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THE FEFFERMAN-STEIN TYPE INEQUALITY FOR THE KAKEYA MAXIMAL OPERATOR IN WOLFF'S RANGE

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Abstract. Let K_δ, $0<\delta\ll 1$, be the Kakeya (Nikodým) maximal operator defined as the supremum of averages over tubes of eccentricity δ. The (so-called) Fefferman-Stein type inequality:

$$\|K_\delta f\|_{L^p(\mathbf{R}^d,w)} \le C(1/\delta)^{d/p-1}(\log(1/\delta))^\alpha \|f\|_{L^p(\mathbf{R}^d,K_\delta w)}$$

is shown in the range $1<p\le (d+2)/2$, where C and α are some constants depending only on p and the dimension d and w is a weight. The result is a sharp bound up to $\log(1/\delta)$-factors.

1. Introduction

In this paper we will discuss the weighted version of the Kakeya problem in Wolff's range.[1] Throughout this paper δ, $0<\delta\ll 1$, will be a small parameter. For a locally integrable function f on $\mathbf{R}^d$, $d>1$, the Kakeya (Nikodým) maximal operator K_δ is defined by

$$K_\delta f(x)=\sup_{T\ni x}\frac{1}{|T|}\int_T |f|\,dy,$$

where the supremum is taken over all tubes T containing $x\in\mathbf{R}^d$ with the property that the eccentricity (the ratio of the radius of the cross section and the length of the axis) is equal to δ. A weight w is a non-negative locally integrable function on $\mathbf{R}^d$. We will represent the norm of the function space $L^p(\mathbf{R}^d,w)$ as

$$\|f\|_{L^p(\mathbf{R}^d,w)}=\left(\int_{\mathbf{R}^d}|f(x)|^p w(x)\,dx\right)^{1/p},\quad p>1.$$

$w(A)$ will denote the $w(x)\,dx$ measure of a set A.

It is conjectured that the (so-called) Fefferman-Stein type inequality

$$\|K_\delta f\|_{L^d(\mathbf{R}^d,w)} \le C(\log(1/\delta))^\alpha \|f\|_{L^d(\mathbf{R}^d,K_\delta w)} \tag{1}$$

holds with some constants C and α depending only on the dimension d. This conjecture is known to be true for $d=2$ (D. Müller and F. Soria, [2]) and for $d>2$

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[1] The developments on the Kakeya problem in recent years are summarized in [4] and [12].

and radial functions (H. Tanaka [6]), but is still open for other cases. This is a weighted version of the long-standing conjecture of A. Córdoba [1].

By using the Fefferman-Stein inequality for the Hardy-Littlewood maximal operator M and the simple majorizations $C_1(Mf) \le K_\delta f \le C_2(1/\delta)^{d-1}(Mf)$, one sees that

$$\lambda\, w(\{x : K_\delta f(x) > \lambda\}) \le C(1/\delta)^{d-1}\|f\|_{L^1(\mathbf{R}^d, K_\delta w)}, \quad \forall\lambda > 0. \tag{2}$$

Interpolating the conjectured bound (1) with the bound (2), one obtains the equivalent conjecture

$$\|K_\delta f\|_{L^p(\mathbf{R}^d, w)} \le C(1/\delta)^{d/p-1}(\log(1/\delta))^\alpha \|f\|_{L^p(\mathbf{R}^d, K_\delta w)}, \quad 1 < p \le d. \tag{3}$$

The main difficulty of this problem lies in making the exponent p as high as possible.

The first achievement in this problem was given by A. Vargas [10], who showed that (3) is true in the range $1 < p \le (d+1)/2$. Her proof was based on the weighted estimate for a "bush". This result was improved by the author [7], who showed that (3) is true in the range $1 < p \le (d^2-2)/(2d-3)$. The proof was based on the weighted estimate for a "hairbrush". In this paper we shall give a further improvement.

Theorem 1. *Equation* (3) *is true in the range* $1 < p \le (d+2)/2$.

