

FRAMES, MODULAR FUNCTIONS FOR SHIFT-INVARIANT SUBSPACES AND FMRA WAVELET FRAMES

QING GU AND DEGUANG HAN

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ABSTRACT. We introduce the concept of the modular function for a shift-invariant subspace that can be represented by normalized tight frame generators for the shift-invariant subspace and prove that it is independent of the selections of the frame generators for the subspace. We shall apply it to study the connections between the dimension functions of wavelet frames for any expansive integer matrix A and the multiplicity functions for general multiresolution analysis (GMRA). Given a frame multiresolution analysis (FMRA), we show that the standard construction formula for orthonormal multiresolution analysis wavelets does not yield wavelet frames unless the underlying FMRA is an MRA. A modified explicit construction formula for FMRA wavelet frames is given in terms of the frame scaling functions and the low-pass filters.

1. PRELIMINARIES

A *frame* for a separable Hilbert space H is a sequence of vectors $\{f_j\}$ in H such that there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|f\|_2^2 \leq \sum_j |\langle f, f_j \rangle|^2 \leq C_2 \|f\|_2^2$$

holds for all $f \in H$. If $C_1 = C_2 = 1$, we say that $\{f_j\}$ is a *normalized tight frame*.

For a $d \times d$ real expansive matrix A (i.e., all the eigenvalues of A are required to have absolute values greater than 1), an *A-dilation (orthogonal) wavelet* is a single function $\psi \in L^2(\mathbb{R}^d)$ with the property that

$$\psi_{m,\ell}^A(t) := \{ |\det A|^{\frac{m}{2}} \psi(A^m t - \ell) : m \in \mathbb{Z}, \ell \in \mathbb{Z}^d \}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$. More generally, ψ will be called a *normalized tight A-dilation wavelet frame* if $\{\psi_{m,\ell}^A\}$ forms a normalized tight frame for $L^2(\mathbb{R}^d)$. In what follows we will use the term “wavelet frame” to denote the normalized tight ones.

The dilation operator δ_A and the translation operator T_ℓ are defined by:

$$(\delta_A f)(t) = |\det A|^{1/2} f(At), \quad (T_\ell f)(t) = f(t - \ell)$$

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for all $f \in L^2(\mathbb{R}^d)$. A closed subspace M of $L^2(\mathbb{R}^d)$ is called *shift-invariant* if $T_\ell M = M$ for any $\ell \in \mathbb{Z}^d$. The Fourier transform of $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined by

$$(\mathcal{F}f)(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle \xi, t \rangle} f(t) d\mu,$$

where μ is the Lebesgue measure on \mathbb{R}^d . Here we include a factor $\frac{1}{(2\pi)^{d/2}}$ in the definition of Fourier transform to guarantee that \mathcal{F} preserves the inner product. Therefore, \mathcal{F} can be extended to a unitary operator, still denoted by \mathcal{F} , on $L^2(\mathbb{R}^d)$. We will write $\hat{f} := \mathcal{F}(f)$ and $\hat{U} = \mathcal{F}U\mathcal{F}^{-1}$ for any $f \in L^2(\mathbb{R}^d)$ and any bounded operator U on $L^2(\mathbb{R}^d)$.

A *general multiresolution analysis* (cf. [Ba], [BMM], [BL], [Pa]) associated with a $d \times d$ expansive real matrix A is a sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying

- (1) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$,
- (2) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$,
- (3) $f \in V_j$ if and only if $\delta_A f \in V_{j+1}$, $j \in \mathbb{Z}$,
- (4) V_0 is invariant under the translation group $\{T_\ell : \ell \in \mathbb{Z}^d\}$.

If we require that V_0 contains a *scaling function* whose translates form an orthonormal basis for V_0 , then (V_j) is called a *multiresolution analysis* (MRA). Similarly, (V_j) is called a *frame multiresolution analysis* (FMRA) if V_0 contains a *frame scaling function* ϕ whose translates form a normalized tight frame for V_0 . Given a scaling function for an MRA, there is a standard way of constructing a wavelet in $V_1 \ominus V_0$ (which is called an MRA wavelet, [Ma]). However, the construction of FMRA wavelet frames is a more complicated issue. The purpose of this paper is twofold: (1) Give an explicit construction formula (Theorem 3.4) for FMRA wavelet frames, and (2) introduce the concept of the modular function for a shift-invariant subspace that can be regarded as a refinement of the multiplicity function (Theorem 2.2) and use it to explore the connections between the dimension function of a GMRA and the multiplicity function of the scaling space V_0 .

