

BLOCH SPACE IN THE UNIT BALL OF \mathbb{C}^n

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ABSTRACT. In this paper we obtain higher-dimensional versions of the Holland-Walsh characterization of the Bloch space and the Stroethoff characterization of the little Bloch space.

1. INTRODUCTION

The purpose of this paper is to generalize the characterizations of the Bloch space and the little Bloch space in the unit disc by Holland-Walsh [6] and Stroethoff [16] to higher dimensions.

Let \mathbb{B} denote the unit ball in \mathbb{C}^n . For any holomorphic function f on \mathbb{B} and any $z \in \mathbb{B}$, set

$$Q_f(z) = \sup_{0 \neq x \in \mathbb{C}^n} \frac{|\langle \nabla f(z), \bar{x} \rangle|}{(H_z(x, x))^{1/2}},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on \mathbb{C}^n , $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z))$ is the complex gradient of f and $H_z(x, x)$ is the Bergman metric in \mathbb{B} , i.e.,

$$H_z(x, x) = \frac{n+1}{2} \frac{(1-|z|^2)|x|^2 + |\langle x, z \rangle|^2}{(1-|z|^2)^2}.$$

As introduced by Timoney in [17] and [18], the Bloch space \mathcal{B} is the set of holomorphic functions f on \mathbb{B} such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{B}} Q_f(z) < \infty,$$

and the little Bloch space \mathcal{B}_0 is the set of holomorphic functions f on \mathbb{B} such that

$$\lim_{|z| \rightarrow 1^-} Q_f(z) = 0.$$

We refer to [17], [18], [13], [7], [5], [4], [10], and [8] for the various characterizations of the Bloch and little Bloch spaces in the unit ball of \mathbb{C}^n . For example, for any holomorphic function f on \mathbb{B} (see [17], [18]),

- (i) $f \in \mathcal{B}$ if and only if $\sup_{z \in \mathbb{B}} (1-|z|^2)|\nabla f(z)| < \infty$;
- (ii) $f \in \mathcal{B}_0$ if and only if $(1-|z|^2)|\nabla f(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$.

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In 1986 Holland and Walsh [6] gave the following characterization for the Bloch space $\mathcal{B}(\mathbb{D})$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem A. *For any holomorphic function f in \mathbb{D} , $f \in \mathcal{B}(\mathbb{D})$ if and only if*

$$(1.1) \quad \sup_{\substack{z, w \in \mathbb{D} \\ z \neq w}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \left| \frac{f(z) - f(w)}{z - w} \right| < \infty.$$

With the Möbius transformation as a main tool, Stroethoff [16] gave an elementary proof of this theorem. Recently, using the same approach, Nowak [9] generalized this result to the Bloch space \mathcal{B} in the unit ball \mathbb{B} of \mathbb{C}^n :

Theorem B. *For any holomorphic function f in \mathbb{B} , $f \in \mathcal{B}$ if and only if*

$$(1.2) \quad \sup_{\substack{z, w \in \mathbb{B} \\ z \neq w}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|w - P_w z - s_w Q_w z|} < \infty,$$

where P_w is the orthogonal projection of \mathbb{C}^n onto the subspace spanned by w , $Q_w = I - P_w$ and $s_w = (1 - |w|^2)^{1/2}$. Note that I is, as usual, the identity operator.

Notice that when $n = 1$, we have $P_w = I$ and $Q_w = 0$, so that the denominator in (1.2) is exactly $|w - z|$. Hence, Theorem B is a generalization of Theorem A.

In view of (1.1) and (1.2), a natural question arises whether the following condition gives the characterization of Bloch space in the unit ball:

$$(1.3) \quad \sup_{\substack{z, w \in \mathbb{B} \\ z \neq w}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

An affirmative answer is obtained in this paper; see Theorem 3.1.

Analogous to the Holland-Walsh characterization for the Bloch space, Stroethoff [16] gave a membership criterion for the little Bloch space, which is analogous to Theorem A. We will also extend the result of Stroethoff to higher dimensions; see Theorem 3.2.