Table 2. *Known results for the upper bound of the range*

$d = 3$	$d = 4$	$d = d$	
2	5/2	$(d+1)/2$	*A. Vargas, 1994,* [10],
7/3	14/5	$(d^2-2)/(2d-3)$	*H. Tanaka, 2001,* [7],
5/2	3	$(d+2)/2$	*our result,*
3	4	d	*conjecture.*

An unweighted case of Theorem 1 (putting $w \equiv 1$) corresponds to Wolff's result (T. Wolff, [11]). However, our result is a sharp bound up to $\log(1/\delta)$-factors.

In [8], the author showed a weaker result than that of Theorem 1. The reason why he could not reach Wolff's range is that the inductive argument in [11], which relies on a simple scaling argument, seems difficult to apply to the weighted setting. Instead of this inductive argument we will use a weighted inequality for the Sogge auxiliary maximal function, which was used in the problem of the estimates for the Nikodým maximal function on three-dimensional curved spaces (C. Sogge, [3]).

Reduction of the problem. We make some reduction of the problem. We write $X \overset{\sim}{<} Y$, $Y \overset{\sim}{>} X$ if there is a constant C such that $X \le CY$. The constant C must depend only on the dimension d but may vary from line to line. We write further $X \sim Y$ if $X \overset{\sim}{<} Y$ and $X \overset{\sim}{>} Y$.

If $\chi_{\mathcal{A}}$ is the indicator function of a compact set $\mathcal{A} \subset \mathbf{R}^d$, define the maximal function $(\chi_{\mathcal{A}})_\delta(x)$ by

$$(\chi_{\mathcal{A}})_\delta(x) = \sup_{T \ni x} \frac{|\mathcal{A} \cap T|}{|T|},$$

where the supremum is taken over all δ-tubes T that contain $x \in \mathbf{R}^d$ and whose axes make an angle less than $\pi/8$ with the d-th coordinate direction e_d. (See [11].) Hereafter, a δ-tube is the tube with length 1 and radius of cross section δ. Let $Q^d = (-1,\ 1)^d$ and $\widetilde{Q}^d = (-1/2,\ 1/2)^d$. We divide Q^d into δ-cubes Q_i centered at $i \in \mathbf{I}$, where $\mathbf{I}$ is the set of lattice points with the δ-separation. Our result can then be reduced to the following (discrete) proposition by using (sieve) arguments and the three-points interpolation lemma. (See [2], [10] and also [6].)

Proposition 3. *Let* $d > 2$ *and* $\mathcal{A} \subset \widetilde{Q}^d$. *Then for any* λ, $0 < \lambda < 1$, *we have*

$$\lambda^{(d+2)/2}\delta^{(d-2)/2}\sum_{i\in I} w(Q_i) \overset{\sim}{<} (\log(1/\delta))^{(d+5)/2} K_\delta w(\mathcal{A}),$$

where $I = \{i \in \mathbf{I} : (\chi_{\mathcal{A}})_\delta(i) > \lambda\}$.

Lemmas. We will use the following lemmas. Lemma 4 corresponds to L^2-bounds for the Sogge auxiliary maximal function.

Given any line L in $\mathbf{R}^d$ and a set $\mathcal{E} \subset \widetilde{Q}^d$, define the maximal function $(\chi_{\mathcal{E}})^L_\delta(x)$ by

$$(\chi_{\mathcal{E}})^L_\delta(x) = \sup_{T\ni x} \frac{|\mathcal{E} \cap T|}{|T|},$$

where the supremum is taken over all δ-tubes T that contain x and whose axes intersect L.

Lemma 4. *Given* σ, $2\delta < \sigma < 1$, *and* ρ, $\delta < \rho < \sigma/2$, *if* $\mathcal{E} \subset \{y : \rho < \operatorname{dist}(y, L) < 2\rho\} \cap \widetilde{Q}^d$, *then for any* λ, $0 < \lambda < 1$, *we have*

$$\lambda^2 \sum_{i\in I} w(Q_i) \overset{\sim}{<} (\log(1/\delta))^2 (\sigma/\rho)^{d-2} K_\delta w(\mathcal{E}),$$

where $I = \{i \in \mathbf{I} \cap \{y : \operatorname{dist}(y, L) < \sigma\} : (\chi_{\mathcal{E}})^L_\delta(i) > \lambda\}$.

