

NON-NORMAL DERIVATION AND ORTHOGONALITY

SALAH MECHERI

(Communicated by Joseph A. Ball)

ABSTRACT. The main purpose of this note is to characterize the operators $T \in B(H)$ which are orthogonal (in the sense of James) to the range of a generalized derivation for non-normal operators $A, B \in B(H)$.

1. INTRODUCTION

Let $B(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex and separable Hilbert space H . For $A, B \in B(H)$, let $\delta_{A,B}$ denote the operator on $B(H)$ defined by $\delta_{AB}(X) = AX - XB$. If $A = B$, δ_A is called the inner derivation induced by $A \in B(H)$. Let

$$B(H) \supset K(H) \supset C_p \supset F(H) \quad (0 < p < \infty)$$

denote, respectively, the class of all bounded linear operators, the class of compact operators, the Schatten p -class, and the class of finite rank operators on H . All operators herein are assumed to be linear and bounded. Let $\|\cdot\|_p, \|\cdot\|_\infty$ denote, respectively, the C_p -norm and the $K(H)$ -norm. Let \mathcal{I} be a proper bilateral ideal of $B(H)$. It is well known that if $\mathcal{I} \neq \{0\}$, then $K(H) \supset \mathcal{I} \supset F(H)$.

In [1, Theorem 1.7], J.Anderson shows that if A is normal and commutes with T , then for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\|.$$

Over the years, Anderson's result has been generalized in various ways. Some results concern elementary operators on $B(H)$ such as $X \rightarrow AXB - X$ or $\delta_{A,B}(X) = AX - XB$; since these are not normal derivations, some extra condition is needed in each case to obtain the orthogonality result. In [2], P.B.Duggal established the orthogonality result for $\Delta_{AB} = AXB - X$ under the hypothesis that (A, B) satisfies a generalized Putnam-Fuglede property (which is one way to generalize normality).

Another way to generalize Anderson's result is to consider the restriction of an elementary operator (e.g., $X \rightarrow AXB - X$) or $\delta_{A,B}(X) = AX - XB$ to a norm ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ of $B(H)$. Among the results in this direction, Duggal [2] has obtained the orthogonality result for $\Delta_{AB}|_{C_p}$ (the restriction to the Schatten p -class C_p) under the Putnam-Fuglede hypothesis on (A, B) , and F.Kittaneh [4]

Received by the editors February 11, 2003 and, in revised form, October 16, 2003.

2000 *Mathematics Subject Classification*. Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words and phrases. Derivation, generalized derivation, orthogonality.

This work was supported by the Research Center Project No. **Math/1422/10**.

proved the orthogonality result for restricted generalized derivations $\delta_{A,B}|_{\mathcal{I}}$ (with the Putnam-Fuglede condition for (A, B)).

In this paper we initiate a different approach to generalize Anderson's theorem, one which does not rely on normality via the Putnam-Fuglede condition.

Let E be a complex Banach space. We first define orthogonality in E . We say that $b \in E$ is orthogonal to $a \in E$ if for all complex λ it holds that

$$(1.1) \quad \|a + \lambda b\| \geq \|a\|.$$

This definition has a natural geometric interpretation, namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint from the open ball $K(0, \|a\|)$, i.e., if and only if this complex line is a tangent one.

Note that if b is orthogonal to a , then a need not be orthogonal to b . If E is a Hilbert space, then from (1.1) it follows that $\langle a, b \rangle = 0$, i.e., orthogonality in the usual sense.

2. MAIN RESULTS

Let \mathcal{A} denote a complex Banach Algebra with identity e and let $\sigma_r(A)$, $\sigma_l(A)$ denote, respectively, the right spectrum and the left spectrum of A .

Recall that

$$a^n x - x b^n = \sum_{i=0}^{n-1} a^{n-i-1} (ax - xb) b^i$$

for all $x \in \mathcal{A}$.

It easy to see that if $ac = cb$, then for all $x \in \mathcal{A}$ we have

$$n c b^{n-1} = a^n x - x b^n - \sum_{i=0}^{n-1} a^{n-i-1} (ax - xb - c) b^i$$

for all $x \in \mathcal{A}$.

Theorem 2.1. *Let a be elements of \mathcal{A} , (k_n) an increasing sequence of positive integers and a^{k_n} converge to $y \in \mathcal{A}$, with $0 \notin \sigma_r(y) \cap \sigma_l(y)$. If there exists a constant α such that $\|a^n\| \leq \alpha$ for all integers n and if b is the left or the right inverse of y , then*

$$\alpha^2 \|b\| \|ax - xa - c\| \geq \|c\|$$

for all $x \in \mathcal{A}$ and for all $c \in \ker \delta_a$.

Proof. If $x \in \mathcal{A}$, we can write

$$(k_n + 1) c a^{k_n} = a^{k_n+1} x - x a^{k_n+1} - \sum_{i=0}^{k_n} a^{k_n-i} (ax - xa - c) a^i.$$

It follows that

$$\|c a^{k_n}\| \leq \frac{2\alpha}{k_n + 1} \|x\| + \alpha^2 \|ax - xa - c\|.$$

Letting n tend to infinity we get

$$\|c y\| \leq \alpha^2 \|ax - xa - c\|.$$

Now, if b is the right or the left inverse of y , we get

$$\|c\| \leq \|b\| \cdot \alpha^2 \|ax - xa - c\|.$$

□

Corollary 2.1. *Let $A \in B(H)$ and assume that there is a constant α such that $\|A^n\| \leq \alpha$ for all integers n and let (k_n) be an increasing sequence of positive integers.*

(i) *If $A^{k_n} \rightarrow C$, with $0 \notin \sigma_r(C) \cap \sigma_l(C)$, then*

$$\alpha^2 \|AX - XA - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

(ii) *If $A^{k_n} \rightarrow C + K$, with K compact and $0 \notin \sigma_r(C) \cap \sigma_l(C)$, then*

$$(1) \quad \alpha^2 \|AX - XA - T - K\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

Proof. It is a simple consequence of the above theorem. \square

Theorem 2.2. *Let $A \in B(H)$ satisfying $A^m = I$ for some integer m . Then*

$$\alpha^2 \|AX - XA - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_A$.