The following theorem and its corollary follow from the *consistency equation* of GMRA in [BMM] and will be used later.

Theorem 1.1. *Let (V_j) be a GMRA with a multi-frame scaling set $\{\phi_\lambda : \lambda \in \Lambda\}$, and let $A\mathbb{Z}^d, A\mathbb{Z}^d + \ell_1, \dots, A\mathbb{Z}^d + \ell_k$ be a basis of the group $\mathbb{Z}^d/A\mathbb{Z}^d$ (let $\ell_0 = 0$). Then there exists a wavelet frame $\psi \in W_0 = V_1 \ominus V_0$ if and only if*

$$\sum_{\lambda \in \Lambda} \sum_{i=0}^k \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\lambda((A^t)^{-1}(\xi + 2\ell_i\pi) + 2\ell\pi)|^2 = \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\lambda(\xi + 2\ell\pi)|^2 + \frac{1}{(2\pi)^d} \chi_E$$

for some measurable set E . In particular, if (V_j) is an FMRA with a frame scaling function ϕ , then there is a (orthogonal) wavelet in W_0 if and only if

$$(1.1) \quad \sum_{i=0}^k \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}((A^t)^{-1}(\xi + 2\ell_i\pi) + 2\ell\pi)|^2 = \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}(\xi + 2\ell\pi)|^2 + \frac{1}{(2\pi)^d},$$

where A^t is the transpose of A .

The set $\{\ell_i\}$ in Theorem 1.1 is called a *complete digit set* for A . For any function $f \in L^2(\mathbb{R}^d)$, we define $\Phi_f(\xi) = \sum_{\ell \in \mathbb{Z}^d} |\hat{f}(\xi + 2\ell\pi)|^2$. Clearly $\text{supp}(\Phi_f) = \bigcup_{\ell \in \mathbb{Z}^d} (\text{supp}(f) + 2\ell\pi)$.

Corollary 1.2. *Let $\{\ell_0, \ell_1, \dots, \ell_k\}$ ($\ell_0 = 0$) be a complete digit set for A . Assume that (V_j) is an FMRA such that $V_0 = \overline{\text{span}}\{T_\ell f : \ell \in \mathbb{Z}^d\}$. Let $F = \text{supp}(\Phi_f)$ and $F_i = A^i F - 2\ell_i \pi$. Then W_0 is generated by a wavelet frame if and only if the following are satisfied:*

- (i) $F \subseteq \bigcup_{i=0}^k F_i$,
- (ii) $F_i \cap F_j \subseteq F$ for $i \neq j$,
- (iii) $F_i \cap F_j \cap F_m$ has measure zero for all different i, j, m .

Moreover, W_0 is generated by a wavelet if and only if the following are satisfied:

- (i') $\bigcup_{i=0}^k F_i = \mathbb{R}^d$,
- (ii') $F = \bigcup_{i>j} (F_i \cap F_j)$,
- (iii) $F_i \cap F_j \cap F_m$ has measure zero for all different i, j, m .

Proof. Let $\phi \in V_0$ be such that $\{T_\ell \phi : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for V_0 . Then

$$F = \text{supp}(\Phi_f) = \text{supp}(\Phi_\phi),$$

and (1.1) becomes the condition that $(\sum_{i=0}^k \chi_{A^i F - 2\ell_i \pi}) - \chi_F = \chi_E$, which is clearly equivalent to (i), (ii) and (iii).

The last statement follows from the fact that (i'), (ii') and (iii) are satisfied if and only if $(\sum_{i=0}^k \chi_{A^i F - 2\ell_i \pi}) - \chi_F = 1$. □

Remark 1.3. Keep all the notation in Corollary 1.2 and assume that (V_j) is associated with a frame wavelet ψ . Then

$$\sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}(\xi + 2\ell\pi)|^2 = \frac{1}{(2\pi)^d} \chi_E,$$

with $E = (E_1 \cup E_2)$, where $E_1 = \bigcup_{i>j} (F_i \cap F_j)$ and $E_2 = (\bigcup_{i=0}^k (F_i \setminus L_i)) \setminus F$ and $L_i = \bigcup_{j \neq i} F_j$.