This lemma is just a minor modification of Theorem 2 in [6]. We postpone the proof until the end of the next section.

Let $\mathcal{B}_{\geq\delta}$ be the class of all rectangles in $\mathbf{R}^d$ with the property that the ratio of the shortest and the longest sidelength is bigger than or equal to δ. The corresponding maximal function associated to this base $\mathcal{B}_{\geq\delta}$ will be denoted by $K_{\geq\delta}$.

Lemma 5 (Theorem 3 in [5])**.** *Given a weight* w, *there exist constants* C_1 *and* C_2 *depending only on the dimension* d *such that for any* $x \in \mathbf{R}^d$

$$C_1 K_\delta w(x) \leq K_{\geq\delta} w(x) \leq C_2 K_\delta w(x).$$

2. PROOF OF PROPOSITION 3

Write $W = K_\delta w$ and $l(\delta) = \log(1/\delta)$. Fix $\mathcal{A} \subset \widetilde{Q}^d$ and λ, $0 < \lambda < 1$. Recall $I = \{i \in \mathbf{I} : (\chi_{\mathcal{A}})_\delta(i) > \lambda\}$. Then for every $i \in I$ we can select a δ-tube T_i, which contains i and whose axis makes an angle less than $\pi/8$ with e_d, such that

$$|\mathcal{A} \cap T_i| > |T_i|\,\lambda. \tag{4}$$

It follows from (4) and the Schwarz inequality that

$$\begin{aligned}(\delta^{d-1}\lambda)^2 &\left(\sum_{i\in I} w(Q_i)\right)^2 \\ &\overset{\sim}{<} \left(\sum_{i\in I} w(Q_i)|\mathcal{A} \cap T_i|\right)^2 \sim \left(\int_{\mathcal{A}} \sum_{i\in I} w(Q_i)\chi_{T_i}\right)^2 \\ &\overset{\sim}{<} \left(\int \left(\sum_{i\in I} w(Q_i)\chi_{T_i}\right)^2 W^{-1}\right) \cdot W(\mathcal{A}).\end{aligned} \tag{5}$$

We shall estimate the quantity

$$\int \left(\sum_{i \in I} w(Q_i) \chi_{T_i} \right)^2 W^{-1} \tag{6}$$

by using the bilinearity introduced in [9].

Let $L(T_i)$ be the axis of T_i. We represent the point $\omega_1(T_i)$ by $\omega_1(T_i) = \Pi \cap L(T_i)$, where $\Pi = \{x \in \mathbf{R}^d : x_d = 0\}$. We represent further the point $\omega_2(T_i)$ by $\omega_2(T_i) = (\xi_1, \cdots, \xi_{d-1}, 0)$, where $(\xi_1, \cdots, \xi_d) \in S^{d-1}$ is the direction of $L(T_i)$. We will identify $\omega_1(T_i)$ and $\omega_2(T_i)$ with the points of $\mathbf{R}^{d-1}$ in the usual manner.

Let $I_1 = \{\omega_1(T_i) : i \in I\}$ and $I_2 = \{\omega_2(T_i) : i \in I\}$. Then without the loss of generality we may assume that I_1 and I_2 have the δ-separations. We will need the following geometric observation:

Lemma 6. *We have*

$$|T_i \cap T_{i'}| \overset{\sim}{<} \frac{\delta^d}{|\omega_2(T_i) - \omega_2(T_{i'})| + \delta}.$$

Moreover,

$$|\omega_2(T_i) - \omega_2(T_{i'})| + \delta \overset{\sim}{>} |\omega_1(T_i) - \omega_1(T_{i'})|$$

if $T_i \cap T_{i'} \neq \emptyset$.

This lemma is proved by using the fact that the axis of T_i makes an angle less than $\pi/8$ with e_d.

