

ON TENSOR PRODUCTS OF k -VERY AMPLE LINE BUNDLES

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ABSTRACT. In this paper, we show that the tensor product of a -very ample and b -very ample line bundles on a complete algebraic variety is $(a + b)$ -very ample.

1. INTRODUCTION

Let X be a complete algebraic variety over an algebraically closed field k and \mathcal{L} a line bundle on X . For a nonnegative integer $k \geq 0$, \mathcal{L} is said to be k -very ample if the restriction map $\Gamma(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{O}_Z)$ is surjective for every 0-dimensional subscheme Z of length $k + 1$. Note that \mathcal{L} is 0-very ample if and only if it is generated by its global sections, and that \mathcal{L} is 1-very ample if and only if it is very ample.

Beltrametti, Sommese, and others have studied “higher order embeddings”, e.g., k -spannedness, k -very ampleness and k -jet ampleness of polarized manifolds (see [2], [3], [4], [5], [6] and so on).

In this paper, we study the tensor products of k -very ample line bundles on X . As is well known, if \mathcal{L}_1 and \mathcal{L}_2 are a -spanned and b -spanned respectively, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $(a + b)$ -spanned (see [3, Lemma 0.5.3] and [1, Lemma 1.1.3]). Also, if \mathcal{L}_1 and \mathcal{L}_2 are a -jet ample and b -jet ample respectively, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $(a + b)$ -jet ample (see [5, Lemma 2.2]). However, it seems to be not clear whether this is the case for k -very ampleness or not (see [6, (0.3)]). The purpose of the paper is to give the affirmative answer for the question. Our main result is as follows.

Theorem 1.1. *Let X be a complete algebraic variety over an algebraically closed field k and $\mathcal{L}_1, \mathcal{L}_2$ line bundles on X . Let a, b be nonnegative integers. If \mathcal{L}_1 is a -very ample and \mathcal{L}_2 is b -very ample, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $(a + b)$ -very ample.*

We prove the theorem by induction on the length of 0-dimensional subschemes of X . The idea is to see the socle in an Artinian ring and to construct 0-dimensional subschemes of smaller length from the given 0-dimensional scheme.

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2. PRELIMINARIES

Let Z be a 0-dimensional subscheme of X . The *support* of Z , denoted by $|Z|$, is the underlying topological space of the reduced scheme Z_{red} . The *length* of Z is defined to be the dimension of the vector space $\Gamma(X, \mathcal{O}_Z)$.

2.1. k -very ampleness and Artinian rings. Let Z be a 0-dimensional subscheme of X of length l and $|Z| = \{x_1, \dots, x_r\}$. Assume that Z is defined by the ideal sheaf $\mathcal{J} \subset \mathcal{O}_X$, and then $\mathcal{O}_X/\mathcal{J} \cong \prod_{i=1}^r \mathcal{O}_{X,x_i}/\mathcal{J}_{x_i}$ and $\sum_{i=1}^r \ell(\mathcal{O}_{X,x_i}/\mathcal{J}_{x_i}) = l$. First note that the restriction map

$$e_{\mathcal{L},Z} : \Gamma(X, \mathcal{L}) \rightarrow \Gamma(X, \mathcal{L} \otimes \mathcal{O}_Z)$$

factors through $\prod_{i=1}^r \mathcal{L}_{x_i}$, that is, we have the following commutative diagram:

$$\begin{CD} \Gamma(X, \mathcal{L}) @>\rho_{\mathcal{L},|Z|}>> \prod_{i=1}^r \mathcal{L}_{x_i} \\ @V e_{\mathcal{L},Z} VV @VV \pi_{\mathcal{L},Z} V \\ \Gamma(X, \mathcal{L} \otimes \mathcal{O}_Z) @= \prod_{i=1}^r \mathcal{L}_{x_i}/\mathcal{J}_{x_i} \mathcal{L}_{x_i}, \end{CD}$$

where $\rho_{\mathcal{L},|Z|}$ sends $s \in \Gamma(X, \mathcal{L})$ to $(s_{x_1}, \dots, s_{x_r}) \in \prod_{i=1}^r \mathcal{L}_{x_i}$. Here note that $\pi_{\mathcal{L},Z}$ is surjective.