Proof. Since $A^n \in \{I, A, A^2, \dots, A^{m-1}\}$ for all integers n , $\|A^n\| \leq \alpha$, $n \in N$ and $A^{k_n} = I$, where $k_n = nm$, $n \in N$. It suffices to apply Corollary 2.1 (i). \square

Corollary 2.2. *Let $A, B \in B(H)$ satisfying $A^m = I$ and $B^m = I$ for some integer m . Then*

$$\|AX - XB - T\| \geq \|T\|,$$

for all $X \in B(H)$ and for all $T \in \ker \delta_{A,B}$.

Proof. It is sufficient to take on $H \oplus H$,

$$N = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

It is clear that $N^m = I$, $NS = SN$, i.e., $S \in \ker \delta_N$ and

$$NY - YN = \begin{pmatrix} 0 & AX - XB \\ 0 & 0 \end{pmatrix}.$$

By applying the above corollary it follows that

$$\|AX - XB - T\| = \|NY - YN\| \geq \|S\| = \|T\|.$$

\square

Remark 2.1. Note that the operators mentioned above are not in general normal or isometric. To see that, it is enough to consider on 2-dimensional H the operator

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix},$$

where $\alpha \in C \setminus \{0\}$. For this $A^2 = I$ and A is not normal or isometric.

About the operators $A^m = I$. Such operators are similar to unitary operators. (Indeed, A^m is a contraction implies A is similar to a contraction C implies $C^m = I$ implies C is unitary.) One can show: If ϕ is a power bounded operator (on a Banach algebra \mathcal{A} , to itself), $S = I - \phi$ and $T = \phi - (\phi)^2$, then $S^{-1}(0) \perp_{\alpha} T(A)$. If the algebra \mathcal{A} is $B(H)$ and a is polynomially bounded, then there exists a quasi-affinity X and a contraction C (on some Hilbert space) such that $XC = aX$.

A simple calculation shows that

$$a^n x b^n - x = \sum_{i=0}^{n-1} a^{n-i-1} (axb - x) b^{n-i-1}$$

for all $x \in \mathcal{A}$.

It is easy to see that if $acb = c$, then for all $x \in \mathcal{A}$ we have

$$-nc = a^n x b^n - x - \sum_{i=0}^{n-1} a^{n-i-1} (axb - c) b^{n-i-1}.$$

Thus, if $\|a^n\| \leq 1$ and $\|b^n\| \leq 1$, then

$$\|axb - x - c\| \geq \|c\|,$$

for all $x \in \mathcal{A}$ and for all $c \in \ker \Delta_{ab}$. We also have

$$\|ax - xb - c\| \geq \|c\|,$$

if $\|a^n\| \leq 1$ and $\|b^n\| \leq 1$.

Note that Theorem 2.1 and its corollary still hold if we consider $axa - x$ instead of $ax - xa$. Also Theorem 2.2 and its corollary still hold if we consider $axb - x$ instead of $ax - xb$. Let $\tilde{\delta}_A$ be the image of δ_A under the canonical projection of $B(H)$ onto the Calkin algebra \mathcal{B} defined by

$$\tilde{\delta}_A(\tilde{X}) = \tilde{A}\tilde{X} - \tilde{X}\tilde{A}.$$

Since in (1) K is an arbitrary compact operator, then

$$\left\| \tilde{A}\tilde{X} - \tilde{X}\tilde{A} - \tilde{T} \right\| \geq \left\| \tilde{T} \right\|.$$

ACKNOWLEDGMENT

I would like to thank Professor B.P.Duggal and the referee for their useful remarks.

REFERENCES

- [1] J. Anderson, On normal derivations, Proc. Amer. Math. Soc. 38-1(1973), 135-140. MR47:875
- [2] B.P. Duggal, A remark on normal derivations, Proc. Amer. Math. Soc. 126-7(1998), 2047-2052. MR98h:47050
- [3] F. Kittaneh, Operators that are orthogonal to the range of a derivation, J. Math. Anal. Appl. 203(1996), 863-873. MR97f:47033
- [4] F. Kittaneh, Normal derivations in norm ideals, Proc. Amer. Math. Soc., 123-6(1995)1779-1785. MR95g:47054
- [5] P.J. Maher, Commutator approximants, Proc. Amer. Math. Soc. 115(1992), 995-1000. MR92j:47059
- [6] S. Mecheri, On minimizing $\|S - (AX - XB)\|_p$, Serdica Math. J. 26 (2000), no. 2, 119-126. MR2001j:47033
- [7] S. Mecheri, On the orthogonality in von Neumann-Shatten classes, Int. Jour. Appl. Math. 8(2002), 441-447. MR2003b:47063
- [8] S. Mecheri and A. Bachir, Generalized derivation modulo the ideal of all compact operators, Int. Jour. Math. Math. Sc., 32.8(2002), 501-506. MR2003i:47038

DEPARTMENT OF MATHEMATICS, KING SAUD UNIVERSITY COLLEGE OF SCIENCE, P.O. BOX 2455, RIYADH 11451, SAUDI ARABIA

E-mail address: mecherisalah@hotmail.com