Let $Q^{2d-2} = (-1, 1)^{2d-2}$. We introduce a Whitney decomposition (see [9]). For each integer $j > 0$ such that $\delta \overset{\sim}{<} 2^{-j}$ we divide Q^{2d-2} into $2^{(2d-2)j}$ dyadic subcubes τ_k^j of sidelength 2^{1-j}. If τ_k^j and τ_l^j are two cubes with the same sidelength that are not adjacent but have adjacent parents, we say that these cubes are close and write $\tau_k^j \sim \tau_l^j$. We partition (6) as

$$\begin{aligned}(6) \quad \sim \textstyle\sum_j \sum_{k,l:\, \tau_k^j \sim \tau_l^j} \int &\left(\textstyle\sum_{i \in I:\, \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^j} w(Q_i) \chi_{T_i} \right) \\ &\cdot \left(\textstyle\sum_{i' \in I:\, \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_l^j} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.\end{aligned}$$

Observing the terms of j and l with the maximum values in the above expression, we can select j_0 and k' so that

$$\begin{aligned}(6) \quad \overset{\sim}{<} l(\delta) \textstyle\sum_k \int &\left(\textstyle\sum_{i \in I:\, \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}} w(Q_i) \chi_{T_i} \right) \\ &\cdot \left(\textstyle\sum_{i' \in I:\, \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0}} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.\end{aligned} \tag{7}$$

We notice that the separation of $\tau_k^{j_0}$ and $\tau_{k'}^{j_0}$ is $O(2^{-j_0})$. Observing the terms with the maximum value again, we can select T_{i_k} so that

$$\begin{aligned}(7) \overset{\sim}{<} \quad & l(\delta) \textstyle\sum_k \sum_{i \in I:\, \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}} w(Q_i) \\ \cdot \quad & \int_{T_{i_k}} \left(\textstyle\sum_{i' \in I:\, \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0}} w(Q_{i'}) \chi_{T_{i'}} \right) W^{-1}.\end{aligned} \tag{8}$$

Set

$$\begin{cases} I_k = \{i \in I : \omega_1(T_i) \times \omega_2(T_i) \in \tau_k^{j_0}\}, \\ I'_k = \{i' \in I : \omega_1(T_{i'}) \times \omega_2(T_{i'}) \in \tau_{k'}^{j_0} \text{ and } T_{i_k} \cap T_{i'} \neq \emptyset\}, \\ s(T_{i_k}) = \inf_{y \in T_{i_k}} W(y), \\ \sigma = 2^{-j_0} \overset{\sim}{>} \delta. \end{cases}$$

Then it follows from Lemma 6 that

$$(8) \overset{\sim}{<} l(\delta)\frac{\delta^d}{\sigma}\sum_k \sum_{i\in I_k} w(Q_i)\left(s(T_{i_k})^{-1}\sum_{i'\in I'_k} w(Q_{i'})\right). \tag{9}$$

The following observation is due to Sogge [3]. This will allow us to avoid the inductive argument in [11].

By the pigeon-hole principle we can select ρ, $\delta < \rho < \sigma/2$, and $I''_k \subset I'_k$, so that

$$(9) \overset{\sim}{<} l(\delta)^2\frac{\delta^d}{\sigma}\sum_k \sum_{i\in I_k} w(Q_i)\left(s(T_{i_k})^{-1}\sum_{i'\in I''_k} w(Q_{i'})\right) \tag{10}$$

and

$$|\mathcal{A}\cap T_{i'}\cap\{y:\, \rho<\operatorname{dist}(y,L(T_{i_k}))<2\rho\}>c_1 l(\delta)^{-1}|T_{i'}|\,\lambda, \quad \forall i'\in I''_k. \tag{11}$$

Let $N_k = s(T_{i_k})^{-1}\sum_{i'\in I''_k} w(Q_{i'})$. If

$$(6) \overset{\sim}{<} l(\delta)^2\frac{\delta^d}{\sigma}\sum_{k:\, N_k \overset{\sim}{<} \delta^{d/2}} N_k \sum_{i\in I_k} w(Q_i),$$

then we have

$$(6) \overset{\sim}{<} l(\delta)^2\delta^{d-1}\delta^{d/2}\sum_{i\in I} w(Q_i).$$

Inserting this expression into (5), we have

$$\lambda^2\delta^{(d-2)/2}\sum_{i\in I} w(Q_i) \overset{\sim}{<} l(\delta)^2 W(\mathcal{A}).$$

