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MULTIPLICITY RESULTS FOR A CLASS OF SUPERLINEAR ELLIPTIC PROBLEMS

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Abstract. We study a class of superlinear elliptic problems $-\Delta u = \lambda f(u)$ under the Dirichlet boundary condition on a bounded smooth domain in $\mathbb{R}^N$. Assuming that the nonlinearity $f(u)$ is *superlinear* in a neighborhood of $u = 0$, we study the dependence of the number of signed and sign-changing solutions on the parameter λ.

Introduction

In this paper we consider the question of multiplicity of both signed and sign-changing solutions for the one-parameter family of elliptic problems (P_λ)

$$\begin{cases} -\Delta u &= \lambda f(u) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega , \end{cases}$$

where $\lambda > 0$ is a parameter, Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, and the nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function of class C^1 satisfying the following conditions:

(f_1) there exists $\gamma \in (2, 2^*)$ such that $\limsup_{|u|\to 0} \frac{f(u)u}{|u|^\gamma} < +\infty$,
(f_2) there exists $\beta \in (2, 2^*)$ such that $\liminf_{|u|\to 0} \frac{F(u)}{|u|^\beta} > 0$,
(f_3) there exists $\mu \in (2, 2^*)$ such that $uf(u) \geq \mu F(u) > 0$ for $0 \neq |u|$ small ,
(f_4) $f(-u) = -f(u)$ $\forall\ |u| \leq \delta$ (for some $\delta > 0$).

Here $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and $F(u) = \int_0^u f(t)dt$. As is well known, if f were assumed to be *superlinear at infinity* (in the sense that (f_1) and (f_2) hold as $|u| \to \infty$ and (f_3) holds for $|u|$ large), then the associated *energy functional*

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega F(u)\, dx \tag{0.1}$$

would be of class C^1 on $H_0^1(\Omega)$ and satisfy the Palais-Smale condition, with its critical points being precisely the solutions of problem (P_λ). Here, the assumptions $(f_1) - (f_3)$ we make on the nonlinearity $f(u)$ refer solely to its behavior in a neighborhood of $u = 0$, and we will show that they suffice for the existence of three solutions of problem (P_λ) when λ is large. In addition, if $f(u)$ is odd near $u = 0$, the number of sign-changing solutions of (P_λ) gets arbitrarily large together with

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λ. With global superlinear conditions these results were known in [1, 3, 4, 5]. More precisely, we will prove the following results.

Theorem 1. *Assume conditions* $(f_1)-(f_3)$ *are satisfied. Then there exists* $\Gamma > 0$ *depending only on* γ *such that, if* $(\beta-\gamma)\Gamma < 1$ *is satisfied, problem* (P_λ) *has at least one positive solution, one negative solution, and a sign-changing solution for all* λ *sufficiently large.*

Theorem 2. *Assume conditions* $(f_1)-(f_4)$ *are satisfied. Then there exists* $\Gamma > 0$ *depending only on* γ *such that, if* $(\beta-\gamma)\Gamma < 1$ *is satisfied then, for any given* $k \geq 1$, *problem* (P_λ) *has* k *pairs of solutions* $\pm v_i$, $i = 1, \ldots, k$, *with* $|v_i|_\infty \leq \delta$, *provided* λ *is sufficiently large. Moreover,* $\pm v_i$ *for* $i = 2, \ldots, k$, *are sign-changing solutions.*

Our approach is inspired by the results of Costa-Tehrani [2] and is based on the fact that we can show an a priori bound of the form

$$|u|_\infty \leq C\lambda^{-\epsilon} \quad , \quad \epsilon > 0 \ ,$$

for a class of solutions of (P_λ) with energy estimates given by minimax methods. The energy estimates for sign-changing solutions rely upon the minimax procedure of Li-Wang [4] for constructing nodal critical points.

The organization of this paper is as follows. Section 1 is reserved for setting the framework and establishing some preliminary results. Theorems 1 and 2 are proved in Section 2.

1. Preliminary results

Throughout the paper we denote the H_0^1-norm by $\|\cdot\|$ and the L^r-norm by $|\cdot|_r$, $1 \leq r \leq \infty$. Also, sometimes we denote various positive constants by the same letter C_*.