Let $A_Z = \prod_{i=1}^r \mathcal{O}_{X,x_i}$, $V_Z := \rho_{\mathcal{L},|Z|}(\Gamma(X, \mathcal{L}))$, and let J_Z be the ideal of A_Z such that $A_Z/J_Z = \prod_{i=1}^r \mathcal{O}_{X,x_i}/\mathcal{J}_{x_i}$. From now on, we will identify $\prod_{i=1}^r \mathcal{L}_{x_i}$ with A_Z and $\prod_{i=1}^r \mathcal{L}_{x_i}/\mathcal{J}_{x_i} \mathcal{L}_{x_i}$ with A_Z/J_Z respectively. Then we obtain the following.

Lemma 2.1. $e_{\mathcal{L},Z}$ is surjective if and only if $V_Z + J_Z = A_Z$.

Proof. If $e_{\mathcal{L},Z}$ is surjective, then for any element $a \in A_Z$ there exists a section $s \in \Gamma(X, \mathcal{L})$ such that $\pi_{\mathcal{L},Z}(a) = e_{\mathcal{L},Z}(s)$. We have $\pi_{\mathcal{L},Z}(a - \rho_{\mathcal{L},|Z|}(s)) = 0$, and $a - \rho_{\mathcal{L},|Z|}(s) \in J_Z$. Therefore $V_Z + J_Z = A_Z$.

Suppose that $V_Z + J_Z = A_Z$. It follows from the above diagram that, for any section $t \in \Gamma(X, \mathcal{L} \otimes \mathcal{O}_Z)$, there exists an element $b \in A_Z$ such that $\pi_{\mathcal{L},Z}(b) = t$. Since $V_Z + J_Z = A_Z$, we can write $b = \rho_{\mathcal{L},|Z|}(s) + f$ for some $s \in \Gamma(X, \mathcal{L})$ and $f \in J_Z$. Then

$$e_{\mathcal{L},Z}(s) = \pi_{\mathcal{L},Z}(\rho_{\mathcal{L},|Z|}(s)) = \pi_{\mathcal{L},Z}(b - f) = \pi_{\mathcal{L},Z}(b) = t.$$

Therefore $e_{\mathcal{L},Z}$ is surjective. □

From the definition of k -very ampleness and the above lemma, we obtain

Proposition 2.2. *With the above notation, \mathcal{L} is k -very ample if and only if $V_Z + J_Z = A_Z$ for any 0-dimensional subscheme $Z \subset X$ of length $k + 1$.*

2.2. k -very ampleness and finite sets. Let $S = \{x_1, \dots, x_r\}$ be a finite set of r points of X , and let Z be a 0-dimensional subscheme of X with the defining ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $|Z| \subseteq S$. We may assume that $|Z| = \{x_1, \dots, x_t\}$ for $t \leq r$. Consider the following commutative diagram (with the identification noted above):

$$\begin{CD} \Gamma(X, \mathcal{L}) @>\rho_{\mathcal{L},|Z|}>> \prod_{i=1}^t \mathcal{O}_{X,x_i} @<\text{pr}_{S,Z}<< \prod_{i=1}^r \mathcal{O}_{X,x_i} \\ @V e_{\mathcal{L},Z} VV @VV \pi_{\mathcal{L},Z} V @VV \pi_{S,Z} V \\ \Gamma(X, \mathcal{L} \otimes \mathcal{O}_Z) @= \prod_{i=1}^t \mathcal{O}_{X,x_i}/\mathcal{I}_{x_i} @= \prod_{i=1}^r \mathcal{O}_{X,x_i}/\mathcal{I}_{x_i}, \end{CD}$$

where $\mathcal{I}_{x_i} = \mathcal{O}_{X,x_i}$ for $i = t + 1, \dots, r$ and $pr_{S,Z}$ is the projection. We define

$$\rho_{\mathcal{L},S} : \Gamma(X, \mathcal{L}) \longrightarrow \prod_{i=1}^r \mathcal{O}_{X,x_i}$$

by sending $s \in \Gamma(X, \mathcal{L})$ to $(s_{x_1}, \dots, s_{x_r}) \in \prod_{i=1}^r \mathcal{O}_{X,x_i}$, and then $pr_{S,Z} \circ \rho_{\mathcal{L},S} = \rho_{\mathcal{L},|Z|}$.

