MULTIPLECTY RESULTS FOR A CLASS OF SUPERLINEAR ELLIPTIC PROBLEMS

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Abstract. We study a class of superlinear elliptic problems $-\Delta u = \lambda f(u)$ under the Dirichlet boundary condition on a bounded smooth domain in $\mathbb{R}^N$. Assuming that the nonlinearity $f(u)$ is superlinear in a neighborhood of $u = 0$, we study the dependence of the number of signed and sign-changing solutions on the parameter $\lambda$.

Introduction

In this paper we consider the question of multiplicity of both signed and sign-changing solutions for the one-parameter family of elliptic problems $(P_\lambda)$

$$
\begin{align*}
-\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\lambda > 0$ is a parameter, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$), and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class $C^1$ satisfying the following conditions:

$(f_1)$ there exists $\gamma \in (2, 2^*)$ such that $\limsup_{|u| \to 0} \frac{f(u)u}{|u|^\gamma} < +\infty$,

$(f_2)$ there exists $\beta \in (2, 2^*)$ such that $\liminf_{|u| \to 0} \frac{F(u)}{|u|^\beta} > 0$,

$(f_3)$ there exists $\mu \in (2, 2^*)$ such that $uf(u) \geq \mu F(u) > 0$ for $0 \neq |u|$ small,

$(f_4)$ $f(-u) = -f(u)$ for some $\delta > 0$.

Here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $F(u) = \int_0^u f(t)dt$. As is well known, if $f$ were assumed to be superlinear at infinity (in the sense that $(f_1)$ and $(f_2)$ hold as $|u| \to \infty$ and $(f_3)$ holds for $|u|$ large), then the associated energy functional

$$(0.1) \quad I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega F(u) \, dx$$

would be of class $C^1$ on $H_0^1(\Omega)$ and satisfy the Palais-Smale condition, with its critical points being precisely the solutions of problem $(P_\lambda)$. Here, the assumptions $(f_1) - (f_3)$ we make on the nonlinearity $f(u)$ refer solely to its behavior in a neighborhood of $u = 0$, and we will show that they suffice for the existence of three solutions of problem $(P_\lambda)$ when $\lambda$ is large. In addition, if $f(u)$ is odd near $u = 0$, the number of sign-changing solutions of $(P_\lambda)$ gets arbitrarily large together with
λ. With global superlinear conditions these results were known in [1, 3, 4, 5]. More precisely, we will prove the following results.

**Theorem 1.** Assume conditions \((f_1) - (f_3)\) are satisfied. Then there exists \(\Gamma > 0\) depending only on \(\gamma\) such that, if \((\beta - \gamma)\Gamma < 1\) is satisfied, problem \((P_\lambda)\) has at least one positive solution, one negative solution, and a sign-changing solution for all \(\lambda\) sufficiently large.

**Theorem 2.** Assume conditions \((f_1) - (f_4)\) are satisfied. Then there exists \(\Gamma > 0\) depending only on \(\gamma\) such that, if \((\beta - \gamma)\Gamma < 1\) is satisfied then, for any given \(k \geq 1\), problem \((P_\lambda)\) has \(k\) pairs of solutions \(\pm v_i, i = 1, \ldots, k\), with \(|v_i|_\infty \leq \delta\), provided \(\lambda\) is sufficiently large. Moreover, \(\pm v_i\) for \(i = 2, \ldots, k\), are sign-changing solutions.

Our approach is inspired by the results of Costa-Tehrani [2] and is based on the fact that we can show an a priori bound of the form

\[ |u|_\infty \leq C\lambda^{-\epsilon}, \quad \epsilon > 0, \]

for a class of solutions of \((P_\lambda)\) with energy estimates given by minimax methods. The energy estimates for sign-changing solutions rely upon the minimax procedure of Li-Wang [4] for constructing nodal critical points.

The organization of this paper is as follows. Section 1 is reserved for setting the framework and establishing some preliminary results. Theorems 1 and 2 are proved in Section 2.

### 1. Preliminary results

Throughout the paper we denote the \(H^1_0\)-norm by \(\|\cdot\|\) and the \(L^r\)-norm by \(|\cdot|_r\), \(1 \leq r \leq \infty\). Also, sometimes we denote various positive constants by the same letter \(C\).