2. NOTATION AND PRELIMINARIES

We shall use real techniques to deal with holomorphic functions. For this reason, we identify \mathbb{C}^n with \mathbb{R}^m ($m = 2n$). In general, for any $x \in \mathbb{R}^m$ we write $x = |x|x'$ in polar coordinates, where $x' \in \partial\mathbb{B}$. Especially when $x = 0$, we set $x' = (1, 0, \dots, 0)$.

By the symmetric lemma,

$$(2.1) \quad \|y|w - y'\| = \|w|y - w'\|, \quad \forall y, w \in \mathbb{R}^m,$$

which can be verified by squaring both sides and expanding through the inner product. The same reasoning leads to

$$\|y|w - (1 - |w|^2)y'\| = \|w|y - (1 - |w|^2)w'\|,$$

so that

$$(2.2) \quad \|y\|^2 w - (1 - |w|^2)y = |y| \|w|y - (1 - |w|^2)w'\|.$$

We regard \mathbb{B} as the real unit ball in \mathbb{R}^m . For any $a \in \mathbb{B}$, denote by φ_a the Möbius transformation in the real unit ball in \mathbb{R}^m . It is an involution automorphism of \mathbb{B} such that $\varphi_a(0) = a$ and $\varphi_a(a) = 0$, which is of the form

$$(2.3) \quad \varphi_a(x) = \frac{|x - a|^2 a - (1 - |a|^2)(x - a)}{||a|x - a'|^2}, \quad a, x \in \mathbb{B}.$$

We refer the reader to [1] for further properties of the Möbius transformations in the real unit ball.

For any $a, x \in \mathbb{B}$, from (2.3) and (2.2) with $w = a$ and $y = x - a$ we have

$$(2.4) \quad |\varphi_a(x)| = \frac{|x - a|}{||a|x - a'|},$$

such that

$$(2.5) \quad 1 - |\varphi_a(x)|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{||a|x - a'|^2}.$$

Combining (2.4) with (2.5), we immediately get the following identity.

Lemma 2.1. *For any $z, w \in \mathbb{B}$ with $z \neq w$,*

$$(2.6) \quad \frac{1 - |\varphi_z(w)|^2}{|\varphi_z(w)|^2} = \frac{(1 - |z|^2)(1 - |w|^2)}{|w - z|^2}.$$

For any $a \in \mathbb{B}$ and $\delta \in (0, 1)$, we denote

$$E(a, \delta) = \{x \in \mathbb{B} : |\varphi_a(x)| < \delta\},$$

$$B(a, \delta) = \{x \in \mathbb{B} : |x - a| < \delta\}.$$

Clearly, $E(a, \delta) = \varphi_a(B(0, \delta))$. It is easy to see that

$$(2.7) \quad B(a, \frac{\delta}{2}(1 - |a|^2)) \subset E(a, \delta).$$

In fact, for any $x \in B(a, \frac{\delta}{2}(1 - |a|^2))$, from (2.4) we have

$$|\varphi_a(x)| = \frac{|x - a|}{||a|x - a'|} \leq \frac{|x - a|}{|a'| - |a||x|} \leq \frac{|x - a|}{1 - |a|} < \delta,$$

which implies $x \in E(a, \delta)$, as desired.

In the real unit ball \mathbb{B} of \mathbb{R}^m , we consider the measure

$$d\tau(w) = (1 - |w|^2)^{-m} dw,$$

where dw is the normalized Lebesgue measure on \mathbb{B} . It is an invariant measure on \mathbb{B} under the Möbius transformations in (2.3) (see [1]). We will apply the fact that $\tau(E(a, \delta))$ is independent of $a \in \mathbb{B}$; indeed,

$$(2.8) \quad \tau(E(a, \delta)) = \tau(B(0, \delta)) = m \int_0^\delta t^{m-1} (1 - t^2)^{-m} dt.$$

As usual, constants are denoted by the same letter C , which will be independent of the particular functions under consideration. We will often indicate variables in the subscript on which C depends.