The following is well known (cf. [RS1], and also [GHa] for a more general version for any group) and will be needed in the rest of paper:

Lemma 1.4. *Let M be a shift-invariant subspace. Then there exists $\{\psi_\lambda : \lambda \in \Lambda\} \subset M$ such that $\{T_\ell \psi_\lambda : \ell \in \mathbb{Z}^d, \lambda \in \Lambda\}$ is a normalized tight frame for M .*

2. TIGHT FRAMES AND MULTIPLICITY FUNCTIONS

Let $\psi \in L^2(\mathbb{R})$ be a dyadic (dilation by 2) orthonormal wavelet. The *dimension function* of ψ is defined by

$$D_\psi(\xi) = \sum_{j \geq 1} \sum_{\ell \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + 2\ell\pi))|^2.$$

The importance of dimension functions for orthonormal wavelets was discovered by Lemarié [Le1], [Le2] and Auscher [Aus]. One of the applications is to use dimension functions to classify orthonormal wavelets. In particular, it is well known that a wavelet is an MRA wavelet if and only if $(2\pi)D_\psi(\xi) = 1, a.e. \xi \in \mathbb{R}$. The reason behind this is that the dimension function of ψ is the *multiplicity function* for the scaling space V_0 associated with ψ (cf. [We] for the definition of multiplicity functions for shift-invariant subspaces). Although some evidence shows that this must be true for any (multi) wavelet frames, it has not been proved yet (cf. [Ba], [BRS]). The purpose of this section is to provide a result concerning the modules for normalized tight frame generators, which seems first to be observed and from

which we will see a more transparent connection between dimension and multiplicity functions.

For any collection $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ of functions in $L^2(\mathbb{R}^d)$, let

$$D_\Psi(\xi) = \sum_{\lambda \in \Lambda} \sum_{j \geq 1} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(B^j(\xi + 2\ell\pi))|^2,$$

where $B = A^t$ is the transpose of A .

Lemma 2.1. *Let M be a shift-invariant subspace. Assume that both $\{T_\ell\psi_\lambda : \ell \in \mathbb{Z}^d, \lambda \in \Lambda\}$ and $\{T_\ell\phi_\gamma : \ell \in \mathbb{Z}^d, \gamma \in \Gamma\}$ are normalized tight frames for M . Then*

$$\sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(\xi + 2\ell\pi)|^2 = \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2.$$

In particular, the above function is equal to the multiplicity function $m_M(\xi)$.

Proof. Let $F \subset \mathbb{T}^d$ be any measurable subset, and let $E = \bigcup_{\ell \in \mathbb{Z}^d} (F + 2\ell\pi)$. Then χ_E is a 2π -periodic function and $\chi_E \hat{M} \subset \hat{M}$ since \hat{M} is invariant under multiplication by $e^{2\pi i \langle \ell, \xi \rangle}$ for all $\ell \in \mathbb{Z}^d$. This would imply that both $\{\frac{1}{(2\pi)^d} \hat{T}_\ell \chi_E \hat{\psi}_\lambda : \ell \in \mathbb{Z}^d, \lambda \in \Lambda\}$ and $\{\frac{1}{(2\pi)^d} \hat{T}_\ell \chi_E \hat{\phi}_\gamma : \ell \in \mathbb{Z}^d, \gamma \in \Gamma\}$ are normalized tight frames for $\chi_E \hat{M}$. Thus

$$\begin{aligned} \int_F \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(\xi + 2\ell\pi)|^2 &= \sum_{\lambda \in \Lambda} \|\chi_E \hat{\psi}_\lambda\|^2 \\ &= \sum_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\langle \chi_E \hat{\psi}_\lambda, \frac{1}{(2\pi)^d} \hat{T}_\ell \chi_E \hat{\phi}_\gamma \rangle|^2 \\ &= \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\langle \frac{1}{(2\pi)^d} \hat{T}_{-\ell} \chi_E \hat{\psi}_\lambda, \chi_E \hat{\phi}_\gamma \rangle|^2 \\ &= \sum_{\gamma \in \Gamma} \|\chi_E \hat{\phi}_\gamma\|^2 \\ &= \int_F \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2. \end{aligned}$$

Since F is an arbitrary measurable set of \mathbb{T}^d , it follows that

$$\sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(\xi + 2\ell\pi)|^2 = \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2.$$