We start by observing that \((f_2)\) and \((f_1)\) imply the existence of constants \(C_0, C_1 > 0\) such that

\[
\begin{align*}
F(u) &\geq C_0|u|^{\beta}, \\
F(u) &\leq C_1|u|^{\gamma},
\end{align*}
\]

for \(|u|\) small. Now, let \(\rho(t)\) be an even cut-off function satisfying \(\rho(t) \equiv 1\) if \(|t| \leq \delta\), \(\rho(t) \equiv 0\) if \(|t| \geq 2\delta\), \(t\rho'(t) \leq 0\), and \(|t\rho'(t)| \leq \frac{2}{\delta}\), where \(0 < \delta < \frac{1}{2}\) is chosen such that \((1.1)\), \((1.2)\) and \((f_3)\) hold for \(|u| \leq 2\delta\).

**Lemma 1.1.** Define \(\tilde{F}(u) = \rho(u)F(u) + (1 - \rho(u))F_\infty(u)\), where \(F_\infty(u) := C_1|u|^{\gamma}\). Then

\[ u\tilde{F}'(u) \geq \theta \tilde{F}(u) > 0 \]

for all \(u \neq 0\), where \(\theta = \min\{\mu, \gamma\}\).

**Proof.** We have

\[
\tilde{F}'(u) = \rho(u)f(u) + (1 - \rho(u))F'_\infty(u) + \rho'(u)F(u) - \rho'(u)F_\infty(u)
\]
so that
\[
\theta \tilde{F}(u) = \theta \rho(u) F(u) + \theta (1 - \rho(u)) F_{\infty}(u) \\
\leq \rho(u) \frac{\theta}{\mu} f(u) u + (1 - \rho(u)) \frac{\theta}{\gamma} F'_{\infty}(u) u \\
+ \rho'(u) u F(u) - \rho'(u) u F_{\infty}(u) - \rho'(u) u F(u) + \rho'(u) u F_{\infty}(u) \\
\leq u F'(u) + \rho'(u) u (F_{\infty}(u) - F(u)) \leq u F'(u) .
\]

Now let us consider the modified equation of \((P_{\lambda})\) given by \((\tilde{P}_{\lambda})\)
\[
\begin{aligned}
-\Delta u &= \lambda \tilde{F}'(u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
The corresponding functional
\[
\tilde{I}_{\lambda}(u) = \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 \, dx - \lambda \int_{\Omega} \tilde{F}(u) \, dx , \quad u \in E := H^1_0(\Omega) ,
\]
is of class \(C^2\), and its critical points are the solutions of \((\tilde{P}_{\lambda})\). We note that critical points of \((\tilde{I}_{\lambda})\) with \(L^\infty\)-norm less than or equal to \(\delta\) are also solutions of the original problem \((P_{\lambda})\).

**Lemma 1.2.** The functional \(\tilde{I}_{\lambda}\) satisfies \((PS)\) on \(E\).

*Proof.* This is standard in view of Lemma 1.1 and the fact that \(\gamma < 2^*\). \(\square\)

**Lemma 1.3.** Let \(u \in E\) be a critical point of \(\tilde{I}_{\lambda}\). Then
\[
||u||^2 \leq \frac{2\theta}{\theta - 2} \tilde{I}_{\lambda}(u) .
\]

*Proof.* This estimate readily follows from Lemma 1.1 and
\[
\frac{1}{2} ||u||^2 - \lambda \int_{\Omega} \tilde{F}(u) \, dx = \tilde{I}_{\lambda}(u) , \quad ||u||^2 - \lambda \int_{\Omega} \tilde{F}'(u) u \, dx = 0 .
\]
\(\square\)

Now, let \(0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots\) and \(\phi_1, \phi_2, \phi_3, \ldots\) denote the eigenvalues and corresponding eigenfunctions of \(-\Delta\) on \(H^1_0(\Omega)\). Also, for \(k = 1, 2, \ldots\), let
\[
Y_k := \text{span} \{ \phi_1, \ldots, \phi_k \} , \quad Z_k := \text{span} \{ \phi_k, \phi_{k+1}, \ldots \} .
\]

**Lemma 1.4.** Let \(b_{k,\lambda} = \sup_{u \in Y_k} \tilde{I}_{\lambda}(u)\). Then
\[
b_{k,\lambda} \leq C_\ast C_k \lambda^{-\frac{\beta - \gamma}{\delta - \gamma}}
\]
where \(C_\ast > 0\) depends only on \(\gamma, \beta\) and \(\Omega\) and \(C_k := \lambda_k^{-\frac{\beta - \gamma}{\delta - \gamma}} + \lambda_k^{-\frac{\beta - \gamma}{\delta - \gamma}}\).

