EXPONENTIAL DECAY OF CORRELATIONS FOR SURFACE SEMI-FLOWS WITHOUT FINITE MARKOV PARTITIONS

VIVIANE BALADI AND BRIGITTE VALLÉE

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ABSTRACT. We extend Dolgopyat’s bounds on iterated transfer operators to suspensions of interval maps with infinitely many intervals of monotonicity.

1. Statement of results

Let $0 < c_1 < \ldots < c_m < c_{m+1} < \ldots < 1$ be a finite or countable partition of $I = [0,1]$ into subintervals, and let $T : I \to I$ be so that $T|_{(c_m,c_{m+1})}$ is $C^2$ and extends to a homeomorphism from $[c_m,c_{m+1}]$ to $I$. We assume that $T$ is piecewise uniformly expanding: there exist $C \geq 1$ and $\rho < 1$ so that $|h(x) - h(y)| \leq C\rho^n|x-y|$ for every inverse branch $h$ of $T^n$ and all $n$. Let $\mathcal{H}$ be the set of inverse branches $h : I \to [c_m,c_{m+1}]$ of $T$. We suppose (Renyi’s condition) that there is a $K > 0$ so that every $h \in \mathcal{H}$ satisfies $|h''| \leq K|h'|$. Let $r : I \to \mathbb{R}_+$ be so that $r|_{(c_m,c_{m+1})}$ is $C^1$, and $\inf r > 0$. Assume that there is $\sigma_0 < 0$ so that $\sum_{h \in \mathcal{H}} \sup \exp(-\sigma(r \circ h))|h'| < \infty$ for all $\sigma > \sigma_0$, and that $|r' \circ h| \cdot |h'| \leq K$ for all $h \in \mathcal{H}$. For $n \geq 1$, write $r^{(n)}(x) = \sum_{k=0}^{n-1} r(T^k)(x)$.

We study the transfer operators, indexed by $s = \sigma + it$,

$$L_s f(x) = \sum_{T(y) = x} e^{-sr(y)} \frac{f(y)}{|T'(y)|} = \sum_{h \in \mathcal{H}} e^{-sr(h(x))}|h'(x)| \cdot (f \circ h)(x),$$

acting on $C^1(I)$, with norm $\|f\| = \sup |f| + \sup |f'|$. Note that the $L_s$ are the transfer operators associated to the (Fourier transform of the correlation function for the) absolutely continuous invariant probability measure of the suspension semiflow defined by $\phi^t(x,s) = (x, s+t)$ on the branched surface $\{(x,s) \in I \times \mathbb{R}_+ | s \leq r(x)\}/\sim$, with $(x, r(x)) \sim (T(x), 0)$. See e.g. [5].

Finally, the following assumption is a translation of Dolgopyat’s “uniform non-integrability of foliations” condition (see [1, 2, 6] for formulations closer to ours): we say that the pair $(T,r)$ satisfies UNI if there exist $D > 0$ and $n_0 \geq 1$ such that, for every integer $n \geq n_0 \geq 1$, there exist two elements $h, k$ of the set $\mathcal{H}_n$ of inverse branches of $T^n$ so that the function on $I$ defined by $\psi_{h,k} : = r^{(n)}(h(x)) - r^{(n)}(k(x))$ satisfies $\inf |\psi_{h,k}'| \geq D$. (See also Remark 2.3.)

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To state our main result, we use the equivalent norms \( \|f\|_{1,t} = \sup |f| + \frac{\sup |f'|}{|t|} \), for \(|t| \geq \epsilon_0 > 0\), on \( C^1(I) \).

**Theorem 1.1.** Let \( T \) and \( r \) satisfy the assumptions above (in particular, UNI for \( D \)). Then there is \( A \geq n_0 \) and \( \gamma < 1 \) so that for all \( \sigma \), close enough to 0, all \(|t| \geq \max(2\pi/D,4)\), and all \( n \geq A \log |t| \), we have \( \|L_n^s\|_{1,t} \leq \gamma^n \).

Theorem 1.1 was proved by Dolgopyat [3] when \( \mathcal{H} \) is finite. In [2], we considered the special case when \( T(x) = \{1/x\} \) (or analogues of the Gauss map) and \( r = \log |T'| \), working with a different version of UNI, adapted to “algebraic” situations.

Note that the present UNI assumption also holds in the setting of [2]: if \( h \in \mathcal{H}_n \) is a linear fraction \((ax + b)/(cx + d)\), then \( h''(x)/h'(x) = -2c/(cx + d) \) so that \(|\psi'_{h,\hat{h}}(x)| = |2c/(cx + d) - c/(\hat{cx} + \hat{d})|\). Write \( \mathcal{F}_n \) for the \( n \)th Fibonacci number and \( \hat{\mathcal{F}}_n \) for the sequence 0, 1, \( \hat{\mathcal{F}}_n = 2\hat{\mathcal{F}}_{n-1} + \hat{\mathcal{F}}_{n-2} \). For \( h \) and \( \hat{h} \) in \( \mathcal{H}_n \) associated to the sequence of digits 1, 1, . . . , 1, and 2, 2, . . . , 2, we get \( a = \mathcal{F}_{n-2}, b = c = \mathcal{F}_{n-1}, \) and \( d = \mathcal{F}_n \), while \( \hat{a} = \hat{\mathcal{F}}_{n-2}, \hat{b} = \hat{c} = \hat{\mathcal{F}}_{n-1}, \) and \( \hat{d} = \hat{\mathcal{F}}_n \). We conclude by using \( \lim_{n \to \infty} \mathcal{F}_n/\mathcal{F}_{n-1} = (1 + \sqrt{5})/2 \) and \( \lim_{n \to \infty} \hat{\mathcal{F}}_n/\hat{\mathcal{F}}_{n-1} = (1 + \sqrt{8})/2 \).