Let $A_S = \prod_{i=1}^r \mathcal{O}_{X,x_i}$, $V_S = \rho_{\mathcal{L},S}(\Gamma(X, \mathcal{L}))$, and let I be the ideal of A_S such that $A_S/I = \prod_{i=1}^r \mathcal{O}_{X,x_i}/\mathcal{I}_{x_i}$. Then we obtain the following lemma by the same argument as in the proof of Lemma 2.1.

Lemma 2.3. *With the above notation, $e_{\mathcal{L},Z}$ is surjective if and only if $V_S + I = A_S$ in A_S .*

Remark 2.4. In the above lemma, note that A_S and V_S do not depend on the 0-dimensional subscheme Z but on the finite set S .

3. PROOF OF THEOREM 1.1

Let Z be a 0-dimensional subscheme of X of length $a + b + 1$, $|Z| = \{x_1, \dots, x_r\}$, and let $\mathcal{J} \subset \mathcal{O}_X$ be the defining ideal sheaf of Z . Put $A = \prod_{i=1}^r \mathcal{O}_{X,x_i}$ and let J be the ideal of A such that $A/J = \prod_{i=1}^r \mathcal{O}_{X,x_i}/\mathcal{J}_{x_i}$, and then

$$\ell(A/J) = \sum_{i=1}^r \ell(\mathcal{O}_{X,x_i}/\mathcal{J}_{x_i}) = a + b + 1.$$

Let $\rho_{\mathcal{L}_i,|Z|} : \Gamma(X, \mathcal{L}_i) \rightarrow A$ for $i = 1, 2$ and $\rho_{\mathcal{L}_1 \otimes \mathcal{L}_2,|Z|} : \Gamma(X, \mathcal{L}_1 \otimes \mathcal{L}_2) \rightarrow A$ be the maps defined above. Then we have the following commutative diagram:

$$\begin{CD} \Gamma(X, \mathcal{L}_1) \otimes \Gamma(X, \mathcal{L}_2) @>\rho_{\mathcal{L}_1,|Z|} \otimes \rho_{\mathcal{L}_2,|Z|}>> A \otimes A \\ @V \text{mult.} VV @VV \text{mult.} V \\ \Gamma(X, \mathcal{L}_1 \otimes \mathcal{L}_2) @>\rho_{\mathcal{L}_1 \otimes \mathcal{L}_2,|Z|}>> A, \end{CD}$$

where each mult. is the multiplication map. Hence we obtain

$$(3.1) \quad V_1 V_2 \subset V,$$

where $V_i := \rho_{\mathcal{L}_i,|Z|}(\Gamma(X, \mathcal{L}_i))$ for $i = 1, 2$, $V := \rho_{\mathcal{L}_1 \otimes \mathcal{L}_2,|Z|}(\Gamma(X, \mathcal{L}_1 \otimes \mathcal{L}_2))$, and $V_1 V_2$ is the set of all elements of the form $\sum_i u_i v_i$ ($u_i \in V_1, v_i \in V_2$).

Since \mathcal{L}_1 is a -very ample, $e_{\mathcal{L}_1,Z'}$ is surjective for any 0-dimensional subscheme $Z' \subseteq Z$ of length $a + 1$, and from Lemma 2.3, we see that $V_1 + I = A$ for any ideal I of A such that $\ell(A/I) = a + 1$. Similarly, $V_2 + I' = A$ for any ideal I' of A such that $\ell(A/I') = b + 1$. In order to prove that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is $(a + b)$ -very ample, by Proposition 2.2, we have to show that $V + J = A$, hence $V_1 V_2 + J = A$ from (3.1). Therefore it is sufficient to prove the following claim.