*Proof.* We recall that \(\delta > 0\) was chosen so that
\[
\tilde{F}(u) \geq F(u) \geq C_0||u||^2 \quad \text{for } ||u|| \leq 2\delta ,
\]
\[
\tilde{F}(u) = F_{\infty}(u) = C_1||u||^\gamma \quad \text{for } ||u|| \geq 2\delta .
\]
For \( u \in E \), denote \( \Omega_1 = \{ x \mid |u| \geq 2\delta \} \), \( \Omega_2 = \{ x \mid |u| < 2\delta \} \), and let \( u_1 = u|\Omega_1 \), \( u_2 = u|\Omega_2 \). In view of Hölder’s inequality we obtain

\[
\int_{\Omega} \bar{F}(u_2) \, dx \geq C_* |u_2|_2^\beta,
\]

so that, for \( u \in Y_k \), it follows that

\[
\bar{t}_\lambda(u) \leq \frac{\lambda_k}{2} |u|^2 - \lambda C_* |u_1|^2 - \lambda C_* |u_2|^\beta,
\]

\[
= \frac{\lambda_k}{2} |u_1|^2 - \lambda C_* |u_1|^2 + \frac{\lambda_k}{2} |u_2|^2 - \lambda C_* |u_2|^\beta.
\]

Since \( 2 < \gamma \leq \beta < 2^* \), we obtain the following estimate:

\[
(1.8) \quad b_{k,\lambda} \leq C_* C_k \lambda^{-\frac{\gamma}{2}}.
\]

where \( C_k := \lambda_k^{\frac{\beta}{2}} + \lambda_k^{\frac{\gamma}{2}} \).

\[\]  

2. Proofs of the theorems

As is well known, \( (\bar{P}_\lambda) \) has a positive solution \( u_{1,\lambda} \) and a negative solution \( u_{2,\lambda} \) obtained through an application of the Mountain-Pass Theorem [1] to the functionals

\[
\bar{I}_{1,\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} \bar{F}_1(u) \, dx, \quad u \in E := H_0^1(\Omega)
\]

and

\[
\bar{I}_{2,\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} \bar{F}_2(u) \, dx, \quad u \in E := H_0^1(\Omega),
\]

respectively, where \( \bar{F}_1(u) = \bar{F}(u) \) if \( u > 0 \), \( \bar{F}_1(u) \equiv 0 \) if \( u \leq 0 \), and \( \bar{F}_2(u) = \bar{F}(u) \) if \( u < 0 \), \( \bar{F}_2(u) \equiv 0 \) if \( u \geq 0 \). The corresponding critical values are given by

\[
\bar{d}_{j,\lambda} = \inf_{h \in \Gamma_j} \sup_{0 \leq t \leq 1} \bar{I}_{j,\lambda}(h(t)), \quad j = 1, 2,
\]

where \( \Gamma_j = \left\{ h \in C([0,1], E) \mid h \neq 0, \bar{I}_{j,\lambda}(h(1)) \leq 0 \right\} \). It is clear that

\[
(2.1) \quad \bar{d}_{j,\lambda} \leq d_{j,\lambda}
\]

where

\[
d_{1,\lambda} := \inf_{u > 0} \sup_{0 \leq t < \infty} \bar{I}_{\lambda}(tu), \quad d_{2,\lambda} := \inf_{u < 0} \sup_{0 \leq t < \infty} \bar{I}_{\lambda}(tu).
\]

Lemma 2.1. \( \bar{d}_{j,\lambda} \leq C_* \lambda^{-\frac{\beta}{2}} \) for \( j = 1, 2 \) and \( \lambda > 0 \) large.