3. MAIN RESULTS

In this section, we give a natural extension of the Holland-Walsh characterization of the Bloch space to the unit ball. An analogous little Bloch version also holds true. Our main results are the following theorems.

Theorem 3.1. *For any holomorphic function f on \mathbb{B} , $f \in \mathcal{B}$ if and only if*

$$(3.1) \quad S(f) := \sup_{\substack{z, w \in \mathbb{B} \\ z \neq w}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

Moreover, the two seminorms $\sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)|$ and $S(f)$ are equivalent.

Theorem 3.2. *For any holomorphic function f on \mathbb{B} , $f \in \mathcal{B}_0$ if and only if*

$$(3.2) \quad \lim_{|z| \rightarrow 1^-} \sup_{\substack{w \in \mathbb{B} \\ w \neq z}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} = 0.$$

Proof of Theorem 3.1. Assume that $f \in \mathcal{B}$. For any $z, w \in \mathbb{B}$, we have

$$\begin{aligned} f(z) - f(w) &= \int_0^1 \frac{df}{dt}(tz + (1 - t)w) dt \\ &= \sum_{k=1}^n (z_k - w_k) \int_0^1 \frac{\partial f}{\partial z_k}(tz + (1 - t)w) dt. \end{aligned}$$

By the Cauchy-Schwarz inequality and the obvious inequality $|\frac{\partial f}{\partial z_k}| \leq |\nabla f|$, we have

$$\begin{aligned} |f(z) - f(w)| &\leq \sqrt{\sum_{k=1}^n |z_k - w_k|^2} \sqrt{\sum_{k=1}^n \left(\int_0^1 \left| \frac{\partial f}{\partial z_k}(tz + (1 - t)w) \right| dt \right)^2} \\ &\leq |z - w| \sqrt{n} \int_0^1 |\nabla f|(tz + (1 - t)w) dt. \end{aligned}$$

By the result of Timoney [17], the seminorms $\sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)|$ and $\|f\|_{\mathcal{B}}$ are equivalent. Thus, $(1 - |tz + (1 - t)w|^2) |\nabla f(tz + (1 - t)w)| \leq C \|f\|_{\mathcal{B}}$ for some absolute constant $C > 0$. This implies

$$\frac{|f(z) - f(w)|}{|z - w|} \leq \sqrt{n} C \|f\|_{\mathcal{B}} \int_0^1 \frac{1}{1 - |tz + (1 - t)w|^2} dt.$$

Now by the triangle inequality, $|tz + (1 - t)w| \leq t|z| + (1 - t)|w|$, we have

$$\begin{aligned} 1 - |tz + (1 - t)w| &\geq 1 - t|z| - (1 - t)|w| \\ &= (1 - t)(1 - |w|) + t(1 - |z|). \end{aligned}$$

Thus for any $0 < t < 1$ and $z, w \in \mathbb{B}$, we have $1 - |tz + (1 - t)w| \geq (1 - t)(1 - |w|)$ and $1 - |tz + (1 - t)w| \geq t(1 - |z|)$, such that

$$1 - |tz + (1 - t)w| \geq \sqrt{(1 - t)(1 - |w|)} \sqrt{t(1 - |z|)}.$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{1}{1 - |tz + (1 - t)w|} dt &\leq \int_0^1 \frac{1}{\sqrt{(1 - t)(1 - |w|)} \sqrt{t(1 - |z|)}} dt \\ &= \pi \frac{1}{(1 - |w|)^{1/2} (1 - |z|)^{1/2}}. \end{aligned}$$

Noticing that $1 - |tz + (1 - t)w|^2 \geq 1 - |tz + (1 - t)w|$ for any $0 < t < 1$ and $z, w \in \mathbb{B}$, we finally obtain

$$(3.3) \quad (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} \leq \pi\sqrt{n}C\|f\|_{\mathcal{B}}.$$

This proves the necessity.