Claim 3.1. *Let (A_i, \mathfrak{m}_i, k) be local rings, let $A = \prod_{i=1}^r A_i$ be the direct product, and let V_1, V_2 be k -vector spaces contained in A . Assume that A contains k and $V_1 + I = A$ (resp. $V_2 + I' = A$) for any ideal I (resp. I') of A such that $\ell(A/I) = a + 1$ (resp. $\ell(A/I') = b + 1$). Then $V_1 V_2 + J = A$ for any ideal J of A such that $\ell(A/J) = a + b + 1$, where $V_1 V_2$ is the set of all elements of the form $\sum_i u_i v_i$ ($u_i \in V_1, v_i \in V_2$).*

Proof. Let 1_i be the identity element of A_i for $i = 1, \dots, r$. We identify 1_i with the element $(0, \dots, 0, 1_i, 0, \dots, 0) \in A$. Then $1_1, \dots, 1_r$ are orthogonal idempotents of A , that is, $1_i^2 = 1_i$ for each i , $1_i 1_j = 0$ if $i \neq j$, and $\sum_{i=1}^r 1_i = 1_A$.

We prove the claim by induction on a and b .

Assume that $a = 0$. If I is an ideal of A such that $\ell(A/I) = 1$, then I is of the form

$$I = A_1 \times \dots \times A_{i-1} \times \mathfrak{m}_i \times A_{i+1} \times \dots \times A_r$$

for some i . Since $V_1 + I = A$ by the assumption, there exist $v_1 \in V_1$, $m_i \in \mathfrak{m}_i$, and $c_i \in \prod_{j \neq i} A_j$ such that $v_1 + m_i + c_i = 1_i$. Hence we obtain $v_1 = 1_i - m_i - c_i$.

If J is any ideal such that $\ell(A/J) = b + 1$, then $V_2 + J = A$ by the assumption, and in particular we may assume that $A_i \subset V_2 + J$ for any i .

Now we show that $V_1 V_2 + J = A$. To do this, it is sufficient to prove $A_i \subset V_1 V_2 + J$ for all i . By the above assumption, we have

$$v_1 A_i \subset v_1 V_2 + v_1 J, \quad \text{and} \quad v_1 A_i = (1_i - m_i - c_i) A_i = A_i$$

since $c_i A_i = 0$ and $1_i - m_i$ is a unit in A_i . Therefore we obtain

$$A_i = v_1 A_i \subset v_1 V_2 + v_1 J \subset V_1 V_2 + J.$$

Similarly we can prove the case $b = 0$.

Next assume that $a > 0$ and $b > 0$. Then the induction assumption is:

(A1) if $a' \leq a$, $b' \leq b$, $a' + b' < a + b$, and J' is any ideal of A such that $\ell(A/J') = a' + b' + 1$, then $V_1 V_2 + J' = A$.

This is equivalent to the assumption:

(A2) if J'' is any ideal of A such that $\ell(A/J'') < a + b + 1$, then $V_1 V_2 + J'' = A$. Indeed, there are nonnegative integers a', b' such that $a' \leq a$, $b' \leq b$, $a' + b' < a + b$, and $\ell(A/J'') = a' + b' + 1$. From the assumption (A1), we see that $V_1 V_2 + J'' = A$. We will use this assumption repeatedly.

Let J be an ideal of A such that $\ell(A/J) = a + b + 1$. Now we write A (resp. \mathfrak{m}) instead of A/J (resp. \mathfrak{m}/J) for simplicity, and we show that, if $\ell(A) = a + b + 1$, then $V_1 V_2 = A$. To do this, we recall the definition of the socle of A . The socle of A is defined to be the sum of all the ideals of length 1 of A and is denoted by $\text{Soc}(A)$.