From Theorem 1.1 one easily gets (see e.g. [2]):

**Corollary 1.2.** For every \( 0 < \alpha < 1 \) there is \( t_0 \) so that for all \(|t| > t_0 \) and \( \sigma \) close to 0, we have \( \|((Id - L_{\sigma})^{-1})_{1,t} \leq |t|^{\alpha} \).

Theorem 1.1 implies [5, section 4] exponential decay of correlations for \( C^1 \) observables and the absolutely continuous invariant probability (SRB) measure of the semi-flow \( \phi^t \). We hope this will be a useful step towards proving exponential decay of correlations for (continuous-time) planar Sinai billiards, using [8]. (For the moment, only open continuous-time billiards have been considered [7], and they admit finite Markov sections.) See Remark 2.1 for extensions to other Gibbs states.

2. PROOF OF THEOREM 1.1

We basically follow Dolgopyat’s proof, as detailed in [5], [6], and [1]. A key point is the Federer property of any absolutely continuous measure \( \nu \) with continuous density bounded from above and from below: There exist \( C, C' > 0 \) so that if \( I, J \) are adjacent intervals with \(|I| \geq |J|/C\), then \( \nu(I) \geq \nu(J)/C' \). To exploit this information when considering \( L_\sigma \) for \( \sigma \neq 0 \), the arguments in [3] (e.g. last lines of p. 367) and [1] (e.g. first lines of p. 43) use that there is \( \alpha_\sigma \to 1 \) when \( \sigma \to 0 \) so that \( \bar{L}_\sigma f(x) \leq \alpha_\sigma \bar{L}_0 f(x) \), for the normalised operators in (2.1) and positive \( f \). The above inequality uses that there are finitely many branches and is for example not true for the Gauss map. To bypass this problem, we exploit carefully the Cauchy-Schwartz decomposition in Lemma 2.3 below (see also [2], Lemma 2).

**Remark 2.1.** Beware that even when there are finitely many branches, the Federer property is not true for arbitrary Gibbs measures \( \nu \), in particular the measures \( \nu_\sigma \) introduced below for \( \sigma \neq 0 \), contrary to what is stated in [3, Proposition 7]; [3, Lemma 6]; and [6, Lemma 4]. (Proposition 7 of [5] is true e.g. if \( T \) is a \( C^2 \) circle map, and if \( r \) is \( C^1 \) on the circle, and not only piecewise \( C^1 \). For a counterexample, take \( T(x) = 2x \) modulo 1 with \( \exp(r) \equiv 3 \) on \([0, 1/2]\) and \( \exp(r) \equiv 3/2 \) on \((1/2, 1]\), and consider the intervals of size \( 1/2^n \) to the right and to the left of \( 1/2 \). By adding \( \epsilon \sin(2\pi x) \) to \( r \), this example can probably be made to satisfy the UNI condition [5, p. 537].) When there are finitely many branches, the Federer property does hold [3].
for Gibbs measures and “most” adjacent intervals from the partitions in [3, 5, 9]:
This is enough e.g. to recover the results in [3], in particular Theorem 1. When \( \mathcal{H} \)
is infinite, the situation is more complicated, but we expect that Theorem 1 will
also hold for more general transfer operators \( L_{s,g} \) associated to suitable positive \( g \).

**Preliminary steps.** Fix from now on \( \rho < \sigma < 1 \). The inverse branches of \( T^n \)
satisfy \(|h''| \leq K|h'|\) for all \( n \) and the distortion constant \( K = K/(1 - \rho) \), and
similarly for \((r(n))'\circ h\). As a consequence it is easy to prove that, for every \( n \geq 1 \), and
each pair \( h, k \) in \( \mathcal{H}_n \), the function \( \psi_{h,k}(x) = r(n)\circ h - r(n)\circ k \) satisfies \( \sup |\psi_{h,k}'| \leq 2K \).
We next recall spectral properties of the \( L_s \) (see e.g. [2] and references therein).
Let \( \sigma > \sigma_0 \) be real. The essential spectral radius \( \sigma^e \) of \( L_\sigma \) is strictly smaller than its
spectral radius \( \lambda_\sigma \) (in fact \( \lambda^e_\sigma \leq \rho \lambda_\sigma \)). Since \( T \) is topologically mixing, the operator \( L_\sigma \) has a unique (simple) eigenvalue \( \lambda_\sigma \) of maximal modulus, for a strictly positive
\( \sigma \) eigenfunction \( f_\sigma \); the rest of the spectrum is in the disc of radius \( \tau_\sigma \lambda_\sigma \) for some
\( \tau_\sigma < 1 \). The eigenvector \( \mu_\sigma \) of \( L_\sigma^* \) for \( \lambda_\sigma \) is Lebesgue measure for \( \sigma = 0 \), and for all
\( \sigma > \sigma_0 \) is a positive Radon measure \( \mu_\sigma \). We may assume \( \mu_\sigma(1) = 1 \) and \( \rho_\sigma(f_\sigma) = 1 \)
so that \( \nu_\sigma = f_\sigma \cdot \mu_\sigma \) is a probability measure. Note that \( L_\sigma : C^1(I) \to C^1(I) \) depends
continuously on \( \sigma \), so that \( \lambda^{\pm 1}_\sigma, \tau_\sigma, f^{\pm 1}_\sigma \), and \( f'_\sigma \) depend continuously on \( \sigma \)
and therefore satisfy uniform bounds in any compact subset \( \Sigma \subset (\sigma_0, \infty) \). Also, \( \sigma \mapsto \lambda_\sigma \)
is a nonincreasing function. Finally, the spectral radius of \( L_{\sigma + it} \) is not larger than \( \rho \lambda_\sigma \) for all \( t \in \mathbb{R} \).