We start by observing that (f_2) and (f_1) imply the existence of constants C_0, $C_1 > 0$ such that

$$F(u) \geq C_0|u|^\beta \ , \tag{1.1}$$

$$F(u) \leq C_1|u|^\gamma \ , \tag{1.2}$$

for $|u|$ small. Now, let $\rho(t)$ be an even cut-off function satisfying $\rho(t) \equiv 1$ if $|t| \leq \delta$, $\rho(t) \equiv 0$ if $|t| \geq 2\delta$, $t\rho'(t) \leq 0$, and $|t\rho'(t)| \leq \frac{2}{\delta}$, where $0 < \delta < \frac{1}{2}$ is chosen such that (1.1), (1.2) and (f_3) hold for $|u| \leq 2\delta$.

Lemma 1.1. *Define* $\widetilde{F}(u) = \rho(u)F(u) + (1-\rho(u))F_\infty(u)$, *where* $F_\infty(u) := C_1|u|^\gamma$. *Then*

$$u\widetilde{F}'(u) \geq \theta\widetilde{F}(u) > 0$$

for all $u \neq 0$, *where* $\theta = \min\{\mu, \gamma\}$.

Proof. We have

$$\widetilde{F}'(u) = \rho(u)f(u) + (1-\rho(u))F'_\infty(u) + \rho'(u)F(u) - \rho'(u)F_\infty(u) \tag{1.3}$$

so that

$$\begin{aligned}\theta\widetilde{F}(u) &= \theta\rho(u)F(u) + \theta(1-\rho(u))F_\infty(u)\\ &\le \rho(u)\frac{\theta}{\mu}f(u)u + (1-\rho(u))\frac{\theta}{\gamma}F'_\infty(u)u\\ &+\rho'(u)uF(u) - \rho'(u)uF_\infty(u) - \rho'(u)uF(u) + \rho'(u)uF_\infty(u)\\ &\le u\widetilde{F}'(u) + \rho'(u)u(F_\infty(u) - F(u)) \le u\widetilde{F}'(u)\ .\end{aligned}$$

□

Now let us consider the modified equation of (P_λ) given by $(\widetilde{P}_\lambda)$

$$\begin{cases} -\Delta u &= \lambda\widetilde{F}'(u) \text{ in } \Omega,\\ \quad u &= 0 \text{ on } \partial\Omega\ .\end{cases}$$

The corresponding functional

$$\widetilde{I}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega \widetilde{F}(u)\,dx\ , \qquad u \in E := H_0^1(\Omega)\ ,$$

is of class C^2, and its critical points are the solutions of $(\widetilde{P}_\lambda)$. We note that critical points of $(\widetilde{I}_\lambda)$ with L^∞-norm less than or equal to δ are also solutions of the original problem (P_λ).

Lemma 1.2. *The functional $\widetilde{I}_\lambda$ satisfies (PS) on E.*

Proof. This is standard in view of Lemma 1.1 and the fact that $\gamma < 2^*$. □

Lemma 1.3. *Let $u \in E$ be a critical point of $\widetilde{I}_\lambda$. Then*

$$\|u\|^2 \le \frac{2\theta}{\theta-2}\widetilde{I}_\lambda(u)\ . \tag{1.4}$$

Proof. This estimate readily follows from Lemma 1.1 and

$$\frac{1}{2}\|u\|^2 - \lambda\int_\Omega \widetilde{F}(u)\,dx = \widetilde{I}_\lambda(u)\ , \|u\|^2 - \lambda\int_\Omega \widetilde{F}'(u)u\,dx = 0\ .$$

□

Now, let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$ and $\phi_1, \phi_2, \phi_3, \ldots$ denote the eigenvalues and corresponding eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$. Also, for $k = 1, 2, \ldots$, let

$$Y_k := span\ \{\phi_1, \ldots, \phi_k\}\ , \quad Z_k := \overline{span}\ \{\phi_k, \phi_{k+1}, \ldots\}\ .$$

Lemma 1.4. *Let $b_{k,\lambda} = \sup_{u\in Y_k}\widetilde{I}_\lambda(u)$. Then*

$$b_{k,\lambda} \le C_* C_k \lambda^{-\frac{2}{\beta-2}} \tag{1.5}$$

where $C_* > 0$ depends only on γ, β and Ω and $C_k := \lambda_k^{\frac{\beta}{\beta-2}} + \lambda_k^{\frac{\gamma}{\gamma-2}}$.

