PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 3, Pages 677–679
S 0002-9939(04)07643-9
Article electronically published on August 4, 2004

EQUIVALENCE OF THE NASH CONJECTURE FOR PRIMITIVE AND SANDWICHED SINGULARITIES

JESÚS FERNÁNDEZ-SÁNCHEZ

(Communicated by Michael Stillman)

ABSTRACT. We show that in order to prove the Nash Conjecture for sandwiched singularities it is enough to prove it for primitive singularities.

Let (X,Q) be a normal surface singularity and $f : \widehat{S} \longrightarrow (X,Q)$ its minimal resolution. An *arc* (an *arc of order* i) on (X,Q) is a local $\mathbb{C}$-morphism $\mathcal{O}_{X,Q} \longrightarrow \mathbb{C}[[t]]$ (a local $\mathbb{C}$-morphism $\mathcal{O}_{X,Q} \longrightarrow \mathbb{C}[[t]]/(t^{i+1})$). If $\{E_u\}_{u\in T_Q}$ are the irreducible exceptional components of $\widehat{S}$, let $\mathcal{F}_u^Q$ denote the set of arcs on (X,Q) such that the lifted arc $\widetilde{\varphi}$ on $\widehat{S}$ intersects E_u. Let $\mathcal{H}$ ($\mathcal{H}_i$ for $i \geq 0$) be the space of arcs (arcs of order i) on (X,Q). Define $Tr(i)$ to be the space of i-truncations of arcs on (X,Q), that is $Tr(i) = \rho_i(\mathcal{H})$, where $\rho_i : \mathcal{H} \longrightarrow \mathcal{H}_i$ is the morphism induced by the projection $\mathbb{C}[[t]] \longrightarrow \mathbb{C}[[t]]/(t^{i+1})$. Then, if $\mathcal{F}_u^Q(i) = \rho_i(\mathcal{F}_u^Q)$, the minimal resolution $f : \widehat{S} \longrightarrow (S,0)$ induces a decomposition of the space of i-truncations $Tr(i) = \bigcup_{u\in T_Q} \mathcal{F}_u^Q(i)$ and, by taking the Zariski closure in $\mathcal{H}_i$,

$$\overline{Tr(i)} = \bigcup_{u\in T_Q} \overline{\mathcal{F}_u^Q(i)}.$$

The Nash Conjecture says that for $i \gg 0$, this is just the decomposition of $\overline{Tr(i)}$ into its irreducible components (see [5]).

A normal surface singularity (X,Q) is said to be *sandwiched* if it dominates birationally a non-singular surface. They arise when a complete $\mathbf{m}$-primary ideal in a local regular $\mathbb{C}$-algebra R of dimension two is blown up. A sandwiched singularity is said to be *primitive* if it can be obtained by blowing up a *simple* ideal, that is, a complete irreducible ideal of R ([7] II.3). It is known that any sandwiched singularity is the birational join of some primitive singularities ([7] Corollary II.1.5). In this note, we prove that the Nash Conjecture for sandwiched singularities and for primitive singularities are equivalent.

First, we need a lemma.

Lemma 1. *Let (X_1,Q_1) be a rational surface singularity, $g : X \longrightarrow (X_1,Q_1)$ a birational dominant morphism and E_p an exceptional component of Q such that E_p also appears in the minimal resolution of Q_1 modulo birational equivalence (see*

Received by the editors August 27, 2003 and, in revised form, November 1, 2003.
2000 *Mathematics Subject Classification.* Primary 14B05, 14E15, 32B10.
Key words and phrases. Sandwiched singularity, arc, infinitely near point.
The author was partially supported by Generalitat de Catalunya 2001SGR00071 and EAGER, European Union contract HPRN-CT-2000-00099.