It will be convenient to work with the normalised operators

\[
\tilde{L}_\sigma(f) = \frac{L_\sigma(f \cdot f)}{\lambda_\sigma f_\sigma}, \quad s = \sigma + it.
\]

If \( s = \sigma > \sigma_0 \), the operator \( \tilde{L}_\sigma \) acting on \( C^1(I) \) has spectral radius 1, essential
spectral radius \( \leq \rho \), and fixes the constant function \( \equiv 1 \). Clearly \( \tilde{L}_\sigma^* \) preserves the
probability measure \( \nu_\sigma = f_\sigma \cdot \mu_\sigma \). Our starting point is a Lasota-Yorke inequality:

**Lemma 2.2** (Lasota-Yorke). For every compact \( \Sigma \subset (\sigma_0, \infty) \), there is a constant
\( C = C(\Sigma, K) > 0 \), so that for all \( n \geq 1 \), all \( s \in \Sigma \) and all \( f \in C^1(I) \):

\[
|\tilde{L}_\sigma^n f'(x)| \leq C(\Sigma, K) |s| \cdot \tilde{L}_\sigma^n(|f'|)(x) + \rho^n \cdot \tilde{L}_\sigma^n(|f'|)(x).
\]

**Lemma 2.2** indicates that we should concentrate on the sup component of the norm, which will be estimated by the \( L^2(d\nu_0) \) norm (see the beginning of the proof of Theorem 1.1 below). Indeed, the crucial estimate (**Lemma 2.8**) will show exponential decay of iterates of the operator in the \( L^2(d\nu_0) \) norm. These \( L^2(d\nu_0) \) integrals are oscillatory integrals in disguise: because of the weights \( \exp(-s(r \circ h)) \) in \( L_s \), the integrand is the absolute value of a sum of complex numbers with strong phase variations for large \( |t| \) (UNI is crucial here). We shall exhibit enough cancellations in the terms, via the key Lemma 2.4.

**Proof.** The Leibniz sum for the derivative of each term \( \exp(-s(r(n) \circ h))|h'| \cdot \frac{1}{\lambda_\sigma f_\sigma} \cdot (f_\sigma f \circ h) \) forming \( \tilde{L}_\sigma^n(f) \) (\( h \in \mathcal{H}_n \)) contains four terms. We can bound the first for all \( s \) using our “distortion” assumption on \( r \) since \( |s||(r(n))' \circ h||h'| e^{-s(r \circ h)} \leq |s| |K e^{-s(r \circ h)} | \). The second one is controlled by the Renyi assumption on \( T \). Compactness of \( \Sigma \) and continuity of \( \sigma \mapsto \lambda_\sigma \) and \( \sigma \mapsto f_\sigma \) imply \( \inf_{\sigma \in \Sigma} f_\sigma > 0 \), so that the third term may be controlled by \( |f'_\sigma| \cdot \frac{1}{\lambda_\sigma f_\sigma} \leq C(\Sigma, K) |s| \cdot \tilde{L}_\sigma^n(|f'|)(x) \).
some $C_2 > 0$. Finally, the last term can be estimated using
\[ |(f_\sigma \cdot f)' \circ h||h'|| \leq \rho^n |[f_\sigma' \cdot f] \circ h + (f_\sigma \cdot |f'|) \circ h| \cdot \rho^n |h| \cdot \rho^n |h| \cdot \rho^n |h| \cdot \rho^n |h|. \]
We can ensure $K|s| + 2C_2 + 2\rho(C_2 \leq C(\Sigma, K)|s|$ (for fixed $\Sigma$, if $|s|$ is large, i.e., if $|t|$ is large enough, then $C(\Sigma, K)$ is close to $K$).

We next state and prove an elementary lemma about complex numbers with almost opposite phases. Note that $2/3 < (\sqrt{7} - 1)/2 < 1$.

**Lemma 2.3 (Calculus lemma).** For each $\eta \in [(\sqrt{7} - 1)/2, 1)$ and every pair of complex numbers, $z_1 = r_1 \exp(i\theta_1)$ and $z_2 = r_2 \exp(i\theta_2)$,
\[ \cos(\theta_1 - \theta_2) \leq 1/2 \Rightarrow |z_1 + z_2| \leq \max(\eta r_1 + r_2, r_1 + \eta r_2). \]

**Proof.** Up to exchanging $z_1$ and $z_2$, we can suppose that $r_1 \leq r_2$ so that $\eta r_1 + r_2 \geq r_1 + \eta r_2$. Our assumption on $\cos(\theta_1 - \theta_2)$ implies
\[ |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2) \leq r_1^2 + r_2^2 + r_1r_2. \]
Since $(\eta r_1 + r_2)^2 = \eta^2 r_1^2 + r_2^2 + 2\eta r_1 r_2$, we must show $r_1^2(1 - \eta^2) + 2r_1r_2(1 - 2\eta) \leq 0$.
Since $-1/2 \geq 1 - \eta^2 \geq 0$ (use $3 \geq \sqrt{7} \geq 2$), we get
\[ r_1^2(1 - \eta^2) + 2r_1r_2(1 - 2\eta) \leq r_1^2(\eta - 1/2) + r_1r_2(1/2 - 2\eta) \leq r_1(\eta - 1/2)(r_1 - r_2) \leq 0. \]