For the last statement, we can decompose M into an orthogonal direct sum of cyclic shift-invariant subspaces S_i . Let $\phi_i \in S_i$ be such that its translations generate a normalized tight frame. Then it is obvious that the multiplicity function m_i is equal to $\sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_i(\xi + 2\ell\pi)|^2$. Hence $m_M(\xi) = \sum_{i \in \mathcal{I}} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_i(\xi + 2\ell\pi)|^2$. Thus, by what we have proved, $m_M(\xi) = \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(\xi + 2\ell\pi)|^2$ for any normalized tight frame $\{T_\ell\psi_\lambda : \ell \in \mathbb{Z}^d, \lambda \in \Lambda\}$ of M . \square

In the case that the multiplicity function is finite we can prove the following much stronger result:

Theorem 2.2. *Let M be a shift-invariant subspace such that its multiplicity function $m_M(\xi)$ is finite almost everywhere. If both $\{T_\ell\psi_\lambda : \lambda \in \Lambda, \ell \in \mathbb{Z}^d\}$ and*

$\{T_\ell \phi_\gamma : \gamma \in \Gamma, \ell \in \mathbb{Z}^d\}$ are normalized tight frames for M , then

$$(2.1) \quad \sum_{\lambda \in \Lambda} |\hat{\psi}_\lambda(\xi)|^2 = \sum_{\gamma \in \Gamma} |\hat{\phi}_\gamma(\xi)|^2, \quad a.e.$$

We call the function represented by (2.1) the *modular function* of M . Clearly the multiplicity function is the periodization of the modular function.

Case I. $\sum_{\lambda \in \Lambda} \|\psi_\lambda\|^2 < \infty$ (equivalently, $m_M(\xi)$ is integrable on \mathbb{T}^d).

Let E be any measurable subset of \mathbb{R}^d .

$$\begin{aligned} \sum_{\lambda \in \Lambda} \|\chi_E \hat{\psi}_\lambda\|^2 &= \sum_{\lambda \in \Lambda} \langle \chi_E \hat{\psi}_\lambda, \hat{\psi}_\lambda \rangle \\ &= \frac{1}{2\pi^d} \sum_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \langle \chi_E \hat{\psi}_\lambda, \sum_{\ell \in \mathbb{Z}^d} \langle \hat{\psi}_\lambda, \hat{T}_\ell \hat{\phi}_\gamma \rangle \hat{T}_\ell \hat{\phi}_\gamma \rangle \\ &= \frac{1}{2\pi^d} \sum_{\lambda \in \Lambda} \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} \langle \chi_E \hat{\psi}_\lambda, \langle \hat{\psi}_\lambda, \hat{T}_\ell \hat{\phi}_\gamma \rangle \hat{T}_\ell \hat{\phi}_\gamma \rangle \\ &= \frac{1}{2\pi^d} \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} \langle \hat{\phi}_\gamma, \hat{T}_{-\ell} \hat{\psi}_\lambda \rangle \langle \hat{T}_{-\ell} \chi_E \hat{\psi}_\lambda, \hat{\phi}_\gamma \rangle \\ &= \frac{1}{2\pi^d} \sum_{\gamma \in \Gamma} \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} \langle \hat{\phi}_\gamma, \hat{T}_{-\ell} \hat{\psi}_\lambda \rangle \langle \hat{T}_{-\ell} \hat{\psi}_\lambda, \chi_E \hat{\phi}_\gamma \rangle \\ &= \sum_{\gamma \in \Gamma} \langle \hat{\phi}_\gamma, \chi_E \hat{\phi}_\gamma \rangle \\ &= \sum_{\gamma \in \Gamma} \|\chi_E \hat{\phi}_\gamma\|^2, \end{aligned}$$

where in the fourth equality we could interchange the summations because of the assumption $\sum_{\lambda \in \Lambda} \|\psi_\lambda\|^2 < \infty$. Since E is arbitrary, it follows that

$$\sum_{\lambda \in \Lambda} |\hat{\psi}_\lambda(\xi)|^2 = \sum_{\gamma \in \Gamma} |\hat{\phi}_\gamma(\xi)|^2, \quad a.e.$$

Case II. The general case.