Thus, we may assume that $N_k \overset{\sim}{>} \delta^{d/2}$. We see also that $N_k \overset{\sim}{<} 1$ by Lemma 5. From these facts and the pigeon-hole principle again, there exists a number N such that

$$(6) \overset{\sim}{<} l(\delta)^3\frac{\delta^d}{\sigma}\sum_{k:\, N\le N_k\le 2N} N_k \sum_{i\in I_k} w(Q_i). \tag{12}$$

Put $K = \{k:\, N \le N_k \le 2N\}$. Then we now have the following claim.

Claim 7. *We have*

$$(6) \overset{\sim}{<} l(\delta)^3\frac{\delta^d}{\sigma}N\sum_{i\in I} w(Q_i). \tag{13}$$

To proceed, we shall estimate the quantity

$$\sum_{k\in K} N_k \sum_{i\in I_k} w(Q_i). \tag{14}$$

Let $T^{\sigma}_{i_k}$ be the tubes concentric with T_{i_k} and with length ~ 1, the radius of the cross section $\sim \sigma$. We now observe the geometric fact that

$$\bigcup_{i\in I_k} Q_i \subset T^{\sigma}_{i_k}.$$

From the definition of N_k, this observation and Lemma 5,

$$\begin{aligned} N_k \sum_{i\in I_k} w(Q_i) & \\ &\overset{\sim}{<} \sigma^{d-1} s(T_{i_k})^{-1} \frac{w(T^{\sigma}_{i_k})}{|T^{\sigma}_{i_k}|} \sum_{i'\in I''_k} w(Q_{i'}) \\ &\overset{\sim}{<} \sigma^{d-1} \sum_{i'\in I''_k} w(Q_{i'}). \end{aligned} \tag{15}$$

Define the cylindric sets

$$D^{\rho}_k = T^{\sigma}_{i_k} \cap \{y :\, \rho < \operatorname{dist}(y, L(T_{i_k})) < 2\rho\}.$$

Applying Lemma 4 with $\mathcal{E} = \mathcal{A} \cap D^{\rho}_k$ and $L = L(T_{i_k})$, we obtain that

$$\lambda^2 \sum_{i'\in I''_k} w(Q_{i'}) \overset{\sim}{<} l(\delta)^4 (\sigma/\rho)^{d-2} W(\mathcal{A} \cap D^{\rho}_k) \tag{16}$$

by (11). From (15) and (16),

$$(14) \overset{\sim}{<} (1/\lambda)^2 l(\delta)^4 \sigma^{d-1} (\sigma/\rho)^{d-2} \sum_{k\in K} W(\mathcal{A} \cap D^{\rho}_k). \tag{17}$$

The inequalities (12), (14) and (17) imply

$$(6) \overset{\sim}{<} (1/\lambda)^2 \delta^d l(\delta)^7 \sigma^{d-2} (\sigma/\rho)^{d-2} \sum_{k\in K} W(\mathcal{A} \cap D^{\rho}_k). \tag{18}$$

The final step is a bush-type analysis.

Rewrite

$$\sum_{k\in K} W(\mathcal{A} \cap D^{\rho}_k) = \int_{\mathcal{A}} \left(\sum_{k\in K} \chi_{D^{\rho}_k} \right) W.$$

Observing the points with the maximum value, we can select a point $a \in \mathcal{A}$ so that

$$W(\mathcal{A})^{-1} \sum_{k\in K} W(\mathcal{A} \cap D^{\rho}_k) \overset{\sim}{<} \operatorname{card}(\{k \in K :\, a \in D^{\rho}_k\}). \tag{19}$$

Putting $s = \min_{k\in K} s(T_{i_k})$, we have

$$Ns \overset{\sim}{<} \sum_{i'\in I''_k} w(Q_{i'}), \quad k \in K. \tag{20}$$

Applying Lemma 4 with $\mathcal{E} = \mathcal{A} \cap D^{\rho}_k \setminus B(a,\ c_1 l(\delta)^{-1}\lambda/3)$ and $L = L(T_{i_k})$, we obtain that

$$\lambda^2 \sum_{i'\in I''_k} w(Q_{i'}) \overset{\sim}{<} l(\delta)^4 (\sigma/\rho)^{d-2} W\left(\mathcal{A} \cap D^{\rho}_k \setminus B\left(a,\ c_1 l(\delta)^{-1}\lambda/3\right)\right) \tag{21}$$

by (11). The inequalities (20) and (21) imply the following claim.