Proof. We recall that $\delta > 0$ was chosen so that

$$\widetilde{F}(u) \ge F(u) \ge C_0|u|^\beta \quad \text{for} \quad |u| \le 2\delta\ , \tag{1.6}$$

$$\widetilde{F}(u) = F_\infty(u) = C_1|u|^\gamma \quad \text{for} \quad |u| \ge 2\delta\ . \tag{1.7}$$

For $u \in E$, denote $\Omega_1 = \{x \mid |u| \geq 2\delta\}$, $\Omega_2 = \{x \mid |u| < 2\delta\}$, and let $u_1 = u|\Omega_1$, $u_2 = u|\Omega_2$. In view of Hölder's inequality we obtain

$$\int_\Omega \widetilde{F}(u_2)\ dx \geq C_*|u_2|_2^\beta\ ,$$
$$\int_\Omega \widetilde{F}(u_1)\ dx \geq C_*|u_1|_2^\gamma\ ,$$

so that, for $u \in Y_k$, it follows that

$$\begin{aligned}\widetilde{I}_\lambda(u) &\leq \frac{\lambda_k}{2}|u|_2^2 - \lambda C_*|u_1|_2^\gamma - \lambda C_*|u_2|_2^\beta \\ &= \frac{\lambda_k}{2}|u_1|_2^2 - \lambda C_*|u_1|_2^\gamma + \frac{\lambda_k}{2}|u_2|_2^2 - \lambda C_*|u_2|_2^\beta \\ &\leq C_*\lambda_k^{\frac{\gamma}{\gamma-2}}\lambda^{-\frac{2}{\gamma-2}} + C_*\lambda_k^{\frac{\beta}{\beta-2}}\lambda^{-\frac{2}{\beta-2}}\ .\end{aligned}$$

Since $2 < \gamma \leq \beta < 2^*$, we obtain the following estimate:

$$b_{k,\lambda} \leq C_* C_k \lambda^{-\frac{2}{\beta-2}}\ , \tag{1.8}$$

where $C_k := \lambda_k^{\frac{\beta}{\beta-2}} + \lambda_k^{\frac{\gamma}{\gamma-2}}$. □

2. Proofs of the theorems

As is well known, $(\widetilde{P}_\lambda)$ has a positive solution $u_{1,\lambda}$ and a negative solution $u_{2,\lambda}$ obtained through an application of the Mountain-Pass Theorem [1] to the functionals

$$\widetilde{I}_{1,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx - \lambda\int_\Omega \widetilde{F}_1(u)\ dx\ , \qquad u \in E := H_0^1(\Omega)$$

and

$$\widetilde{I}_{2,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx - \lambda\int_\Omega \widetilde{F}_2(u)\ dx\ , \qquad u \in E := H_0^1(\Omega)\ ,$$

respectively, where $\widetilde{F}_1(u) = \widetilde{F}(u)$ if $u > 0$, $\widetilde{F}_1(u) \equiv 0$ if $u \leq 0$, and $\widetilde{F}_2(u) = \widetilde{F}(u)$ if $u < 0$, $\widetilde{F}_2(u) \equiv 0$ if $u \geq 0$. The corresponding critical values are given by

$$\widetilde{d}_{j,\lambda} = \inf_{h\in\Gamma_j}\sup_{0\leq t\leq 1}\widetilde{I}_{j,\lambda}(h(t))\ , \quad j = 1, 2\ ,$$

where $\Gamma_j = \left\{h \in C([0,1], E) \mid h \neq 0\ ,\ \widetilde{I}_{j,\lambda}(h(1)) \leq 0\right\}$. It is clear that

$$\widetilde{d}_{j,\lambda} \leq d_{j,\lambda} \tag{2.1}$$

where

$$d_{1,\lambda} := \inf_{u>0}\sup_{0\leq t<\infty}\widetilde{I}_\lambda(tu)\quad ,\quad d_{2,\lambda} := \inf_{u<0}\sup_{0\leq t<\infty}\widetilde{I}_\lambda(tu)\ .$$