Definition 2.1 *of* [4]*). Keeping the notation as above, assume that for some* $i > 0$*,* $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ *and that the projection by* g *of any element of* $\mathcal{F}_q^Q(i)$ *is in* $\mathcal{F}_u^{Q_1}(i)$*, for some* $u \in T_{Q_1}$*. Then,*

$$\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}.$$

Proof. The morphism $g : X \longrightarrow (X_1, Q_1)$ is the blow-up of a complete ideal $J = (g_1, ..., g_m) \subset A = \mathcal{O}_{X_1,Q_1}$, and we may assume that $Q \in U_0$, where U_0 is an affine open set of X of the form $SpecA[g_1/g_0, ..., g_m/g_0] \subset \mathbb{A}_A^m$. Now, if $Spec(A) \subset \mathbb{A}_{\mathbb{C}}^n$, any arc γ on (X_1, Q_1) is written in the form $\gamma = (x_1, ..., x_n)$, $x_k \in \mathbb{C}[[t]]$. Thus, the lifting $\widetilde{\gamma}$ of γ on X is given by

$$(x_1(t), ..., x_n(t), \overline{g_1}(t)/\overline{g_0}(t), ..., \overline{g_m}(t)/\overline{g_0}(t))$$

where $\overline{g_k}(t) = g_k(x_1(t), ..., x_n(t))$, $k = 1, ..., m$.

If $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$, the i-truncation of any arc of $\mathcal{F}_p^Q$ can be approximated by the i-truncations of arcs of $\mathcal{F}_q^Q$. By taking the projections of these i-truncations on (X, Q), we see that $\mathcal{F}_p^{Q_1}(i) \subset \overline{g_*(\mathcal{F}_q^Q(i))} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$ and hence, $\overline{\mathcal{F}_p^{Q_1}(i)} \subset \overline{\mathcal{F}_u^{Q_1}(i)}$. □

Remark 1. A similar result has been proved independently by Camille Plénat [6].

From now on, assume that (X, Q) is a sandwiched singularity and that $X = Bl_I(R)$ is the surface obtained by blowing up a complete **m**-primary ideal I in a regular local two-dimensional $\mathbb{C}$-algebra R. Assume also that I satisfies the conditions of Corollary 1.14 of [7]. In particular, if

$$I = \prod_{j=1}^{N} I_j$$

is the factorization of I into simple complete ideals, we have that $I_j \neq I_k$ for $j \neq k$, and for $j = 1, \dots, N$ the surface $X_j = Bl_{I_j}(R)$ has only one singularity, that will be denoted by Q_j.

Following the notation introduced in [3], we denote by $\mathcal{K} = BP(I)$ and $\mathcal{K}_j = BP(I_j)$ the clusters of base points of I and the simple ideals I_j, for $j = 1, \dots, N$. We also denote by S_K and S_j the surfaces obtained by blowing up all the points of the clusters $\mathcal{K}$ and $\mathcal{K}_j$. The reader is referred to [2] for the connection between sandwiched singularities and clusters, and to chapters 3 and 4 of [1] for conventions and definitions about clusters.

It is known that the morphisms $f : S_K \longrightarrow X$ and $f_j : S_j \longrightarrow X_j$ induced by the universal property of the blowing up are the minimal resolutions of X and $X_j, j = 1, \dots, N$ respectively. Moreover, X and S_K are the birational join of the surfaces X_j and S_j for $j = 1, \dots, N$ ([7] Proposition 3.6). If we write $\alpha_j : X \longrightarrow X_j$ for the blow up of $I\mathcal{O}_{X_j}$ in X_j and $\sigma_j : S_K \longrightarrow S_j$ for the induced morphism, we have commutative diagrams of birational morphisms:

$$\begin{array}{ccc} S_K & \xrightarrow{f} & X \\ \downarrow \sigma_j & & \downarrow \alpha_j \\ S_j & \xrightarrow{f_j} & X_j \end{array}$$

By abuse of notation, we will write $\{E_u\}_{u \in T_{Q_j}}$ for the exceptional components of $f_j : S_j \longrightarrow X_j$.