**Preparatory lemmas in view of $C^2$ contraction.** In the next lemma, we combine UNI and Lemma 2.3 to obtain cancellation-type estimates on terms appearing when applying iterates of $L_\sigma$ to a suitable pair $(u, v)$ of initial test functions in $C^1(I)$. We first introduce the “cone” condition that $(u, v)$ must satisfy: there exist $C > 0$ and $t \in \mathbb{R}$ so that
\[ (2.4) \quad u > 0, \quad 0 \leq |v| \leq u, \quad \max(|u'(x)|, |v'(x)|) \leq 2C|t|u(x). \]

**Lemma 2.4 (Exhibiting cancellations).** Assume that UNI holds for $D$ and $n_0$. Then, for all $C > 0$, there exist $n_1 \geq n_0$, $\delta > 0$ and $\Delta > 0$, so that for any $|t| > 2\pi/D$, and all $u, v \in C^1(I)$ satisfying (2.4) for $C$ and $t$, we have the following:
Fix $n \geq n_1$, and let $h, k \in H_n$ be the branches from UNI. For every $x_0 \in I$, there is $x_1 \in I$ with $|x_0 - x_1| < \Delta/|t|$, so that the function
\[ F(x) := e^{-(\sigma + it)r(n)(h(x))}|h'(x)|(\langle v, f_\sigma \rangle \circ h)(x) + e^{-(\sigma + it)r(n)(k(x))}|k'(x)|(\langle u, f_\sigma \rangle \circ k)(x) \]
satisfies for all $x$ s.t. $|x - x_1| < \delta/|t|$, all $\sigma > \sigma_0$, and all $\eta > (\sqrt{7} - 1)/2$,
\[ (2.5) \quad |F(x)| \leq \max\left[\eta e^{-\sigma r(n)(h(x))}|h'(x)|(u, f_\sigma) \circ h(x) \right. \]
\[ + e^{-\sigma r(n)(k(x))}|k'(x)|(u, f_\sigma) \circ k(x), \]
\[ e^{-\sigma r(n)(h(x))}|h'(x)|(u, f_\sigma) \circ h(x) + \eta e^{-\sigma r(n)(k(x))}|k'(x)|(u, f_\sigma) \circ k(x) \right]. \]

When the maximum in (2.5) is attained by the expression where the $\eta$ factor is attached to branch $h$ we say we are “in case $h$,” and otherwise “in case $k$.”

It follows from the proof that $n_1 \geq n_0$ so that $3 \times 16C\rho^{n_1} < 1/24$ works. In the application of Lemma 2.4 in Lemma 2.7 we require $C \geq C(\Sigma, K)$ from Lemma 2.2.
Proof. Let us fix $x_0 \in I$. Assume first (this case does not require UNI) that
\begin{equation}
|v(h(x_0))| \leq \max(u(h(x_0))/2, u(k(x_0))/2).
\end{equation}

Let us suppose the maximum is realised for $u \circ h$ (the other case is symmetric). Then it is easy to see that for any $\epsilon > 0$, if $|x - x_0| < \delta_1/|t|$, with $\delta_1(2C \rho^0) = \epsilon$, we have $\exp(-\epsilon) \leq \frac{u(h(x))}{u(h(x_0))} \leq \exp(\epsilon)$. (Use exp(\log u(h(x)) - \log u(h(x_0))) dx \leq \exp \int_{h(x_0)}^{h(x)} \|(\log u(y))'\| dy, the assumed bound on |u'|/u from (2.4) and $n \geq n_0$.)

To prove (2.5), it is then enough to check that $|x - x_0| < \delta_1/|t|$ implies $|v(h(x))| \leq \eta' u(h(x_0))$ for some $\eta' > 2/3$ with $\eta' \exp(\epsilon) \leq \eta$: indeed, we would then have $|v(h(x)| \leq \eta' \exp(\epsilon) u(h(x)) \leq \eta u(h(x))$ whenever $|x - x_0| < \delta_1/|t|$, so that (2.6) would hold. Assume for a contradiction that no such $\eta'$ exists, i.e., for each $2/3 < \eta' \leq \eta \exp(-\epsilon)$ there is $x_1$ with $|x_1 - x_0| \leq \delta_1/|t|$ and $|v(h(x_1))| \geq \eta'(u(h(x_0)))$, so that (use (2.4)) $|v(h(x_0)) - v(h(x_1))| \geq (\eta' - 1/2) u(h(x_0))$. On the other hand, (2.4) and the choice of $\epsilon$ imply that there is $y$ with $|y - x_0| \leq \delta_1/|t|$ so that
\[ |v(h(x_0)) - v(h(x_1))| \leq u(h(y))2C|t|\rho^0\delta_1 |t| \leq u(h(x_0))\epsilon^2 2C \rho^0 \delta_1 = u(h(x_0))\epsilon^2, \]
a contradiction if $\epsilon \exp(\epsilon) < 1/6$. This ends the easy case, where we can take $x_1 = x_0$ (i.e. $\Delta_1 = 0$) and $\delta_1 = \epsilon/(2C \rho^0)$ for small (independently of $u, v, C, \text{etc.}$) $\epsilon > 0$. (The dependence of $\delta_1$ on $C$ can be removed by taking large enough $n$.)