Lemma 2.1. *$\widetilde{d}_{j,\lambda} \leq C_*\lambda^{-\frac{2}{\beta-2}}$ for $j = 1, 2$ and $\lambda > 0$ large.*

Proof. From Lemma 1.3 and (2.1), we have the estimates

$$\|u_{j,\lambda}\| \leq C_*\sqrt{d_{j,\lambda}}\quad ,\quad j = 1, 2\ . \tag{2.2}$$

Let us consider $u_{1,\lambda}$ since the same argument applies to $u_{2,\lambda}$. Recalling from (1.6) that $\widetilde{F}(u) \geq C_0|u|^\beta$ for $|u| \leq 2\delta$ and defining

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\ dx - \lambda\int_\Omega C_0|u|^\beta\ dx\ ,$$

we have

$$\begin{aligned} d_{1,\lambda} &= \inf_{u>0} \sup_{0\le t<\infty} \widetilde{I}_\lambda(tu) &&\le \inf_{0<u\le 2\delta} \sup_{0\le t<\infty} \widetilde{I}_\lambda(tu) \\ &\le \inf_{0<u\le 2\delta} \sup_{0\le t<\infty} J_\lambda(tu) &&= \inf_{u\neq 0} \sup_{0\le t<\infty} J_\lambda(tu) \ , \end{aligned} \tag{2.3}$$

where the last equality holds provided $\lambda > 0$ is sufficiently large so that the *ground states* of $J_\lambda(u)$ have L^∞-norm less than 2δ. In fact, we obtain through straightforward calculations as in [2, Thm. 1.1] that

$$\inf_{u\neq 0} \sup_{0\le t<\infty} J_\lambda(tu) \le C_* \lambda^{-\frac{2}{\beta-2}} \ .$$

Therefore, keeping in mind (2.3) and (2.2) above, we conclude that

$$d_{1,\lambda} \le C_* \lambda^{-\frac{2}{\beta-2}} \tag{2.4}$$

and

$$\|u_{1,\lambda}\| \le C_* \lambda^{-\frac{1}{\beta-2}} \tag{2.5}$$

for $\lambda > 0$ large. It follows by Sobolev's inequality that

$$|u_{1,\lambda}|_{2^*} \le C_* \lambda^{-\frac{1}{\beta-2}} \tag{2.6}$$

for $\lambda > 0$ large. The proof is complete in view of (2.4) and (2.1). □

Lemma 2.2. *Let u_λ be solutions of (P_λ) satisfying $\widetilde{I}_\lambda(u_\lambda) \le C_* \lambda^{-\frac{2}{\beta-2}}$. Then there is an integer m depending only on γ such that for all $\lambda \ge 1$,*

$$|u_\lambda|_\infty \le C_* \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2} \sum_{i=0}^{m-1} (\gamma-1)^i}.$$

Proof. Since the cut-off function $\rho(t)$ satisfies $|\rho'(t)t| \le 2/\delta$, we obtain from (1.3) that

$$\left|\widetilde{F}'(u_\lambda)\right| \le C_* |u_\lambda|^{\gamma-1} \ ;$$

hence

$$\widetilde{F}'(u_\lambda) \in L^{\frac{2^*}{\gamma-1}}(\Omega) \ .$$

Now, if $r_1 = \frac{2^*}{\gamma-1}$, then L^{r_1}-estimates give

$$\begin{aligned} \|u_\lambda\|_{W^{2,r_1}} &\le C_* \left(|u_\lambda|_{r_1} + \lambda \left|\widetilde{F}'(u_\lambda)\right|_{r_1} \right) \\ &\le C_* \left(C_* |u_\lambda|_{2^*} + \lambda C_* |u_\lambda|_{2^*}^{\gamma-1} \right) \\ &\le C_* \left(\lambda^{-\frac{1}{\beta-2}} + \lambda \lambda^{-\frac{\gamma-1}{\beta-2}} \right) \\ &\le C_* \left(\lambda^{-\frac{1}{\beta-2}} + \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} \right) \ ; \end{aligned}$$

hence

$$|u_\lambda|_{s_1} \le C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} \right) \ ,$$

where $\frac{1}{s_1} = \frac{1}{r_1} - \frac{2}{N}$. Note that $s_1 > 2^*$ since $\gamma < 2^*$. Next, letting $r_2 = \frac{s_1}{\gamma - 1}$ and using L^{r_2}-estimates, we obtain