Proof. From Lemma [1.3] and (2.1), we have the estimates

\[
(2.2) \quad \|u_{j,\lambda}\| \leq C_* \sqrt{\bar{d}_{j,\lambda}}, \quad j = 1, 2.
\]

Let us consider \( u_{1,\lambda} \) since the same argument applies to \( u_{2,\lambda} \). Recalling from (1.6) that \( \bar{F}(u) \geq C_0 |u|^\beta \) for \( |u| \leq 2\delta \) and defining

\[
J_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \lambda \int_{\Omega} C_0 |u|^\beta \, dx,
\]
we have
\[
\begin{aligned}
&d_{1,\lambda} = \inf_{u > 0} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu) \\
&\leq \inf_{0 < u \leq 2\delta} \sup_{0 \leq t < \infty} \tilde{I}_\lambda(tu) \\
&\leq \inf_{0 < u \leq 2\delta} \sup_{0 \leq t < \infty} J_\lambda(tu) = \inf_{u \neq 0} \sup_{0 \leq t < \infty} J_\lambda(tu),
\end{aligned}
\]
(2.3)

where the last equality holds provided \( \lambda > 0 \) is sufficiently large so that the ground states of \( J_\lambda(u) \) have \( L^\infty \)-norm less than \( 2\delta \). In fact, we obtain through straightforward calculations as in [2, Thm. 1.1] that
\[
\inf_{u \neq 0} \sup_{0 \leq t < \infty} J_\lambda(tu) \leq C_* \lambda^{- \frac{\beta}{2-\gamma}}.
\]

Therefore, keeping in mind (2.3) and (2.2) above, we conclude that
\[
d_{1,\lambda} \leq C_* \lambda^{- \frac{\beta}{2-\gamma}}
\]
(2.4)

and
\[
\|u_{1,\lambda}\| \leq C_* \lambda^{- \frac{\beta}{2-\gamma}}
\]
(2.5)

for \( \lambda > 0 \) large. It follows by Sobolev’s inequality that
\[
|u_{1,\lambda}|_{2,\ast} \leq C_* \lambda^{- \frac{\beta}{2-\gamma}}
\]
(2.6)

for \( \lambda > 0 \) large. The proof is complete in view of (2.4) and (2.1). \( \square \)

**Lemma 2.2.** Let \( u_\lambda \) be solutions of \((P_\lambda)\) satisfying \( \tilde{I}_\lambda(u_\lambda) \leq C_* \lambda^{- \frac{\beta}{2-\gamma}} \). Then there is an integer \( m \) depending only on \( \gamma \) such that for all \( \lambda \geq 1 \),
\[
|u_\lambda|_{\infty} \leq C_* \lambda^{- \frac{\beta}{2-\gamma}} \sum_{n=0}^{m-1} (\gamma - 1)^n.
\]

**Proof.** Since the cut-off function \( \rho(t) \) satisfies \( |\rho'(t)|t \leq 2/\delta \), we obtain from (1.3) that
\[
|\tilde{F}'(u_\lambda)| \leq C_* |u_\lambda|^{\gamma - 1};
\]

hence
\[
\tilde{F}'(u_\lambda) \in L^{\frac{\gamma}{\gamma - 1}}(\Omega).
\]

Now, if \( r_1 = \frac{2\gamma}{\gamma - 1} \), then \( L^{r_1} \)-estimates give
\[
\begin{aligned}
|u_\lambda|_{s_1} &\leq C_* \left( |u_\lambda|_{r_1} + \lambda \left| \tilde{F}'(u_\lambda) \right|_{r_1} \right) \\
&\leq C_* \left( |u_\lambda|_{2,\ast} + \lambda C_* |u_\lambda|_{2,\ast}^{\gamma - 1} \right) \\
&\leq C_* \left( \lambda^{- \frac{\beta}{2-\gamma}} + \lambda^{- \frac{\beta}{2-\gamma}} \lambda^{- \frac{\beta}{2-\gamma}} \right) \\
&\leq C_* \left( \lambda^{- \frac{\beta}{2-\gamma}} \lambda^{- \frac{\beta}{2-\gamma}} \right); \\
\end{aligned}
\]

hence
\[
|u_\lambda|_{s_1} \leq C_* \left( \lambda^{- \frac{\beta}{2-\gamma}} \right).
\]
where $\frac{1}{q_1} = \frac{1}{r_1} - \frac{2}{q_2}$. Note that $s_1 > 2^*$ since $\gamma < 2^*$. Next, letting $r_2 = \frac{r_1}{q_1}$ and using $L^{q_2}$-estimates, we obtain

