration $F \hookrightarrow E \setminus p^{-1}(A) \xrightarrow{p} M \setminus A$ is:

$$\cdots \to H_{n-1}(F) \to H_{n-1}\big(E \setminus p^{-1}(A)\big) \longrightarrow H_{n-1}(M \setminus A) \to H_{n-2}(F) \to \cdots .$$

Because $H_{n-1}(\mathcal{F}) \simeq H_{n-1}(E \setminus p^{-1}(A))$ and $H_{n-1}(\mathcal{F})$, $H_{n-2}(F)$, are finitely generated, it follows that $H_{n-1}(M \setminus A)$ must be finitely generated, which is a contradiction with Corollary 2.2. In order to prove that $p^{-1}(A)$ is not $CS^\infty(E, S^1)$-critical, under the additional hypothesis of simply connectedness of E, we assume that there exists $g \in CS^\infty(E, S^1)$ such that $C(g) = p^{-1}(A)$ and consider a lifting $\tilde{g} : E \to \mathbf{R}$ that does not belong to $CS^\infty(E, \mathbf{R})$ since $C(\tilde{g}) = C(g) = p^{-1}(A)$. Using Remark 1.1 one can deduce that $g = exp \circ \tilde{g} \notin CS^\infty(E, S^1)$.

(ii) Because $H_{n-1}(M) \simeq 0$, this implies that h^M_{n-1} is obviously surjective. According to Corollary 2.5 and Remark 2.6 (ii), the natural group homomorphism

$$h_{n-1} : \pi_{n-1}(E \setminus p^{-1}(A)) \to H_{n-1}(E \setminus p^{-1}(A))$$

is an isomorphism, while $h^{M \setminus A}_{n-1} : \pi_{n-1}(M \setminus A) \to H_{n-1}(M \setminus A)$ is surjective. Therefore, $h^{-1}_{n-1} \circ H_{n-1}(\mathcal{F}) : H_{n-1}(\mathcal{F}) \to \pi_{n-1}(E \setminus p^{-1}(A))$ is an isomorphism.

The homotopy sequence of the fibration $F \hookrightarrow E \setminus p^{-1}(A) \xrightarrow{p} M \setminus A$ is

$$\cdots \to \pi_{n-1}(F) \to \pi_{n-1}\big(E \setminus p^{-1}(A)\big) \longrightarrow \pi_{n-1}(M \setminus A) \to \pi_{n-2}(F) \to \cdots ,$$

and it forces $\pi_{n-1}(M \setminus A)$ to be finitely generated since $H_{n-1}(\mathcal{F}) \simeq \pi_{n-1}\big(E \setminus p^{-1}(A)\big)$ and $\pi_{n-2}(F)$, $H_{n-1}(\mathcal{F})$ are finitely generated. But this is impossible because Remark 2.6 (iii) ensures us that $\pi_{n-1}(M \setminus A)$ is not finitely generated. The $CS^\infty(E, S^1)$-non-criticality of $p^{-1}(A)$ can be proved similiarly. □

Corollary 3.2. *If* $F \hookrightarrow E \xrightarrow{p} M$ *is one of the following fibrations:*
(*i*) $Spin(n) \hookrightarrow Spin(n+1) \to S^n$, $n \geq 3$,
(*ii*) $SU(n) \hookrightarrow SU(n+1) \to S^{2n+1}$, $n \geq 2$,
(*iii*) $Sp(n) \hookrightarrow Sp(n+1) \to S^{4n+3}$, $n \geq 1$,
(*iv*) $S^3 \hookrightarrow S^{4n+3} \to P\mathbf{H}^n$, $n \geq 1$,
(*v*) $S^1 \hookrightarrow S^{2n+1} \to P\mathbf{C}^n$, $n \geq 2$,
and if A *is a closed countable subset of the base space, then* $p^{-1}(A)$ *is neither* $CS^\infty(E, \mathbf{R})$*-critical nor* $CS^\infty(E, S^1)$*-critical.*

Proof. Indeed, (i), (ii) and (iii) follow immediately from Theorem 3.1 (i) taking into account that S^k is $(k-1)$-connected, $Spin(n)$, $U(n)$, $Sp(n)$ are compact and $H_i(Spin(n))$, $H_i(SU(n))$, $H_i(Sp(n))$, $i \in \{1, 2\}$ are trivial because $\pi_i(Spin(n))$, $\pi_i(SU(n))$, $\pi_i(Sp(n))$, $i \in \{1, 2\}$ are trivial [1, p. 368], [2, p. 224].

(iv) and (v) follow from Theorem 3.1 (ii) since $H_{4n-1}(P\mathbf{H}^n) \simeq 0 \simeq H_{2n-1}(P\mathbf{C}^n)$ while $\pi_{2n-2}(S^1)$, $\pi_{4n-2}(S^3)$ are finitely generated, $\pi_{2n-2}(S^1)$ being actually trivial for $n \geq 2$, while $\pi_{4n-2}(S^3)$ is even finite [3, p. 318]. Finally the natural group homomorphisms $h_{4n-1}^{S^{4n+3}}$, $h_{4n}^{S^{4n+3}}$, $h_{2n-1}^{S^{2n+1}}$, $h_{2n}^{S^{2n+1}}$ are obviously isomorphisms, because $\pi_q(S^{4n+3})$, $H_q(S^{4n+3})$, $q \in \{4n-1, 4n\}$ and $\pi_r(S^{2n+1})$, $H_r(S^{2n+1})$, $r \in \{2n-1, 2n\}$ are trivial. □

Remark 3.3. The quaternionic Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$ can also be treated by means of Theorem 3.1 (i).

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