Define $F_n = \{\xi \in \mathbb{T}^d : m_M(\xi) \leq n\}$, and let $E_n = \bigcup_{\ell \in \mathbb{Z}^d} (F_n + 2\ell\pi)$. Then, by assumption, $\bigcup_{n=1}^\infty F_n = \mathbb{T}^d$ and $\bigcup_{n=1}^\infty E_n = \mathbb{R}^d$ modulo null sets. As discussed in the proof of Lemma 2.1, both $\{\frac{1}{(2\pi)^d} \hat{T}_\ell \chi_{E_n} \hat{\psi}_\lambda : \ell \in \mathbb{Z}^d, \lambda \in \Lambda\}$ and $\{\frac{1}{(2\pi)^d} \hat{T}_\ell \chi_{E_n} \hat{\phi}_\gamma : \ell \in \mathbb{Z}^d, \gamma \in \Gamma\}$ are normalized tight frames for $\chi_{E_n} \hat{M}$. Moreover,

$$\sum_{\lambda} \|\chi_{E_n} \hat{\psi}_\lambda\|^2 \leq n(2\pi)^d < \infty.$$

So by applying Case I to the shift-invariant subspace that is the image of the inverse Fourier transform of $\chi_{E_n} \hat{M}$, we get

$$\sum_{\lambda \in \Lambda} |\chi_{E_n}(\xi) \hat{\psi}_\lambda(\xi)|^2 = \sum_{\gamma \in \Gamma} |\chi_{E_n}(\xi) \hat{\phi}_\gamma(\xi)|^2, \quad a.e.$$

Hence

$$\sum_{\lambda \in \Lambda} |\hat{\psi}_\lambda(\xi)|^2 = \sum_{\gamma \in \Gamma} |\hat{\phi}_\gamma(\xi)|^2, \quad a.e.,$$

since $\bigcup_{n=1}^\infty E_n = \mathbb{R}^d$.

Corollary 2.3. *Let M be a shift-invariant subspace such that its multiplicity function $m_M(\xi)$ is finite almost everywhere. If both $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ and $\Phi = \{\phi_\gamma : \gamma \in \Gamma\}$ generate normalized tight frames for M , then*

$$D_\Psi(\xi) = D_\Phi(\xi).$$

Proof. By Theorem 2.2, we have that for any $j \in \mathbb{Z}$,

$$\sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(B^j(\xi + 2\ell\pi))|^2 = \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(B^j(\xi + 2\ell\pi))|^2.$$

Summing over $j > 0$ on both sides, we get $D_\Psi(\xi) = D_\Phi(\xi)$. □

We conjecture that the condition concerning the multiplicity function of M in Theorem 2.2 and Corollary 2.3 can be dropped.

Proposition 2.4. *Let V_0 and M be shift-invariant subspaces such that $M \perp V_0$ and $\delta_A V_0 = V_0 \oplus M$. If $m_{V_0} \in L^1(\Omega)$, then*

$$m_{V_0}(\xi) = D_\Psi(\xi),$$

where $\Omega = [0, 2\pi)^d$ and $\Psi := \{\psi_\lambda : \lambda \in \Lambda\} \subset M$ such that its translates form a normalized tight frame for M .

Proof. Let $\{T_\ell \phi_\gamma : \gamma \in \Gamma, \ell \in \mathbb{Z}^d\}$ be a normalized tight frame for V_0 . By the assumption we have $\sum_{\gamma \in \Gamma} \|\phi_\gamma\|^2 < \infty$.

Let $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ generate a normalized tight frame for M . In this case $m_M(\xi) < \infty$ (in fact $\sum_\lambda \|\psi_\lambda\|^2 < \infty$). Let $q = |\det A|$ and $\{\ell_0, \dots, \ell_{q-1}\}$ be a full digit set for B , i.e. $\{AZ^d + \ell_k : 1 \leq k \leq d-1\}$ is a basis for the quotient group $\mathbb{Z}^d/A\mathbb{Z}^d$. Then both

$$\{T_\ell \phi_\gamma : \gamma \in \Gamma, \ell \in \mathbb{Z}^d\} \cup \{T_\ell \psi_\lambda : \lambda \in \Lambda, \ell \in \mathbb{Z}^d\}$$

and

$$\{T_\ell \delta_A T_{\ell_k} \phi_\gamma : \gamma \in \Gamma, \ell \in \mathbb{Z}^d, 1 \leq k \leq q-1\}$$

are normalized tight frames for $V_0 \oplus M$. Let $g_{k,\gamma} = \delta_A T_{\ell_k} \phi_\gamma$. Then $\hat{g}_{k,\gamma}(\xi) = \frac{1}{|\det A|^{1/2}} e^{i\langle B^{-1}\xi, \ell_k \rangle} \hat{\phi}_\gamma(B^{-1}\xi)$. Applying Theorem 2.2 to $V_0 \oplus M$ and these two normalized tight frames, we have