Let us now move to the more interesting situation when
\begin{equation}
|v(h(x_0))| > \max(u(h(x_0))/2, u(k(x_0))/2).
\end{equation}

We shall use UNI to show that we are in a position to apply Lemma 2.4 to the sum forming $F(x)$, for $x$ in a $\delta_2/|t|$-interval around a point $x_1$ that is $\Delta_2/|t|$ close to $x_0$. Since $f_0$ is real and positive, the difference $\theta(x)$ between the argument of the two terms of $F(x)$ can be decomposed as $\theta(x) = tv_{h,k}(x) + \arg(v(h(x)) - \arg(v(k(x)))$.

Let us first show the claim by assuming that we found $\delta_2, \Delta_2$ so that $\cos \theta(x) \leq 1/2$, for all $x$ with $|x - x_1| \leq \delta_2/|t|$, for some $x_1$ with $|x_1 - x_0| < \Delta_2/|t|$, leaving the (nontrivial) proof of this fact for the end. We have
\[ r_1(x) = e^{-\sigma r(h(x))} |h'(x)| [v \cdot f_\sigma](h(x)) \]

and
\[ r_2(x) = e^{-\sigma r(k(x))} |k'(x)| [v \cdot f_\sigma](k(x)). \]

Fix $x$ with $|x - x_1| \leq \delta_2/|t|$, and assume (the other case is analogous) that $r_1(x) \leq r_2(x)$. Lemma 2.4 then yields the claim:
\[ |F(x)| \leq \eta e^{-\sigma r(h(x))} |h'(x)| [v \cdot f_\sigma](h(x)) + e^{-\sigma r(k(x))} |k'(x)| [v \cdot f_\sigma](k(x)) \]
\[ \leq \eta e^{-\sigma r(h(x))} |h'(x)|(u \cdot f_\sigma)(h(x)) + e^{-\sigma r(k(x))} |k'(x)|(u \cdot f_\sigma)(k(x)). \]

It remains to prove that $\cos \theta(x) \leq 1/2$ for $x$ as above and some $\delta_2, \Delta_2$. For this, the following consequence of (2.7) and (2.4) will be helpful: for all $y, z$ with $|z - x_0| \leq |y - x_0| \leq \xi/|t|$, $|v(h(y))| \geq |v(h(x_0))| - |v(h(x_0)) - v(h(y))|$
\begin{equation}
\geq (1/2 - \exp(\epsilon)\rho^0 \xi 2C) \geq u(h(x_0))/4.
\end{equation}
Next observe that, because of (2.9), \( V(x) = \arg(v(h(x)) - \arg(v(k(x))) \) does not vary too much around \( x_0 \). More precisely:

\[
|V(x) - V(x_0)| = |\log|v(h(x))/v(k(x))| - \log|v(h(x_0))/v(k(x_0))| |
\]
(2.9)

and, if \( |x - x_0| \leq \xi/|t| \),

\[
|\log|v(h(x))/v(h(x_0))| - \log|v(k(y))/v(h(y))|| \leq |h(x) - h(x_0)| \leq \rho^n \frac{\xi}{|t|} 8C|\epsilon|e^{|u(h(x_0))|/u(h(x_0))} \leq \xi 8C e^{|\rho^n|}.
\]

(We used \( |y - x_0| \leq |x - x_0| \) and (2.8), (2.7)). We may control \( |\log|v(k(y))/v(h(x))|| \), mutatis mutandis, and we have for \( |x - x_0| < \xi/|t| \):

(2.10)

\[
|V(x) - V(x_0)| \leq \xi 16C \exp(\epsilon) \rho^n.
\]

Recall that we have to show cos \( \theta(x) \leq 1/2 \) in a suitable interval. We first find \( x_1 \) with \( |x_1 - x_0| < \Delta_2/|t| \) such that \( |\theta(x_1) - \pi| \leq \pi/24 \). For this, we use UNI which ensures that, since \( t(\psi(z) - \psi(x_0)) = t(z - x_0)\psi'(y) \) for \( y \in I \), if \( \Delta_2 = 2\pi/2D \), then \( \{|t(\psi(z) - \psi(x_0))| \leq 2\pi/|t| \} = [0, 2\pi] \). (We use here \(|t| > 2\pi/2D \).)

In particular, there is \( z = x_1 \) so that \( t(\psi(x_1) - \psi(x_0)) = \pi - \theta(x_0) \) (mod 2\pi).

Applying (2.10) to \( x = x_1, \xi = \Delta_2 \), we find

(2.11)

\[
|\theta(x_1) - \pi| = |\theta(x_0) + t(\psi(x_1) - \psi(x_0)) + (V(x_1) - V(x_0)) - \pi| \leq |V(x_1) - V(x_0)| < \Delta_2 16C \exp(\epsilon) \rho^n < \pi/24,
\]
if \( n \) is large enough (depending on \( C \) and, via \( \Delta_2 \), on \( D \)).