Case 1. Assume that $\dim_k(\text{Soc}(A)) \geq 2$. Let $\mathfrak{n}_1, \mathfrak{n}_2$ be any two distinct minimal ideals in $\text{Soc}(A)$. Since $\ell(A/\mathfrak{n}_2) < a + b + 1$, it follows from assumption (A2) that $V_1 V_2 + \mathfrak{n}_2 = A$. For any nonzero element x of \mathfrak{n}_1 , there exist elements $v \in V_1 V_2$ and $y \in \mathfrak{n}_2$ such that $v + y = x$. Here note that $v \neq 0$ since $\mathfrak{n}_1 \cap \mathfrak{n}_2 = 0$. Let \mathfrak{n}' be the ideal of A generated by v . Then $\ell(A/\mathfrak{n}') < a + b + 1$ and $\mathfrak{n}' = A \cdot v = \mathfrak{k} \cdot v \subset V_1 V_2$ since $v = x - y \in \text{Soc}(A)$. Therefore by assumption (A2), we obtain

$$A = V_1 V_2 + \mathfrak{n}' = V_1 V_2.$$

Case 2. Assume that $\dim_k(\text{Soc}(A)) = 1$. Then A is a local ring and has the unique minimal ideal \mathfrak{n} . Since $\ell(A/\mathfrak{n}) < a + b + 1$, by assumption (A2), we obtain

$$V_1 V_2 + \mathfrak{n} = A.$$

It follows from [8, Theorem 221] that A is a 0-dimensional Gorenstein ring. For any ideal \mathfrak{a} of the 0-dimensional Gorenstein ring A , we obtain $(0 : (0 : \mathfrak{a})) = \mathfrak{a}$ by [7, Exercise 3.2.15], and

$$(3.2) \quad \ell((0 : \mathfrak{a})) = \ell(A/\mathfrak{a}) \quad \text{and} \quad \ell(\mathfrak{a}) = \ell(A/(0 : \mathfrak{a}))$$

by considering the length of the composition series.

Let $\mathfrak{b}, \mathfrak{c}$ be any ideals of A such that $\mathfrak{b} \supset \mathfrak{c}$, $\ell(\mathfrak{b}) = b + 1$, and $\ell(\mathfrak{c}) = b$. From (3.2), we obtain $\ell(A/(0 : \mathfrak{b})) = b + 1$ and $\ell(A/\mathfrak{c}) = a + 1$. By the assumption of the claim, $V_2 + (0 : \mathfrak{b}) = A$, and for any element $x \in (0 : \mathfrak{c})$, there exist $v_2 \in V_2$ and $z \in (0 : \mathfrak{b})$ such that $v_2 + z = x$. Similarly, $V_1 + \mathfrak{c} = A$, and for any element $y \in \mathfrak{b}$, there exist $v_1 \in V_1$ and $w \in \mathfrak{c}$ such that $v_1 + w = y$. Then

$$v_1 v_2 = (y - w)(x - z) = yx - yz - wx + wz = xy.$$

Indeed, $yz \in \mathfrak{b} \cdot (0 : \mathfrak{b}) = 0$, $wx \in \mathfrak{c} \cdot (0 : \mathfrak{c}) = 0$, and $wz \in \mathfrak{c} \cdot (0 : \mathfrak{b}) \subset \mathfrak{b} \cdot (0 : \mathfrak{b}) = 0$. Hence we obtain

$$(3.3) \quad (0 : \mathfrak{c}) \cdot \mathfrak{b} \subset V_1 V_2.$$

Moreover $(0 : \mathfrak{c}) \supsetneq (0 : \mathfrak{b})$, and then $(0 : \mathfrak{c}) \cdot \mathfrak{b} \neq 0$. We obtain

$$(3.4) \quad \mathfrak{n} \subset (0 : \mathfrak{c}) \cdot \mathfrak{b}$$

because \mathfrak{n} is the unique minimal ideal of A . Therefore by (3.3) and (3.4) we obtain $\mathfrak{n} \subset V_1 V_2$, and then

$$A = V_1 V_2 + \mathfrak{n} = V_1 V_2.$$

This completes the proof of the claim, hence Theorem 1.1. □

We obtain the following corollary inductively.

