NON-NORMAL DERIVATION AND ORTHOGONALITY

SALAH MECHERI

(Communicated by Joseph A. Ball)

Abstract. The main purpose of this note is to characterize the operators $T \in B(H)$ which are orthogonal (in the sense of James) to the range of a generalized derivation for non-normal operators $A, B \in B(H)$.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex and separable Hilbert space $H$. For $A, B \in B(H)$, let $\delta_{A,B}$ denote the operator on $B(H)$ defined by $\delta_{A,B}(X) = AX - XB$. If $A = B$, $\delta_A$ is called the inner derivation induced by $A \in B(H)$. Let

$$B(H) \supset K(H) \supset C_p \supset F(H) (0 < p < \infty)$$

denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten $p$-class, and the class of finite rank operators on $H$. All operators herein are assumed to be linear and bounded. Let $\| \cdot \|_p$, $\| \cdot \|_\infty$ denote, respectively, the $C_p$-norm and the $K(H)$-norm. Let $I$ be a proper bilateral ideal of $B(H)$. It is well known that if $I \neq \{0\}$, then $K(H) \supset I \supset F(H)$.

In [1, Theorem 1.7], J. Anderson shows that if $A$ is normal and commutes with $T$, then for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\|.$$

Over the years, Anderson’s result has been generalized in various ways. Some results concern elementary operators on $B(H)$ such as $X \rightarrow AXB - X$ or $\delta_{A,B}(X) = AX - XB$; since these are not normal derivations, some extra condition is needed in each case to obtain the orthogonality result. In [2], P.B. Duggal established the orthogonality result for $\Delta_{AB} = AXB - X$ under the hypothesis that $(A, B)$ satisfies a generalized Putnam-Fuglede property (which is one way to generalize normality).

Another way to generalize Anderson’s result is to consider the restriction of an elementary operator (e.g., $X \rightarrow AXB - X$) or $\delta_{A,B}(X) = AX - XB$ to a norm ideal $(I, \| \cdot \|_I)$ of $B(H)$. Among the results in this direction, Duggal [2] has obtained the orthogonality result for $\Delta_{AB}\mid_{C_p}$ (the restriction to the schatten $p$-class $C_p$) under the Putnam-Fuglede hypothesis on $(A, B)$, and F.Kittaneh [4]...
proved the orthogonality result for restricted generalized derivations $\delta_{A,B}|_{I}$ (with the Putnam-Fuglede condition for $(A,B)$).

In this paper we initiate a different approach to generalize Anderson’s theorem, one which does not rely on normality via the Putnam-Fuglede condition.

Let $E$ be a complex Banach space. We first define orthogonality in $E$. We say that $b \in E$ is orthogonal to $a \in E$ if for all complex $\lambda$ it holds that

$$\|a + \lambda b\| \geq \|a\|.$$  \hfill (1.1)

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint from the open ball $K(0, \|a\|)$, i.e., if and only if this complex line is a tangent one.

Note that if $b$ is orthogonal to $a$, then $a$ need not be orthogonal to $b$. If $E$ is a Hilbert space, then from (1.1) it follows that $\langle a, b \rangle = 0$, i.e., orthogonality in the usual sense.

2. Main results

Let $A$ denote a complex Banach Algebra with identity $e$ and let $\sigma_r(A), \sigma_l(A)$ denote, respectively, the right spectrum and the left spectrum of $A$.

Recall that

$$a^n x - x b^n = \sum_{i=0}^{n-1} a^{n-i-1} (ax - xb)b^i$$

for all $x \in A$.

It easy to see that if $ac = cb$, then for all $x \in A$ we have

$$ncb^{n-1} = a^n x - xb^n - \sum_{i=0}^{n-1} a^{n-i-1} (ax - xb - c)b^i$$

for all $x \in A$.

**Theorem 2.1.** Let $a$ be elements of $A$, $(k_n)$ an increasing sequence of positive integers and $a^{k_n}$ converge to $y \in A$, with $0 \notin \sigma_r(y) \cap \sigma_l(y)$. If there exists a constant $\alpha$ such that $\|a^n\| \leq \alpha$ for all integers $n$ and if $b$ is the left or the right inverse of $y$, then

$$\alpha^2 \|b\| \|ax - xa - c\| \geq \|c\|$$

for all $x \in A$ and for all $c \in \ker \delta_a$.

**Proof.** If $x \in A$, we can write

$$(k_n + 1)c a^{k_n} = a^{k_n + 1} x - xa^{k_n + 1} - \sum_{i=0}^{k_n} a^{k_n-i} (ax - xa - c)a^i.$$  

It follows that

$$\|ca^{k_n}\| \leq \frac{2\alpha}{k_n + 1} \|x\| + \alpha^2 \|ax - xa - c\|.$$  

Letting $n$ tend to infinity we get

$$\|cy\| \leq \alpha^2 \|ax - xa - c\|.$$  

Now, if $b$ is the right or the left inverse of $y$, we get

$$\|c\| \leq \|b\| \cdot \alpha^2 \|ax - xa - c\|.$$  

$\square$
Corollary 2.1. Let $A \in B(H)$ and assume that there is a constant $\alpha$ such that $\|A^n\| \leq \alpha$ for all integers $n$ and let $(k_n)$ be an increasing sequence of positive integers.