To conclude, we apply (2.10) and the “distortion” upper bound, using \( |x - x_0| < |x - x_1| + |x_1 - x_0| < (\delta_2 + \Delta_2)/|t| \) and \( |x - x_1| < \Delta_2/|t| \) to get, if \( n \) is large enough (depending on \( C \) and \( D \) and \( 0 < \delta_2 \leq \Delta_2 \) is small enough (depending on \( K \)):

(2.12)

\[
|\theta(x) - \pi| \leq \pi/24 + |\theta(x) - \theta(x_1)| \leq \pi/24 + |t||\psi(x) - \psi(x_1)| + |V(x) - V(x_0)| + |V(x_1) - V(x_0)| \leq \pi/24 + 2K|t|\delta_2 |t| \leq 16C \exp(\epsilon) \rho^n D|t| \delta_2 + \Delta_2/|t| < \pi/12.
\]

Taking \( \delta = \min(\delta_1, \delta_2) \) and \( \Delta = \Delta_2 \), we have proved the lemma. \( \square \)

**Remark 2.5.** If we replace UNI by the assumption that there exist \( D > 0, n_0 \), and two inverse branches \( h \) and \( k \) of \( T^{n_0} \) so that inf \(|\psi'_{h,k} \geq D \), then for every \( n \geq n_0 \) there are \( \hat{h}, \hat{k} \in \mathcal{H}_n \) so that inf \(|\psi'_{\hat{h},\hat{k}} \geq \rho^{n-n_0}D \). (Take \( \hat{h} = h \circ \ell, \hat{k} = k \circ \ell \), for \( \ell \in \mathcal{H}_{n-n_0} \) and observe that \( \psi_{\hat{h},\hat{k}}(x) = \psi_{h,k}(\ell(x)) \). However, this is not enough.

In (2.11) we would get (in view of the definition of \( \Delta_2 \)) \( D \frac{2\pi}{\rho^{n-n_0}} 8C \exp(\epsilon) \rho^n = \frac{16C}{\rho^{n-n_0}} \exp(\epsilon) \rho^n \), which is independent of \( n \) and not necessarily smaller than \( \pi/24 \).

(The constant 16 can be reduced, but not below 1.) Unfortunately, the strategy presented on p. 545 of [3] seems to suffer from the same problem.

The following consequence of Lemma 2.4 will be instrumental in arriving at Lemma 2.8

**Corollary 2.6.** Let \( T \) satisfy UNI for \( D \). Let \( C > 0 \), and let \( n_1 = n_1(C), \delta = \delta(C), \Delta = \Delta(C) \) be given by Lemma 2.4. Fix \( n \geq n_1 \), let \( h, k \in \mathcal{H}_n \) come from UNI, and let \( \rho_{n,C} = \min(\min|h'|, \min|k'|) \) (we have \( 0 < \rho_{n,C \leq \rho^n} \)).
Corollary 2.6 says that

\begin{equation}
\eta \leq \chi(x) \leq 1, \quad |\chi'| \leq \frac{3(1-\eta)}{\rho \cdot \sigma \cdot 3} |t|,
\end{equation}

\begin{align*}
type(I_1) = h \text{ and } x \in I_1 & \Rightarrow \chi_h(x) = \eta, \\
type(I_2) = k \text{ and } x \in I_2 & \Rightarrow \chi_k(x) = \eta, \\
\chi(y) < 1 & \Rightarrow |y = h(x), x \in I_1, type(I_1) = h| \text{ or } |y = k(x), x \in I_2, type(I_2) = k|.
\end{align*}

Finally, for \( s = \sigma + it \) with \( \sigma > \sigma_0 \), we have \( |\hat{L}_\sigma^n(v)(x)| \leq \hat{L}_\sigma^n(\chi u)(x), \forall x \in I \).

Note that \( \sup |\chi'|/|t| \) can be made arbitrarily small by taking \( \eta < 1 \) close to 1, once \( C \) and \( h, k, n \) are fixed. To exploit Corollary 2.6 we shall use the following:

**Lemma 2.7 (Invariance of “cone condition”).** Let \( T \) satisfy \( UNI \) for \( D \) and fix \( \Sigma \) a compact subset of \((\sigma_0, \infty)\). Let \( C(\Sigma, \hat{K}) \) be from Lemma 2.2 and fix \( C > 1 \) so that:

\[ C \geq C(\Sigma, \hat{K}) \cdot \max(1, \sup_{\sigma \in \Sigma} |\sigma|/D/(2\pi)) \]

Then, there is \( n_2 \geq n_1 \) (\( n_1 \) from Lemma 2.4) so that for every large enough \( |t| > 2\pi/D \), each \( u, v, \) satisfying (2.4) for \( C \) and \( t \), and all \( n \geq n_2 \), taking \( \eta = \eta(n) < 1 \) close enough to 1, and \( \chi = \chi(u, v, \eta) \) from Corollary 2.6 the pair \( u = L_\sigma^n(\chi u), \hat{v} = \hat{L}_\sigma^n(v), \) satisfies (2.4), for the same \( |t| \) and \( C \), and for all \( s = \sigma + it \) with \( \sigma \in \Sigma \).

**Proof.** Corollary 2.6 says that \( |\hat{v}(x)| = |\hat{L}_\sigma^n(v)(x)| \leq \hat{L}_\sigma^n(\chi u)(x) = \hat{u}(x) \) for all \( x \in I \). We also have inf \( \hat{u} > 0 \) since inf \( \chi(\hat{u}) > 0 \) and \( \hat{L}_\sigma \) preserves the cone of strictly positive functions. To check the condition on \( \max(|\hat{u}'|, |\hat{v}'|) \) we shall (finally!) invoke the Lasota-Yorke inequality from Lemma 2.2 (recalling also that \( \hat{L}_\sigma \) is normalised so that \( \sup \hat{L}_\sigma/f \leq \sup |f| \)). We first consider \( \hat{u}' \) and get, using \( |\hat{u}'| \leq 2C|t|u, \chi \geq \eta \) and \( |\chi' | \leq 1 \) (\( \eta = \eta(C, n) \) is close to 1):

\[ \left| \frac{d}{dx} \hat{L}_\sigma^n(\chi u)(x) \right| \leq C(\Sigma, \hat{K}) \sigma \hat{L}_\sigma^n(\chi u)(x) + \rho^n \hat{L}_\sigma^n(|\chi' u + \chi u'|)(x) \]

\[ \leq \left( C(\Sigma, \hat{K}) |t| + \rho^n \left( \frac{1}{\eta} + 2C|t| \right) \right) \hat{L}_\sigma^n(\chi u)(x) \leq 2C|t|\hat{L}_\sigma^n(\chi u)(x), \]

if \( n \geq n_2 \geq n_1 \) and \( C \geq C(\Sigma, \hat{K}) \).