Claim 8. *We have*

$$Ns \overset{\sim}{<} (1/\lambda)^2 l(\delta)^4 (\sigma/\rho)^{d-2} W\left(\mathcal{A} \cap D^{\rho}_k \setminus B\left(a,\ c_1 l(\delta)^{-1}\lambda/3\right)\right), \quad k \in K. \tag{22}$$

From (19), (22) and the fact that the set $\{D_k^\rho : a \in D_k^\rho,\ k \in K\}$ has the σ-separation in the direction sides

$$\begin{aligned} &W(\mathcal{A})^{-1} \sum_{k\in K} W(\mathcal{A} \cap D_k^\rho) \\ &\quad \overset{\sim}{<} (Ns)^{-1}(1/\lambda)^2 l(\delta)^4 (\sigma/\rho)^{d-2} \sum_{k\in K:\, a\in D_k^\rho} W\left(\mathcal{A} \cap D_k^\rho \setminus B\left(a,\ c_1 l(\delta)^{-1}\lambda/3\right)\right) \\ &\quad \overset{\sim}{<} (Ns)^{-1}(1/\lambda)^{d+1} l(\delta)^{d+3} (\rho/\sigma)\, W(\mathcal{A}). \end{aligned}$$

Hence we reach that

$$\sum_{k\in K} W(\mathcal{A} \cap D_k^\rho) \overset{\sim}{<} (Ns)^{-1}(1/\lambda)^{d+1} l(\delta)^{d+3} (\rho/\sigma)\, W(\mathcal{A})^2, \tag{23}$$

and we have the following claim by (18) and (23).

Claim 9. *We have*

$$(6) \overset{\sim}{<} (Ns)^{-1}(1/\lambda)^{d+3}\delta^d l(\delta)^{d+10} \sigma^{d-2} (\sigma/\rho)^{d-3}\, W(\mathcal{A})^2. \tag{24}$$

Completion of the proof. First, we notice the fact that $\sigma/\rho \overset{\sim}{<} l(\delta)(1/\lambda)$ by (11) and a simple geometric observation.

<u>CASE 1:</u> If $\sigma \geq s^{-1/(d-1)} \left(\sum_{i\in I} w(Q_i)\right)^{1/(d-1)}$, then (5), (13) and (22) yield

$$\sigma s \lambda^{d+2} \delta^{d-2} \sum_{i\in I} w(Q_i) \overset{\sim}{<} l(\delta)^{d+5} W(\mathcal{A})^2,$$

and hence

$$\lambda^{(d+2)/2} \delta^{(d-2)/2} s^{(d-2)/(2d-2)} \left(\sum_{i\in I} w(Q_i)\right)^{d/(2d-2)} \overset{\sim}{<} l(\delta)^{(d+5)/2} W(\mathcal{A}). \tag{25}$$

<u>CASE 2:</u> If $\sigma \leq s^{-1/(d-1)} \left(\sum_{i\in I} w(Q_i)\right)^{1/(d-1)}$, then (13) and (24) yield

$$(6) \overset{\sim}{<} s^{-1/2} \left(\sum_{i\in I} w(Q_i)\right)^{1/2} (1/\lambda)^d \delta^d l(\delta)^{d+5} \sigma^{(d-3)/2} W(\mathcal{A}),$$

and hence we have the same inequality as (25) by (5).

Lastly, we notice the fact that $\sum_{i\in I} w(Q_i) \overset{\sim}{<} s$ by Lemma 5. Thus, we obtain the desired inequality

$$\lambda^{(d+2)/2} \delta^{(d-2)/2} \sum_{i\in I} w(Q_i) \overset{\sim}{<} l(\delta)^{(d+5)/2} W(\mathcal{A}). \qquad \square$$

Remark 10. If we put $w \equiv 1$ in (25), then we recover the $L^p \to L^q$ bound of Wolff. But, our result is a sharp bound up to $\log(1/\delta)$-factors.