(i) If $A^{k_n} \to C$, with $0 \notin \sigma_r(C) \cap \sigma_l(C)$, then

$$\alpha^2 \|AX - XA - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

(ii) If $A^{k_n} \to C + K$, with $K$ compact and $0 \notin \sigma_r(C) \cap \sigma_l(C)$, then

$$\alpha^2 \|AX - XA - T - K\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

Proof. It is a simple consequence of the above theorem. \hfill \Box

Theorem 2.2. Let $A \in B(H)$ satisfying $A^m = I$ for some integer $m$. Then

$$\alpha^2 \|AX - XA - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

Proof. Since $A^n \in \{I, A, A^2, \ldots, A^{n-1}\}$ for all integers $n$, $\|A^n\| \leq \alpha$, $n \in N$ and $A^{k_n} = I$, where $k_n = nm$, $n \in N$. It suffices to apply Corollary 2.1 (i). \hfill \Box

Corollary 2.2. Let $A, B \in B(H)$ satisfying $A^m = I$ and $B^m = I$ for some integer $m$. Then

$$\|AX - XB - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_{A,B}$.

Proof. It is sufficient to take on $H \oplus H$,

$$N = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

It is clear that $N^m = I$, $NS = SN$, i.e., $S \in \ker \delta_N$ and

$$NY - YN = \begin{pmatrix} 0 & AX - XB \\ 0 & 0 \end{pmatrix}.$$

By applying the above corollary it follows that

$$\|AX - XB - T\| = \|NY - YN\| \geq \|S\| = \|T\|.$$

\hfill \Box

Remark 2.1. Note that the operators mentioned above are not in general normal or isometric. To see that, it is enough to consider on 2-dimensional $H$ the operator

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & -1 \end{pmatrix},$$

where $\alpha \in C \setminus \{0\}$. For this $A^2 = I$ and $A$ is not normal or isometric.

About the operators $A^m = I$. Such operators are similar to unitary operators. (Indeed, $A^m$ is a contraction implies $A$ is similar to a contraction $C$ implies $C^m = I$ implies $C$ is unitary.) One can show: If $\phi$ is a power bounded operator (on a Banach algebra $A$, to itself), $S = I - \phi$ and $T = \phi - (\phi)^2$, then $S^{-1}(0) \perp \ker T(A)$. If the algebra $A$ is $B(H)$ and $\alpha$ is polynomially bounded, then there exists a quasi-affinity $X$ and a contraction $C$ (on some Hilbert space) such that $XC = aX$.
A simple calculation shows that
\[ a^n x b^n - x = \sum_{i=0}^{n-1} a^{n-i-1}(axb - x)b^{n-i-1} \]
for all \( x \in \mathcal{A} \).

It is easy to see that if \( acb = c \), then for all \( x \in \mathcal{A} \) we have
\[ -nc = a^n x b^n - x - \sum_{i=0}^{n-1} a^{n-i-1}(axb - c)b^{n-i-1}. \]

Thus, if \( \|a^n\| \leq 1 \) and \( \|b^n\| \leq 1 \), then
\[ \|axb - x - c\| \geq \|c\|, \]
for all \( x \in \mathcal{A} \) and for all \( c \in \ker \Delta_{xb} \). We also have
\[ \|ax - xb - c\| \geq \|c\|, \]
if \( \|a^n\| \leq 1 \) and \( \|b^n\| \leq 1 \).

Note that Theorem 2.1 and its corollary still hold if we consider \( axa - x \) instead of \( ax - xa \). Also Theorem 2.2 and its corollary still hold if we consider \( axb - x \) instead of \( ax - xb \). Let \( \tilde{\delta}_A \) be the image of \( \delta_A \) under the canonical projection of \( B(H) \) onto the Calkin algebra \( \mathcal{B} \) defined by
\[ \tilde{\delta}_A (\tilde{X}) = \tilde{AX} - \tilde{XA}. \]
Since in (1) \( K \) is an arbitrary compact operator, then
\[ \|\tilde{AX} - \tilde{XA} - \tilde{T}\| \geq \|\tilde{T}\|. \]

ACKNOWLEDGMENT

I would like to thank Professor B.P. Duggal and the referee for their useful remarks.

REFERENCES


DEPARTMENT OF MATHEMATICS, KING SAUD UNIVERSITY COLLEGE OF SCIENCE, P.O. BOX 2455, RIYADH 11451, SAUDI ARABIA
E-mail address: mecherisalah@hotmail.com