The computation for \( |\hat{v}'| \) is similar:

\begin{equation}
\left| \frac{d}{dx} \hat{L}_\sigma^n(v)(x) \right| \leq C(\Sigma, \hat{K}) |s| \hat{L}_\sigma^n(|v|)(x) + \rho^n \hat{L}_\sigma^n(|v'|)(x)
\end{equation}

\[ \leq \frac{C(\Sigma, \hat{K}) |s| + 2C|t|\rho^n \hat{L}_\sigma^n(\chi u)(x)}{\eta} \leq 2C|t|\hat{L}_\sigma^n(\chi u)(x), \]

if \( n \geq n_2 \geq n_1 \) and \( C(\Sigma, \hat{K}) |s| \leq C|t| \). \( \square \)

**Proof of the \( L^2 \) contraction and proof of Theorem 1.1.** We shall see below that the case \( \sup |f'| > 2C|t| \sup |f| \) is easy. We next prove the key “\( L^2 \) contraction lemma” (see [3] Lemma 4) to handle the other case.
Lemma 2.8 ($L^2(\nu_1)$ contraction). Assume UNI. Let $\Sigma$, $C$, $n \geq n_2$, $|t| > 2\pi/D$, be as in Lemma 2.7. There is $\beta < 1$ so that for all $\sigma$ close enough to 0, and for all $0 \neq f \in C^1$ with $\sup_{x} |f| \leq 2C|\sigma| \sup_{x} |f|$, 
\begin{equation}
\int |\tilde{L}_{\sigma+t}^m f|^2 \, dv_0 < \beta^m \sup_{x} |f|^2, \forall m \geq 1.
\end{equation}

Proof. Recall $\eta < 1$ was taken close to 1 in Lemma 2.7. For $s = \sigma + it$ with $\sigma \in \Sigma$, define a sequence of pairs $(u_m, v_m)$, $m \geq 0$, of functions in $C^1(I)$:
\begin{align*}
 u_0 \equiv 1, \quad v_0 &= \frac{f}{\sup_{x} |f|}, \quad \chi_0 = \chi_{u_0,v_0,0}, \quad u_m = \tilde{L}_{\sigma}^n (\chi_{m-1}u_{m-1}), \quad v_m = \tilde{L}_{\sigma}^n (v_{m-1}).
\end{align*}

Lemma 2.7 implies that all $(u_m, v_m)$ satisfy (2.4) for $C$, $t$, and all $m$. (Note also that $u_m \leq 1$ for all $m$.) In particular, $|\tilde{L}_{\sigma}^m (f / \sup_{x} |f|)| = |v_m| \leq u_m$, and to prove the lemma, it is enough to show that there is $\beta_1 < 1$, so that $\int u_{m+1}^2 \, dv_0 \leq \beta_1 \int u_m^2 \, dv_0$ for all $m$ (note that $\int u_0^2 \, dv_0 = 1$).

The definition of $u_{m+1}$ and the Cauchy-Schwartz inequality imply
\begin{align*}
\lambda_{\sigma}^n f_0^2(x) u_{m+1}^2(x) &= \left( \sum_{\ell \in H_n} e^{-\sigma \gamma^n (\ell(x))} |f_0'(x)| \chi_{m} \cdot f_{\sigma} \cdot u_m(\ell(x)) \right)^2 \\
&\leq \max_{I} \frac{f_\sigma}{f_0} \sum_{\ell \in H_n} |f_0'(x)| (f_0 \cdot u_m(\ell(x)) \cdot \max_{I} \frac{f_\sigma}{f_2\sigma} \sum_{\ell \in H_n} e^{-2\sigma \gamma^n (\ell(x))} |f_0'(x)| \chi_{m} \cdot f_{2\sigma}(\ell(x)).
\end{align*}

Now, if $x \in \hat{I}_j$ for $\chi_m$, of type $h$, say (type $k$ is similar), we get
\begin{align*}
\frac{1}{\lambda_{2\sigma}^n f_{2\sigma}(x)} \sum_{\ell \in H_n} e^{-2\sigma \gamma^n (\ell(x))} |h'(x)| (\chi_{\sigma} \cdot f_{2\sigma} \cdot h(\ell(x)) \\
&\leq 1 - (1 - \eta^2) e^{-2\sigma \gamma^n (h(x))} |h'(x)| |f_{2\sigma} h(x)| \lambda_{2\sigma}^n f_{2\sigma}(x) \leq 1 - \epsilon (1 - \eta^2) = \eta' < 1
\end{align*}

we used $e^{-2\sigma \gamma^n (h(x))} |h'(x)| |f_{2\sigma} h(x)| / (\lambda_{2\sigma}^n f_{2\sigma}(x)) \geq \epsilon > 0$ if $n$ and $h$ are fixed; obviously, $\eta'$ depends on $n$). Denote
\begin{align*}
\xi(\sigma, n) &= \lambda_{2\sigma}^n f_0(x) f_{2\sigma}(x) \cdot \max_{I} \frac{f_\sigma}{f_{2\sigma}} \cdot \max_{I} \frac{f_\sigma}{f_0},
\end{align*}