$$\begin{aligned} & \sum_{j=1}^\infty \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(B^j(\xi + 2\ell\pi))|^2 + \sum_{j=1}^\infty \sum_{\lambda \in \Lambda} \sum_{\ell \in \mathbb{Z}^d} |\hat{\psi}_\lambda(B^j(\xi + 2\ell\pi))|^2 \\ &= \sum_{k=0}^{q-1} \sum_{j=1}^\infty \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |(\hat{g}_{k,\gamma}(B^j(\xi + 2\ell\pi)))|^2 \\ &= (q/q) \sum_{j=1}^\infty \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(B^{j-1}(\xi + 2\ell\pi))|^2 \\ &= \sum_{j=1}^\infty \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(B^j(\xi + 2\ell\pi))|^2 + \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2, \end{aligned}$$

that is,

$$D_\Phi(\xi) + D_\Psi(\xi) = D_\Phi(\xi) + \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2.$$

From the assumption, $D_\Phi(\xi) < \infty$, a.e., which implies that

$$m_{V_0}(\xi) = \sum_{\gamma \in \Gamma} \sum_{\ell \in \mathbb{Z}^d} |\hat{\phi}_\gamma(\xi + 2\ell\pi)|^2 = D_\psi(\xi), \quad a.e.$$

□

Applying Proposition 2.4 to wavelet frames, we have

Corollary 2.5. *Assume that (V_j) is a GMRA and $\Phi = \{\psi_\lambda : \lambda \in \Lambda\}$ generates a normalized tight frame for $V_1 \ominus V_0$. If $m_{V_0}(\xi) \in L^1(\Omega)$, then $m_{V_0}(\xi) = D_\Psi(\xi)$.*

Remark. Various forms of Proposition 2.4 have appeared in different contexts in the literature. In particular, in a recent paper [RS2] Ron and Shen also proved this result. (We thank the referee for bringing our attention to Ron and Shen’s new paper.) However, our treatment here seems more transparent and elementary. Moreover, Theorem 2.2 is a much stronger statement about the multiplicity functions for shift-invariant subspaces, and it seems to be new in the literature.

3. THE CONSTRUCTION OF FMRA WAVELET FRAMES

In this section we discuss the constructions of FMRA wavelet frames. We focus on the case $A = 2$ and $d = 1$. It is well known that if ϕ is a scaling function for an MRA, then there is a unique 2π -periodic function $m_0 \in L^2(\mathbb{T})$ satisfying

$$(3.1) \quad \hat{\phi}(\xi) = m_0(\xi/2)\hat{\phi}(\xi/2),$$

and ψ defined by

$$(3.2) \quad \hat{\psi}(\xi) = e^{i\xi/2} \overline{m_0(\xi/2 + \pi)} \hat{\phi}(\xi/2)$$

is a wavelet. In the frame scaling function case, m_0 is not necessarily unique. We will first prove that for any 2π -periodic function $m_0 \in L^2(\mathbb{T})$ satisfying (3.1), then ψ defined by (3.2) will never be a wavelet frame unless the associated FMRA is an MRA. To prove this we need the following Lemma 3.1. For any two sets E and F , we write $E\Delta F = (E \setminus F) \cup (F \setminus E)$.

Lemma 3.1. *Let (V_j) be an FMRA with a frame scaling function ϕ , and let m_0 be as in (3.1). Write $F = \bigcup_{k \in \mathbb{Z}} (\text{supp}(\phi) + 2k\pi)$. Assume that W_0 is generated by a wavelet frame. Then*

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = \begin{cases} 1, & \xi \in F \cap (F - \pi), \\ 0, & \xi \in (F\Delta(F - \pi)) \setminus (F/2). \end{cases}$$

Proof. Using the connection between m_0 and $\hat{\phi}$, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi + 2k\pi)|^2 &= |m_0(\xi/2)|^2 \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi/2 + 2k\pi)|^2 \\ &\quad + |m_0(\xi/2 + \pi)|^2 \sum_{k \in \mathbb{Z}} |\hat{\phi}(\xi/2 + \pi + 2k\pi)|^2. \end{aligned}$$