Now suppose that f is holomorphic and (3.1) is satisfied. We will show that $f \in \mathcal{B}$.

We identify \mathbb{C}^n with \mathbb{R}^{2n} and adopt the notation from section 2. Fix $\delta \in (0, 1)$. Since f is harmonic, it follows from a well-known result for harmonic functions (see [14, Appendix C. 3] or [11, p. 504]) that

$$(1 - |z|^2)|\nabla f(z)| \leq C \int_{B(z, \frac{\delta}{2}(1 - |z|^2))} |f(w)|d\tau(w)$$

for any $z \in \mathbb{B}$. Combining this result with (2.7), we have

$$(1 - |z|^2)|\nabla f(z)| \leq C \int_{E(z, \delta)} |f(w)|d\tau(w), \quad \forall z \in \mathbb{B}.$$

Now fixing $z \in \mathbb{B}$ and replacing f by $f - f(z)$, we get

$$(3.4) \quad (1 - |z|^2)|\nabla f(z)| \leq C \int_{E(z, \delta)} |f(w) - f(z)|d\tau(w).$$

Therefore, from (2.8),

$$(1 - |z|^2)|\nabla f(z)| \leq C \sup_{\substack{w \in E(z, \delta) \\ w \neq z}} |f(w) - f(z)| = C \sup_{\substack{w \in E(z, \delta) \\ w \neq z}} |f(w) - f(z)|.$$

Notice that for any $w \in E(z, \delta)$, we have $|\varphi_z(w)| \leq \delta$, such that

$$\frac{\sqrt{1 - |\varphi_z(w)|^2}}{|\varphi_z(w)|} \geq \frac{\sqrt{1 - \delta^2}}{\delta}.$$

It follows from Lemma 2.1 that

$$(3.5) \quad \frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|w - z|} \geq \frac{\sqrt{1 - \delta^2}}{\delta}, \quad \forall w \in E(z, \delta).$$

Consequently,

$$(3.6) \quad (1 - |z|^2)|\nabla f(z)| \leq C \sup_{\substack{w \in E(z, \delta) \\ w \neq z}} \frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|z - w|} |f(z) - f(w)|.$$

Since the supremum in (3.6) can be controlled by the quantity in (3.1), we have

$$(1 - |z|^2)|\nabla f(z)| \leq C S(f) < \infty$$

for any $z \in \mathbb{B}$. This fact implies that $f \in \mathcal{B}$.

From the above proof we can easily see that the two seminorms of the Bloch space \mathcal{B} , $\sup_{z \in \mathbb{B}} (1 - |z|^2)|\nabla f(z)|$ and $S(f)$, are equivalent. This completes the proof of Theorem 3.1. \square

Proof of Theorem 3.2. Assume that $f \in \mathcal{B}_0$. Let $f_t(z) = f(tz)$, $t \in (0, 1)$. By (3.3), we have

$$(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|(f - f_t)(z) - (f - f_t)(w)|}{|z - w|} \leq C\|f - f_t\|_{\mathcal{B}}$$

and

$$\begin{aligned} & (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f_t(z) - f_t(w)|}{|z - w|} \\ &= t \frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{(1 - |tz|^2)^{1/2}(1 - |tw|^2)^{1/2}} (1 - |tz|^2)^{1/2}(1 - |tw|^2)^{1/2} \frac{|f(tz) - f(tw)|}{|tz - tw|} \\ &\leq C \frac{t}{1 - t^2} (1 - |z|^2)^{1/2} \|f\|_{\mathcal{B}}. \end{aligned}$$

By the triangle inequality, we thus obtain

$$\sup_{\substack{w \in \mathbb{B} \\ w \neq z}} (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} \leq C \frac{t}{1 - t^2} (1 - |z|^2)^{1/2} \|f\|_{\mathcal{B}} + \|f - f_t\|_{\mathcal{B}}.$$

In the above inequality, first letting $|z| \rightarrow 1^-$, the first term on the right side converges to 0, and then letting $t \rightarrow 1^-$, the second term on the right side also converges to 0 (see [18]).