$$
\|u_\lambda\|_{W^{2,r_2}} \leq C_* \left( |u_\lambda|_{r_2} + \lambda^{1-\frac{2}{r_2}} \|\tilde{F}'(u_\lambda)\|_{r_2} \right) \leq C_* \left( C_* |u_\lambda|_{r_1} + \lambda C_1 |u_\lambda|_{r_1}^{\gamma-1} \right) \leq C_* \left( \lambda^{\frac{1}{q_1}} \lambda^{\frac{2}{r_2}} + \lambda \lambda^{\frac{2}{r_2}} \right) \left( \gamma - 1 \right)^{r_1 - 1}
$$

Iterating $m \geq 1$ times yields

$$(2.7) \quad \|u_\lambda\|_{W^{2,r_m}} \leq C_* \lambda^{\frac{1}{q_1}} \lambda^{\frac{2}{r_2}} \sum_{i=0}^{m-1} (\gamma - 1)^i,$$

and the result follows by taking $m$ so that $r_m > \frac{N}{2}$.

**Remark 2.3.** The number $m$ of iterations needed to have $W^{2,r_m}(\Omega) \subset L^\infty(\Omega)$ in Lemma 2.2 depends only on $N$ and $\gamma$. In other words, since the space dimension $N$ is given, the positive number

$$
\Gamma := \sum_{i=0}^{m-1} (\gamma - 1)^i
$$

above depends solely on $\gamma$. Therefore, if $\beta \geq \gamma$ is such that

$$(\beta - \gamma) \Gamma < 1,$$

we get a negative exponent for $\lambda$ in (2.7).

**Proof of Theorem 1.** Let $2 < \gamma \leq \beta < 2^*$ be such that $(\beta - \gamma) \Gamma < 1$, where $\Gamma > 0$ was defined in Remark 2.3. Then

$$
|u_{j,\lambda}|_{\infty} \leq C_* \lambda^{1-\frac{(\beta - \gamma) \Gamma}{\beta - 2}},
$$

where the exponent of $\lambda$ is negative, so that there exists $\Lambda_0 > 0$ such that

$$
C_* \lambda^{1-\frac{(\beta - \gamma) \Gamma}{\beta - 2}} \leq \delta
$$

for all $\lambda \geq \Lambda_0$. It follows that $u_{1,\lambda} > 0$ and $u_{2,\lambda} < 0$ are solutions of our original problem $(P_\lambda)$ for all $\lambda \geq \Lambda_0$.

Since $f$ is assumed to be of class $C^1$ and $\inf_{s \in \mathbb{R}} \tilde{F}'(s) > -\infty$ by construction, we also prove that there is a sign-changing solution for all $\lambda$ large. We employ the method in [4]. On $E$, let us define

$$
P_E = \{ u \in E \mid u(x) \geq 0, \text{ a.e. in } \Omega \},
$$

which is a closed convex cone. Then, the Banach space $X = C^1_0(\Omega)$ is densely embedded in $E$, and

$$
P = P_E \cap X
$$

is a closed convex cone in $X$. Furthermore, $P = \tilde{P} \cup \partial P$ under the topology of $X$, i.e., there exist interior points in $P$. So, as in [4, Section 3], we may define a partial order relation in $X : u, v \in X, u > v \iff u - v \in P \setminus \{0\}, u \gg v \iff u - v \in \tilde{P}$. We also define $W = P \cup (-P)$.

We follow the arguments of Example 3.2 and Corollary 3.2 in [4]. On $Y_2$ consider $Q = \{ u = s \phi_1 + t \phi_2 \mid |s| \leq R, 0 \leq t \leq R, \|u\| \leq R \}$ and for $0 < r < R$, $T = \{ u \in Z_2 \mid \|u\| = r \}$. Note that the boundary of $Q$ contains two parts: $Q_1$ is
the part contained in $Y_1$ and $Q_2$ is the part such that $||u|| = R$. From the conditions on $\hat{F}$ we may assume $R > 0$ is large enough so that $\bar{I}_\lambda(u) \leq 0$ for all $u \in Y_2$ with $||u|| = R$ and for all $\lambda \geq 1$. Also we may assume $r > 0$ is small enough so that $\bar{I}_\lambda(u) > 0$ for all $u \in T$ and for all $\lambda \geq 1$. Define