Theorem 2. *Let $p, q \in T_Q$, $p \neq q$, be such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$. Then there exists some $j \in \{1, \ldots, N\}$ with $p \in T_{Q_j}$ and some $u \in T_{Q_j}, u \neq p$, such that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_u^{Q_j}(i)}$.*

Proof. Assume that $p, q \in T_Q$ are points such that $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $i \gg 0$. By Theorem 3.4 of [3], we know that q is not infinitely near to p. Let $\mathcal{K}_j$ be any irreducible subcluster of $\mathcal{K}$ containing p.

If p is infinitely near to q, then q is also in $\mathcal{K}_j$ and hence, the exceptional components E_p and E_q also appear in the minimal resolution S_j of Q_j modulo birational equivalence. By Lemma 1 applied to $\alpha_j : X \longrightarrow X_j$, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_q^{Q_j}(i)}$.

If p is not infinitely near to q, let u_0 be the maximal point of $\mathcal{K}_j$ to which q is infinitely near. Then the projection by α_j of any element in $\mathcal{F}_q^Q$ gives an arc γ on (X_j, Q_j) whose lifting to S_j intersects (transversally or not) the exceptional component E_{u_0} and therefore, $\gamma \in \mathcal{F}_{u_0}^{Q_j}$. Thus, the projection of any element of $\mathcal{F}_q^Q(i)$ is in $\mathcal{F}_{u_0}^{Q_1}(i)$ and by Lemma 1 again, we deduce that $\overline{\mathcal{F}_p^{Q_j}(i)} \subset \overline{\mathcal{F}_{u_0}^{Q_j}(i)}$.

In any case, we see that an inclusion of spaces of arcs on the sandwiched singularity (X, Q) implies a non-trivial inclusion of some spaces of arcs on the primitive singularity (X_j, Q_j). The claim follows. □

As a direct consequence, we have the following corollary.

Corollary 3. *If the Nash Conjecture is true for primitive singularities, then it is also true for sandwiched singularities.*

Remark 3. Notice that if (X, Q) is a primitive singularity and we have $\overline{\mathcal{F}_p^Q(i)} \subset \overline{\mathcal{F}_q^Q(i)}$ for $p, q \in T_Q$ with $p \neq q$, then by 3.4 of [3], p must be infinitely near to q and moreover, $e_O(I_p) > e_O(I_q)$ where $e_O(J)$ is the minimum of the multiplicities of the curves defined by elements of the complete ideal J.

References

[1] E. Casas-Alvero, *Singularities of plane curves*, volume 276 of *London Math. Soc. Lecture Note Series.* Cambridge University Press, 2000. MR1782072 (2003b:14035)

[2] J. Fernández-Sánchez, On sandwiched singularities and complete ideals. *J. Pure Appl. Algebra*, **185**: 165–175, 2003. MR2006424

[3] J. Fernández-Sánchez, Nash families of smooth arcs on a sandwiched singularity. To appear in *Math. Proc. Cambridge. Philos. Soc.*

[4] S. Ishii, J. Kollár, The Nash problem on arc families of singularities. *Preprint.* math.AG/0207171. *Duke Math. J.*, **120**: 601–620, 2003. MR2030097

[5] J. Nash, Arc structure of singularities. *Duke Math. J.*, **81**:31–38, 1996. MR1381967 (98f:14011)

[6] C. Plénat, A propos de la conjecture de Nash. *Preprint.* math.AG/0301358.

[7] M. Spivakovsky, Sandwiched singularities and desingularization of surfaces by normalized transformations. *Ann. of Math.*, **131**:411–491, 1990. MR1053487 (91e:14013)

Departament d'Àlgebra i Geometria, Universitat de Barcelona, Gran Via, 585, E–08007, Barcelona, Spain

E-mail address: jsan@cerber.mat.ub.es