$$\begin{aligned} \|u_\lambda\|_{W^{2,r_2}} &\leq C_* \left(|u_\lambda|_{r_2} + \lambda \left| \widetilde{F}'(u_\lambda) \right|_{r_2} \right) \\ &\leq C_* \left(C_* |u_\lambda|_{s_1} + \lambda C_* |u_\lambda|_{s_1}^{\gamma-1} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} + \lambda \lambda^{-\frac{\gamma-1}{\beta-2}} \lambda^{\frac{(\gamma-1)(\beta-\gamma)}{\beta-2}} \right) \\ &\leq C_* \left(\lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2}} + \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\gamma(\beta-\gamma)}{\beta-2}} \right) \ . \end{aligned}$$

Iterating $m \geq 1$ times yields

$$\|u_\lambda\|_{W^{2,r_m}} \leq C_* \lambda^{-\frac{1}{\beta-2}} \lambda^{\frac{\beta-\gamma}{\beta-2} \sum_{i=0}^{m-1} (\gamma-1)^i} \ , \tag{2.7}$$

and the result follows by taking m so that $r_m > \frac{N}{2}$. □

Remark 2.3. The number m of iterations needed to have $W^{2,r_m}(\Omega) \subset L^\infty(\Omega)$ in Lemma 2.2 depends only on N and γ. In other words, since the space dimension N is given, the positive number

$$\Gamma := \sum_{i=0}^{m-1} (\gamma - 1)^i$$

above depends solely on γ. Therefore, if $\beta \geq \gamma$ is such that

$$(\beta - \gamma)\Gamma < 1 \ ,$$

we get a negative exponent for λ in (2.7).

Proof of Theorem 1. Let $2 < \gamma \leq \beta < 2^*$ be such that $(\beta - \gamma)\Gamma < 1$, where $\Gamma > 0$ was defined in Remark 2.3 . Then

$$|u_{j,\lambda}|_\infty \leq C_* \lambda^{-\frac{1-(\beta-\gamma)\Gamma}{\beta-2}} \ ,$$

where the exponent of λ is negative, so that there exists $\Lambda_0 > 0$ such that

$$C_* \lambda^{-\frac{1-(\beta-\gamma)\Lambda}{\beta-2}} \leq \delta$$

for all $\lambda \geq \Lambda_0$. It follows that $u_{1,\lambda} > 0$ and $u_{2,\lambda} < 0$ are solutions of our original problem (P_λ) for all $\lambda \geq \Lambda_0$.

Since f is assumed to be of class C^1 and $\inf_{s \in \mathbb{R}} \widetilde{F}'(s) > -\infty$ by construction, we also prove that there is a sign-changing solution for all λ large. We employ the method in [4]. On E, let us define

$$P_E = \{u \in E \mid u(x) \geq 0, \text{ a.e. in } \Omega\} \ ,$$

which is a closed convex cone. Then, the Banach space $X = C_0^1(\Omega)$ is densely embedded in E, and

$$P = P_E \cap X$$

is a closed convex cone in X. Furthermore, $P = \overset{\circ}{P} \cup \partial P$ under the topology of X, i.e., there exist interior points in P. So, as in [4, Section 3], we may define a partial order relation in X : $u, v \in X$, $u > v \iff u - v \in P \setminus \{0\}$; $u \gg v \iff u - v \in \overset{\circ}{P}$. We also define $W = P \cup (-P)$.

We follow the arguments of Example 3.2 and Corollary 3.2 in [4]. On Y_2 consider $Q = \{u = s\phi_1 + t\phi_2 \mid |s| \leq R, 0 \leq t \leq R, ||u|| \leq R\}$ and for $0 < r < R$, $T = \{u \in Z_2 \mid ||u|| = r\}$. Note that the boundary of Q contains two parts: Q_1 is the part contained in Y_1 and Q_2 is the part such that $||u|| = R$. From the conditions on $\widetilde{F}$ we may assume $R > 0$ is large enough so that $\widetilde{I}_\lambda(u) \leq 0$ for all $u \in Y_2$ with $||u|| = R$ and for all $\lambda \geq 1$. Also we may assume $r > 0$ is small enough so that $\widetilde{I}_\lambda(u) > 0$ for all $u \in T$ and for all $\lambda \geq 1$. Define