This implies that

$$\chi_F(\xi) = |m_0(\xi/2)|^2 \chi_{2F}(\xi) + |m_0(\xi/2 + \pi)|^2 \chi_{2F-2\pi}(\xi),$$

i.e.,

$$\chi_{F/2}(\xi) = |m_0(\xi)|^2 \chi_F(\xi) + |m_0(\xi + \pi)|^2 \chi_{F-\pi}(\xi).$$

Now let $\xi \in F \cap (F - \pi)$. By Corollary 1.2, $F \cap (F - \pi) \subset F/2$. Thus $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$ by the above identity. Similarly

$$|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 0$$

when $\xi \in (F \Delta (F - \pi)) \setminus (F/2)$. \square

Proposition 3.2. *Let (V_j) be an FMRA with a frame scaling function ϕ and ψ be defined by (3.2). Then the following are equivalent:*

- (i) (V_j) is an MRA,
- (ii) $\psi \in W_0$ is a normalized tight frame wavelet.

Proof. We only need to verify the implication of (ii) \Rightarrow (i). Let $F = \text{supp}(\Phi_\phi)$. Since the frame scaling function ϕ is an (orthogonal) scaling function if and only if $F = \mathbb{R}$ (modulo a null set), it suffices to check that $F = \mathbb{R}$ (modulo a null set). By (3.2), we can obtain the following identity:

$$(3.3) \quad (2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |m_0(\xi/2 + \pi)|^2 \chi_{2F}(\xi) + |m_0(\xi/2)|^2 \chi_{2F-2\pi}(\xi).$$

On the other hand, by Corollary 1.2 and Remark 1.3, we also have

$$(2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = \chi_E(\xi)$$

where $E = (2F \cap 2(F - \pi)) \cup ((2F \Delta 2(F - \pi)) \setminus F)$. But from Lemma 3.1 and (3.5), we have that when $\xi \in (2F \Delta 2(F - \pi)) \setminus F$,

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 0.$$

This implies that $(2F \Delta 2(F - \pi)) \setminus F$ must have measure 0. Hence $(2F \Delta 2(F - \pi)) \subset F$ (modulo a null set). By Corollary 1.2, we also have $2F \cap 2(F - \pi) \subset F$. So $2F \cup (2F - 2\pi) \subset F$ and so $2F \subset F$. A standard Lebesgue point argument (cf. [LMS], the proof of Lemma 2) shows that $\bigcup_{\ell \in \mathbb{Z}} (F + 2\ell\pi) = \mathbb{R}$ (modulo a null set). Thus $F = \mathbb{R}$ (modulo a null set) since F is 2π -translation invariant. \square

Lemma 3.3. *Suppose that $\{T_\ell \phi : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for a shift-invariant subspace M . Let $\phi_0 \in M$ be such that $\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $\overline{\text{span}}\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\}$. Then $\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for M if and only if*

$$\sum_{\ell \in \mathbb{Z}^d} |\phi(\xi + 2\ell\pi)|^2 = \sum_{\ell \in \mathbb{Z}^d} |\phi_0(\xi + 2\ell\pi)|^2$$

holds a.e.

Proof. Let P be the orthogonal projection of M onto $\overline{\text{span}}\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\}$. Then $\{T_\ell P\phi : \ell \in \mathbb{Z}^d\}$ is a normalized tight frame for $\overline{\text{span}}\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\}$. Thus, by Lemma 2.1, $\|\phi_0\| = \|P\phi\|$. However, by $\sum_{\ell \in \mathbb{Z}^d} |\phi(\xi + 2\ell\pi)|^2 = \sum_{\ell \in \mathbb{Z}^d} |\phi_0(\xi + 2\ell\pi)|^2$, we also have $\|\phi\| = \|\phi_0\|$, which implies $\|P\phi\| = \|\phi\|$. Thus $\overline{\text{span}}\{T_\ell \phi_0 : \ell \in \mathbb{Z}^d\} = M$ since the ϕ translation generates M . \square