Corollary 3.2. *Let X be a complete algebraic variety over an algebraically closed field k and $\mathcal{L}_1, \dots, \mathcal{L}_m$ line bundles on X . Let a_1, \dots, a_m be nonnegative integers. If \mathcal{L}_i is a_i -very ample for $i = 1, \dots, m$, then $\mathcal{L}_1 \otimes \dots \otimes \mathcal{L}_m$ is $(a_1 + \dots + a_m)$ -very ample.*

ADDED IN PROOF

After the paper was accepted for publication, we found that there is an error in the proof of Case 1 in Claim 3.1. In Case 1, we do not have $A \cdot v = k \cdot v$, in general. We have to change Case 1 and Case 2 into the following:

Case 1. Assume that $\dim_k(\text{Soc}(A_i)) \geq 2$ for some i . Let $\mathfrak{n}_1, \mathfrak{n}_2$ be any two distinct minimal ideals in $\text{Soc}(A_i)$. Since $\ell(A/\mathfrak{n}_2) < a + b + 1$, it follows from assumption (A2) that $V_1 V_2 + \mathfrak{n}_2 = A$. For any nonzero element x of \mathfrak{n}_1 , there exist elements $v \in V_1 V_2$ and $y \in \mathfrak{n}_2$ such that $v + y = x$. Here note that $v \neq 0$, since $\mathfrak{n}_1 \cap \mathfrak{n}_2 = 0$. Let \mathfrak{n}' be the ideal generated by v . Then $\ell(A/\mathfrak{n}') < a + b + 1$ and $\mathfrak{n}' = A \cdot v = k \cdot v \subset V_1 V_2$ since $v = x - y \in \text{Soc}(A_i)$. Therefore by assumption (A2), we obtain

$$A = V_1 V_2 + \mathfrak{n}' = V_1 V_2.$$

Case 2. Assume that $\dim_k(\text{Soc}(A_i)) = 1$ for all i . Then A_i itself is a local ring, and A_i has a unique minimal ideal \mathfrak{n}_i . It follows from [8, Theorem 221] that A_i is a 0-dimensional local Gorenstein ring. For any ideal \mathfrak{a} of the direct product A of 0-dimensional local Gorenstein rings A_i , we obtain $(0 : (0 : \mathfrak{a})) = \mathfrak{a}$ by [7, Exercise 3.2.15], and

$$(3.2) \quad \ell((0 : \mathfrak{a})) = \ell(A/\mathfrak{a}) \quad \text{and} \quad \ell(\mathfrak{a}) = \ell(A/(0 : \mathfrak{a}))$$

by considering the length of the composition series. Let $\mathfrak{b}, \mathfrak{c}$ be any ideals of A such that $\mathfrak{b} \supset \mathfrak{c}$, $\ell(\mathfrak{b}) = b + 1$, and $\ell(\mathfrak{c}) = b$. From (3.2), we obtain $\ell(A/(0 : \mathfrak{b})) = b + 1$ and $\ell(A/\mathfrak{c}) = a + 1$. By the assumption of the claim, $V_2 + (0 : \mathfrak{b}) = A$, and for any element $x \in (0 : \mathfrak{c})$, there exist $v_2 \in V_2$ and $z \in (0 : \mathfrak{b})$ such that $v_2 + z = x$.

Similarly, $V_1 + \mathfrak{c} = A$, and for any element $y \in \mathfrak{b}$, there exist $v_1 \in V_1$ and $w \in \mathfrak{c}$ such that $v_1 + w = y$. Then

$$v_1 v_2 = (y - w)(x - z) = xy - yz - xw + zw = xy.$$

Indeed, $yz \in \mathfrak{b} \cdot (0 : \mathfrak{b}) = 0$, $wx \in \mathfrak{c} \cdot (0, \mathfrak{c}) = 0$, and $wz \in \mathfrak{c} \cdot (0 : \mathfrak{b}) \subset \mathfrak{b} \cdot (0 : \mathfrak{b}) = 0$. Hence we obtain

$$(3.3) \quad (0 : \mathfrak{c}) \cdot \mathfrak{b} \subset V_1 V_2.$$

Moreover $(0 : \mathfrak{c}) \supsetneq (0 : \mathfrak{b})$, and then $(0 : \mathfrak{c}) \cdot \mathfrak{b} \neq 0$. Therefore by assumption (A2), we have

$$A = V_1 V_2 + (0 : \mathfrak{c}) \cdot \mathfrak{b} = V_1 V_2.$$

This completes the proof of the claim, hence Theorem 1.1.

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