Proof of Lemma 4. Let $F = \{y : \rho < \operatorname{dist}(y, L) < 2\rho\} \cap \widetilde{Q}^d$ and for μ, $\delta < \mu < \sigma/2$, let $J = \{i \in \mathbf{I} \cap \{y : \mu < \operatorname{dist}(y, L) < 2\mu\} : (\chi_{\mathcal{E}})^L_\delta(i) > \lambda\}$. We first notice that the lemma clearly follows from the dyadic estimate

$$\lambda^2 \sum_{i \in J} w(Q_i) \overset{\sim}{<} l(\delta)(\sigma/\rho)^{d-2} K_\delta w(\mathcal{E}). \tag{26}$$

Just the same as at the beginning of this section, for every $i \in J$ we can select a δ-tube T_i, which contains i and whose axis intersects L, such that $|\mathcal{E} \cap T_i| > |T_i|\,\lambda$ and we have

$$(\delta^{d-1}\lambda)^2 \left(\sum_{i \in J} w(Q_i)\right)^2 \overset{\sim}{<} \left(\int_F \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right)^2 (K_\delta w)^{-1}\right) \cdot K_\delta w(\mathcal{E}). \tag{27}$$

We shall estimate the quantity

$$\int_F \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right)^2 (K_\delta w)^{-1}. \tag{28}$$

Observing the terms with the maximum value, we can select T_{i_0} so that

$$(28) \overset{\sim}{<} \left(\sum_{i \in J} w(Q_i)\right) \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right) (K_\delta w)^{-1}. \tag{29}$$

Putting $s_0 = \inf_{y \in F \cap T_{i_0}} K_\delta w(y)$, we see

$$\int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right) (K_\delta w)^{-1} \overset{\sim}{<} s_0^{-1} \int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right). \tag{30}$$

Now we claim that

$$\int_{F \cap T_{i_0}} \left(\sum_{i \in J} w(Q_i)\chi_{T_i}\right) \overset{\sim}{<} \delta^{2d-2} l(\delta)(\sigma/\rho)^{d-2} s_0. \tag{31}$$

Clearly (31) with (27)–(30) imply (26). We shall divide the proof into two cases.

THE CASE $\mu \geq \rho$: Choose two-planes Π_k containing L so that any point $i \in J$ belongs to at most ~ 1 Π_k^δ's, where Π_k^δ is the δ-neighbourhood of Π_k. We partition the left-hand side of (31) as

$$\sum_k \int_{F \cap T_{i_0}} \left(\sum_{i \in J \cap \Pi_k^\delta} w(Q_i)\chi_{T_i}\right).$$

By a simple geometric observation for every k and $i \in J \cap \Pi_k^\delta$, we can select $\widetilde{\Pi}_k^\delta$ of the ($\sim \delta$)-neighbourhood of Π_k so that $F \cap T_i \subset \widetilde{\Pi}_k^\delta$ and $Q_i \subset \widetilde{\Pi}_k^\delta$. We can also select a δ-tube $\widetilde{T}_{i_0}$ so that $F \cap T_{i_0} \subset \widetilde{T}_{i_0} \subset \widetilde{\Pi}_k^\delta$. We see further that the multiplicity of $\widetilde{\Pi}_k^\delta$'s on $F \cap T_{i_0}$ is at most $\sim (\mu/\rho)^{d-2}$.