Now assume that (V_j) is an FMRA with a frame scaling function ϕ , and assume that m_0 satisfies (3.1). Let $K_1 = F \cap (F - \pi)$, $K_2 = (F\Delta(F - \pi)) \cap F/2$ and $K_3 = (F\Delta(F - \pi)) \setminus (F/2)$. Then K_1, K_2 and K_3 are disjoint and π -translation stable. We define the 2π -periodic function m_1 by

$$m_1(\xi) = \begin{cases} e^{i\xi\overline{m_0(\xi + \pi)}}, & \xi \in K_1, \\ 1, & \xi \in K_3, \\ 0, & \text{everywhere else,} \end{cases}$$

and define ψ by

$$(3.4) \quad \hat{\psi}(\xi) = m_1(\xi/2)\hat{\phi}(\xi/2).$$

Theorem 3.4. *Assume that (V_j) is associated with a wavelet frame (equivalently, $2F \cap 2(F - \pi) \subset F \subset 2F \cup 2(F - \pi)$ by Corollary 1.2). Then ψ defined by (3.4) is an FMRA wavelet frame.*

Proof. By assumption there exists a function $f \in W_0$ such that $\{T_\ell f : \ell \in \mathbb{Z}\}$ is a normalized tight frame for W_0 . Therefore, by Corollary 1.2, $\sum_{k \in \mathbb{Z}} |\hat{f}(\xi + 2k\pi)|^2 = \frac{1}{2\pi}\chi_E$, where $E = 2(K_1 \cup K_3)$. By Lemma 3.3, it suffices to prove that (i) $\psi \in W_0$ and (ii) $\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = \frac{1}{2\pi}\chi_E$.

In order that $\psi \in W_0$, by Theorem 5.1 in [BL], we need to verify the following:

$$(3.5) \quad \overline{m_0(\xi)}m_1(\xi)\chi_F(\xi) + \overline{m_0(\xi + \pi)}m_1(\xi + \pi)\chi_{F-\pi} = 0.$$

Suppose that $\xi \in K_1$. Then $m_1(\xi) = e^{i\xi\overline{m_0(\xi + \pi)}}$ and $m_1(\xi + \pi) = -e^{i\xi\overline{m_0(\xi)}}$. Hence (3.5) holds in this case. If $\xi \in K_2$, then $\xi + \pi \in K_2$ (note that $F/2$ is π -translation stable). Hence $m_1(\xi) = m_1(\xi + \pi) = 0$, which implies (3.5) holds. If $\xi \in K_3$, then by Lemma 3.1 $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 0$, which also implies that (3.5) holds. Therefore we proved (i).

Now we verify (ii). From (3.4), we have

$$(3.6) \quad (2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |m_1(\xi/2)|^2\chi_{2F}(\xi) + |m_1(\xi/2 + \pi)|^2\chi_{2F-2\pi}(\xi).$$

By Lemma 3.1, $|m_0(\xi/2)|^2 + |m_0(\xi/2 + \pi)|^2 = 1$ when $\xi \in 2F \cap 2(F - \pi) (= 2K_1)$. Thus

$$\begin{aligned} (2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 &= |m_1(\xi/2)|^2 + |m_1(\xi/2 + \pi)|^2 \\ &= |m_0(\xi/2 + \pi)|^2 + |m_0(\xi/2)|^2 = 1. \end{aligned}$$

If $\xi \in (2F\Delta 2(F - \pi)) \setminus F (= 2K_3)$, then

$$m_1(\xi/2) = m_1(\xi/2 + \pi) = 1$$

by the definition of m_1 . Hence

$$(2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |m_1(\xi/2)|^2\chi_{2F}(\xi) + |m_1(\xi/2 + \pi)|^2\chi_{2F-2\pi}(\xi) = 1.$$

Similarly

$$(2\pi) \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = |m_1(\xi/2)|^2\chi_{2F}(\xi) + |m_1(\xi/2 + \pi)|^2\chi_{2F-2\pi}(\xi) = 0$$

when $\xi \notin E$, since $\chi_{2F}(\xi) = \chi_{2F-2\pi}(\xi) = 0$ if $\xi \notin 2F \cup 2(F - \pi)$, and $m_1(\xi/2) = m_1(\xi/2 + \pi) = 0$ if $\xi \in 2K_2$. Considering all the cases, we obtain

$$\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = \frac{1}{2\pi} \chi_E,$$

as claimed. \square

Remark. Since this paper was submitted, a more general result on the construction of wavelet frames from filters and frame scaling functions for an FMRA has been proved by Baggett, Courter and Merrill (Corollary 3.5, [BCM]). The result in [BCM] deals with general dilation matrices. However, it is evident that our construction formula (3.4) (for the special dilation $d = 2$) is more explicit and may be more accessible than the general result.

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DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI, PEOPLES REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FLORIDA 32816
E-mail address: `dhan@pegasus.cc.ucf.edu`