$$\Gamma = \{ h \in C(Q, X) \mid h(Q_1) \in W, h(u) = u, \text{ for } u \in Q_2 \},$$

$$c_\lambda = \inf_{h \in \Gamma} \sup_{h(Q) \in W} \bar{I}_\lambda.$$

Then it follows from [4] that $c_\lambda > 0$ is a critical value of $\bar{I}_\lambda$ having a sign-changing critical point $u_\lambda$ at this critical value. From the construction of $c_\lambda$ and Lemma [1,4] we have $c_\lambda \leq b_{2,\lambda} \leq C_\lambda \lambda^{-\frac{n-2}{2}}$. From Lemma 2.2 and for $\lambda$ large, the $u_\lambda$'s are solutions of the original problem $(P_\lambda)$. □

**Proof of Theorem 2.** To prove Theorem 2 we shall use the arguments in [3] and [4] to get solutions for $(\hat{P}_\lambda)$ first. In order to get estimates on the critical values we shall use the proofs in [3] for the existence and the proofs in [4] for the nodal property of the solutions. Fix an integer $k$. We shall show $(P_\lambda)$ has $k$ pairs of solutions for large $\lambda$, including $k - 1$ pairs of nodal solutions.

Choose $R > 0$ such that $\bar{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $||u|| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$. Define $G = \{ h \in C(D, E) \mid h \text{ is odd and } h(u) = u, \text{ for } ||u|| = R \}$. We denote the genus of a symmetric subset $A$ by $i(A)$ and, for $j = 1, \ldots, k$, we let

$$\Gamma_j = \{ h(D \setminus B) \mid h \in G, \ k \geq j, \ i(B) \leq k - j \} ,$$

$$c_{j,\lambda} = \inf_{\lambda \in \Gamma_j} \sup_{u \in A} \bar{I}_\lambda(u).$$

Then, by [3] (Proposition 9.30, p. 58), and under our conditions on $f$, we have that $0 < c_{1,\lambda} \leq c_{2,\lambda} \leq \cdots \leq c_{k,\lambda}$ are all critical values of $\bar{I}_\lambda$ and there are at least $k$ pairs of critical points at these critical values. Since $Id \in G$ we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{n-2}{2}},$$

where $b_{k,\lambda}$ is as in Lemma [1,4]. Then, by Lemma 2.2 and for $\lambda$ large, these $k$ pairs of critical points are also solutions of the original problem.

Finally, we make use of the fact that $\inf_{s \in \mathbb{R}} \hat{F}'(s) > -\infty$ (by construction) and follow the idea in [4] by restricting the above minimax procedure to $X$ and taking the maximum outside $W$. Choose $R > 0$ such that $\bar{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $||u|| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$ and define $G = \{ h \in C(D, X) \mid h \text{ is odd and } h(u) = u, \text{ for } ||u|| = R \}$. We also denote the genus of a symmetric subset $A$ in $X$ by again $i(A)$ and, for $j = 2, \ldots, k$, let

$$\Gamma_j = \{ h(D \setminus B) \mid h \in G, \ k \geq j, \ i(B) \leq k - j \} ,$$

$$c_{j,\lambda} = \inf_{\lambda \in \Gamma_j} \sup_{A \in W} \bar{I}_\lambda(u).$$

Then, by [3] (Theorem 2.3, p. 3214)) and, again, by the properties of $\hat{F}'$, we have that $0 < c_{2,\lambda} \leq c_{3,\lambda} \leq \cdots \leq c_{k,\lambda}$ are all critical values of $\bar{I}_\lambda$ and there are at least $(k - 1)$ pairs of sign-changing critical points at these critical values. Also, since $Id \in G$, we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{n-2}{2}} ,$$
with $b_{k, \lambda}$ as in Lemma 1.4. Therefore, it follows from Lemma 2.2 and for $\lambda$ large that these $(k - 1)$ pairs of sign-changing critical points are also solutions of the original problem. □

**Remark.** We point out that all the above results are still true when $N = 1, 2$.

**References**


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