Now suppose that f is holomorphic and (3.2) is satisfied. We will show that $f \in \mathcal{B}_0$. By (3.2), for any given $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\sup_{\substack{w \in \mathbb{B} \\ w \neq z}} (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} < \epsilon,$$

whenever $|z| > \delta$. In particular,

$$\sup_{\substack{w \in E(z, \delta) \\ w \neq z}} (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} < \epsilon,$$

whenever $|z| > \delta$. Combining this with (3.6), we get

$$(1 - |z|^2)|\nabla f(z)| < C\epsilon$$

for any $|z| > \delta$, which means $(1 - |z|^2)|\nabla f(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$. This completes the proof. \square

4. EXTENSIONS OF THEOREMS 3.1 AND 3.2

We conclude the paper by giving extensions of Theorems 3.1 and 3.2.

Theorem 4.1. *Let f be a continuously differentiable function on \mathbb{B} satisfying*

$$|\nabla f(z)| \leq C_n \frac{1}{(1 - |z|^2)^{2n+1}} \int_{B(z, \frac{1}{4}(1 - |z|^2))} |f(w)| dw$$

for any $z \in \mathbb{B}$. Then

$$\sup_{z \in \mathbb{B}} (1 - |z|^2)|\nabla f(z)| < \infty$$

if and only if

$$\sup_{\substack{z, w \in \mathbb{B} \\ z \neq w}} (1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

Theorem 4.2. *Let f be a continuously differentiable function on \mathbb{B} satisfying*

$$|\nabla f(z)| \leq C_n \frac{1}{(1 - |z|^2)^{2n+1}} \int_{B(z, \frac{1}{4}(1 - |z|^2))} |f(w)| dw$$

for any $z \in \mathbb{B}$. Then

$$\lim_{|z| \rightarrow 1^-} \sup_{z \in \mathbb{B}} (1 - |z|^2) |\nabla f(z)| = 0$$

if and only if

$$\lim_{|z| \rightarrow 1^-} \sup_{\substack{w \in \mathbb{B} \\ w \neq z}} (1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2} \frac{|f(z) - f(w)|}{|z - w|} = 0.$$

Since the proofs of Theorems 4.1 and 4.2 are similar to the proofs of Theorems 3.1 and 3.2, respectively, they are here omitted.

Example 4.3. Any \mathcal{M} -harmonic function or more generally $\Delta_{\alpha, \beta}$ -harmonic function satisfies the condition of Theorems 4.1 and 4.2 ([3, p. 675]).

Remark 4.4. One anonymous referee suggested another approach to the proof of the necessity of Theorem 3.1. More precisely, from the maximum modulus theorem he showed that for any function f holomorphic in \mathbb{B} and $z \in \mathbb{B}$,

$$\begin{aligned} & (1 - |z|^2) |\nabla f(z)| \\ & \leq C_n \max\{|f(w) - f(z)| : w \in B(z, \frac{1}{4}(1 - |z|^2))\} \\ & = C_n \max\{|f(w) - f(z)| : \text{for all } w \text{ so that } |w - z| = \frac{1}{4}(1 - |z|^2)\} \\ & \leq 8C_n \max\{|f(w) - f(z)| \frac{(1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2}}{|w - z|} : |w - z| = \frac{1}{4}(1 - |z|^2)\} \\ & \leq 8C_n \max\{|f(w) - f(z)| \frac{(1 - |z|^2)^{1/2} (1 - |w|^2)^{1/2}}{|w - z|} : \text{for all } w \in \mathbb{B}\}. \end{aligned}$$

Thus (3.1) implies $f \in \mathcal{B}$.

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