$$\Gamma = \{h \in C(Q, X) \mid h(Q_1) \in W, h(u) = u, \text{ for } u \in Q_2\},$$

$$c_\lambda = \inf_{h\in\Gamma} \sup_{h(Q)\setminus W} \widetilde{I}_\lambda.$$

Then it follows from [4] that $c_\lambda > 0$ is a critical value of $\widetilde{I}_\lambda$ having a sign-changing critical point u_λ at this critical value. From the construction of c_λ and Lemma 1.4, we have $c_\lambda \leq b_{2,\lambda} \leq C_* \lambda^{-\frac{2}{\beta-2}}$. From Lemma 2.2 and for λ large, the u_λ's are solutions of the original problem (P_λ). □

Proof of Theorem 2. To prove Theorem 2 we shall use the arguments in [3] and [4] to get solutions for $(\widetilde{P}_\lambda)$ first. In order to get estimates on the critical values we shall use the proofs in [3] for the existence and the proofs in [4] for the nodal property of the solutions. Fix an integer k. We shall show (P_λ) has k pairs of solutions for large λ, including $k-1$ pairs of nodal solutions.

Choose $R > 0$ such that $\widetilde{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $||u|| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$. Define $G = \{h \in C(D, E) \mid h \text{ is odd and } h(u) = u, \text{ for } ||u|| = R\}$. We denote the genus of a symmetric subset A by $i(A)$ and, for $j = 1, \dots, k$, we let

$$\Gamma_j = \{h(\overline{D \setminus B}) \mid h \in G,\ k \geq j,\ i(B) \leq k - j\ \},$$

$$c_{j,\lambda} = \inf_{A\in\Gamma_j} \sup_{u\in A} \widetilde{I}_\lambda(u).$$

Then, by [3] (Proposition 9.30, p. 58), and under our conditions on f, we have that $0 < c_{1,\lambda} \leq c_{2,\lambda} \leq \cdots \leq c_{k,\lambda}$ are all critical values of $\widetilde{I}_\lambda$ and there are at least k pairs of critical points at these critical values. Since $Id \in G$ we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{2}{\beta-2}}\ ,$$

where $b_{k,\lambda}$ is as in Lemma 1.4. Then, by Lemma 2.2 and for λ large, these k pairs of critical points are also solutions of the original problem.

Finally, we make use of the fact that $\inf_{s\in\mathbb{R}} \widetilde{F}'(s) > -\infty$ (by construction) and follow the idea in [4] by restricting the above minimax procedure to X and taking the maximum outside W. Choose $R > 0$ such that $\widetilde{I}_\lambda(u) \leq 0$ for all $u \in Y_k$ with $||u|| \geq R$, and for all $\lambda \geq 1$. Let $D = B_R \cap Y_k$ and define $G = \{h \in C(D, X) \mid h \text{ is odd and } h(u) = u, \text{ for } ||u|| = R\}$. We also denote the genus of a symmetric subset A in X by again $i(A)$ and, for $j = 2, \dots, k$, let

$$\Gamma_j = \{h(\overline{D \setminus B}) \mid h \in G,\ k \geq j,\ i(B) \leq k - j\ \},$$

$$c_{j,\lambda} = \inf_{A\in\Gamma_j} \sup_{A\setminus W} \widetilde{I}_\lambda(u).$$

Then, by [4] (Theorem 2.3, p. 3214)) and, again, by the properties of $\widetilde{F}'$, we have that $0 < c_{2,\lambda} \leq c_{3,\lambda} \leq \cdots \leq c_{k,\lambda}$ are all critical values of $\widetilde{I}_\lambda$ and there are at least $(k-1)$ pairs of sign-changing critical points at these critical values. Also, since $Id \in G$, we have

$$c_{k,\lambda} \leq b_{k,\lambda} \leq C_k \lambda^{-\frac{2}{\beta-2}}\ ,$$

with $b_{k,\lambda}$ as in Lemma 1.4. Therefore, it follows from Lemma 2.2 and for λ large that these $(k-1)$ pairs of sign-changing critical points are also solutions of the original problem. $\square$

Remark. We point out that all the above results are still true when $N = 1, 2$.

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