For $l, m \in \mathbf{Z} \cap [1,\ \nu]$, $\nu \sim (1/\delta)$, let $\gamma_l = \{y \in \Pi_k^\delta : (l-1)\delta \leq \mathrm{dist}(y, \widetilde{T}_{i_0}) < l\delta\}$ and $\Gamma_m = \{y \in \widetilde{\Pi}_k^\delta : \mathrm{dist}(y, \widetilde{T}_{i_0}) < m\delta\}$. Then we have

$$\begin{aligned}
&\int_{F\cap T_{i_0}} \left(\sum_{i \in J \cap \Pi_k^\delta} w(Q_i)\chi_{T_i} \right) \\
\sim\ & \sum_{l=1}^{\nu} \int_{F\cap T_{i_0}} \left(\sum_{i \in J\cap\gamma_l} w(Q_i)\chi_{T_i} \right) \\
\overset{\sim}{<}\ & \sum_{l=1}^{\nu} (\delta^{d-1}/l) \sum_{i\in J\cap\gamma_l} w(Q_i) \\
\sim\ & \delta^{d-1} \sum_{l=1}^{\nu} \left(\sum_{m=l}^{\nu} \frac{1}{m(m+1)} + \frac{1}{\nu+1} \right) \sum_{i \in J\cap\gamma_l} w(Q_i) \\
\overset{\sim}{<}\ & \delta^{2d-2} \left(\sum_{m=1}^{\nu} \frac{1}{m+1} \frac{w(\Gamma_m)}{|\Gamma_m|} + \frac{w(\Gamma_\nu)}{|\Gamma_\nu|} \right) \\
\overset{\sim}{<}\ & \delta^{2d-2} l(\delta) s_0,
\end{aligned}$$

where we used Lemma 5 and Córdoba's observation: $|\widetilde{T}_{i_0} \cap T_i| \overset{\sim}{<} \frac{\delta^d}{\mathrm{dist}(i, \widetilde{T}_{i_0}) + \delta}$. Thus, we obtain (31) in this case.

THE CASE $\mu < \rho$: Let $\widetilde{\Pi}_\delta$ be the ($\sim \delta$)-neighbourhood of the two-plane determined by L and the axis of T_{i_0}. By a simple geometric observation we notice that only the set that contributes to the left-hand side of (31) is $\{i : i \in J \cap \widetilde{\Pi}_\delta\}$. Thus, we have (31) by using the same inequality as (32).

References

[1] A. Córdoba, *The Kakeya maximal function and the spherical summation multiplier*, Amer. J. Math., **99**(1977), 1–22. MR 56:6259

[2] D. Müller and F. Soria, *A double-weight L^2 inequality for the Kakeya maximal function*, Fourier Anal. Appl., Kahane Special Issue (1995), 467–478. MR 96k:42026

[3] C. Sogge, *Concerning Nikodým-type sets in 3-dimensional curved spaces*, J. Amer. Math. Soc., **12**(1999), 1–31. MR 99h:42037

[4] T. Tao, *From rotating needles to stability of waves: emerging connections between combinatorics, analysis, and PDE*, Notices Amer. Math. Soc. **48**(2001), no3, 294–303. MR 2002b:42021

[5] H. Tanaka, *Some weighted inequalities for the Kakeya maximal operator on functions of product type*, J. Math. Sci. Univ. Tokyo, **6**(1999), 315–333. MR 2001e:42029a

[6] H. Tanaka, *A weighted inequality for the Kakeya maximal operator with a special base*, Tokyo J. Math., **23**(2000), 255–267. MR 2002g:42027

[7] H. Tanaka, *The Fefferman-Stein type inequality for the Kakeya maximal operator*, Proc. Amer. Math. Soc., **129**(2001), 2373–2378. MR 2002e:42024

[8] H. Tanaka, *The Fefferman-Stein type inequality for the Kakeya maximal operator II*, Acta Mathematica Sinica, English Series, **18**(2002) no3, 447–454. MR 2003j:42026

[9] T. Tao, A. Vargas, L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc., **11**(1998), 967–1000. MR 99f:42026

[10] A. M. Vargas, *A weighted inequality for the Kakeya maximal operator*, Proc. Amer. Math. Soc., **120**(1994), 1101–1105. MR 94f:42023

[11] T. Wolff, *An improved bound for Kakeya type maximal functions*, Rev. Mat. Iberoamericana, **11**(1995), 651–674. MR 96m:42034

[12] T. Wolff, *Recent work connected with the Kakeya problem*, Prospects in Mathematics (Princeton, NJ, 1996), 129–162, Amer. Math. Soc., Providence, RI, 1999. MR 2000d:42010

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