We showed that for some $\eta' < 1$ and all $x \in \bigcup_{j} \hat{I}_j$ (recall $\lambda_0 = 1$),
\begin{align*}
 u_{m+1}^2(x) \leq \eta' \xi(\sigma, n) \tilde{L}_{0}^n (u_m^2)(x).
\end{align*}

If $x \notin \bigcup_{j} \hat{I}_j$, the Cauchy-Schwartz inequality just gives, since $\chi_m \leq 1$,
\begin{align*}
 u_{m+1}^2(x) \leq \xi(\sigma, n) \tilde{L}_{0}^n (u_m^2)(x).
\end{align*}

We claim that there is $\delta$, independent of $m$, $n$, and $t$, so that if $\hat{J}_j$ is the union of the rightmost third of $I_j$, $J_j$, and the leftmost third of $\hat{I}_{j+1}$, then
\begin{equation}
\int_{\hat{I}_j} \tilde{L}_{0}^n (u_m^2) \, dv_0 \geq \delta \cdot \int_{J_j} \tilde{L}_{0}^n (u_m^2) \, dv_0.
\end{equation}
We finish the proof assuming (2.15): if \( \hat{\delta}(\beta_2 - \eta') \geq (1 - \beta_2) \) (e.g. \( \beta_2 =: \frac{1 + \eta' \hat{\delta}}{1 + \hat{\delta}} < 1 \)),

\[
\int u_{m+1}^2 \, dv_0 \leq \xi(\sigma, n) \sum_{j} (\eta' \int f \, \tilde{L}_0^n(u_m^2) \, dv_0 + \int f \, \tilde{L}_0^n(u_m^2) \, dv_0)
\]

\[
\leq \xi(\sigma, n) \beta_2 \sum_{j} \int f \, \tilde{L}_0^n(u_m^2) \, dv_0 + \int f \, \tilde{L}_0^n(u_m^2) \, dv_0
\]

\[
= \xi(\sigma, n) \beta_2 \int f \, \tilde{L}_0^n(u_m^2) \, dv_0 = \xi(\sigma, n) \beta_2 \int u_m^2 \, dv_0.
\]

(2.16)

In the last line we used that the dual of \( \tilde{L}_0^n \) leaves \( \nu_0 \) fixed.) By taking \( \sigma \) sufficiently close to 0 (depending on \( n \), which is fixed) we can assume that \( \xi(\sigma, n) \beta_2 < 1 \).

It remains to show (2.15). It suffices to prove that \( \int f \, w^2 \, dv_0 \geq \hat{\delta} \int f \, w^2 \, dv_0 \) for all \( C^1 \) functions \( w \) with \( \sup w \leq 1 \) and \( |w'(z)| \leq 2C|t|w(z) \) (recall Lemma 2.7 and use Lemma 2.2 and \( \tilde{L}_0 \equiv 1 \)). Note that such \( w \) satisfy, for all \( x \in \tilde{I}_j, y \in \tilde{J}_j \):

\[
\frac{w^2(y)}{w^2(x)} = \exp(2(\log w(x) - \log w(y))) = \exp\left(2 \int_x^y \frac{w'/w)(z) \, dz\right) \leq \exp(4C(2\Delta + \hat{\delta})).
\]

Applying the above inequality, and making use of the Federer property (for intervals with length-ratio \( 3\Delta \)), of \( \nu_0 \) which has density \( f_0 \) (bounded from above and from below) with respect to Lebesgue measure, we find

\[
\int_{\tilde{I}_j} w^2 \, dv_0 \geq \nu_0(\tilde{I}_j) \min_{\tilde{I}_j} w^2 \geq \hat{\delta} e^{-(4C(2\Delta + \hat{\delta}))} \nu_0(\tilde{I}_j) \max_{\tilde{I}_j} w^2 \geq \hat{\delta} \int_{\tilde{I}_j} w^2 \, dv_0.
\]

\[\square\]

We are finally ready to prove the theorem:

**Proof.** Since there exists \( B \) so that \( (\lambda_\sigma \text{ is simple}) \|L^n_\sigma\|_{1,t} \leq B\lambda_\sigma^n \|\tilde{L}^n_\sigma\|_{1,t} \) for all \( n \geq 1 \), and since \( \lambda_0 = 1 \) and \( \sigma \) is in a neighbourhood of 0, it is enough to show that there exist \( \tilde{A} \) and \( \tilde{\gamma} < 1 \) so that \( \|\tilde{L}_\sigma^n\|_{1,t} \leq \tilde{\gamma}^n \) for \( n \geq \tilde{A} \log |t| \). Clearly, this will follow from the existence of \( n_4 \) and \( \tilde{A} \) so that \( \|\tilde{L}_\sigma^n\|_{1,t} \leq \tilde{\gamma}^{n_4} \) for all \( m \geq \tilde{A} \log |t| \) (write \( n = qn_4 + r \), with \( q, r \in \mathbb{Z}^+ \) and \( 0 \leq r < n_4 \), and use \( \|\tilde{L}^n_\sigma\|_{1,t} \leq \tilde{B} \)).

Let (see Lemma 2.7) \( C = \max(3/2, C(\Sigma, \tilde{K}) \cdot \max(1, D/(2\pi))) \), and let \( n_2 \) be given by Lemma 2.2. Let \( n_3 \geq n_2 \) be so that \( \rho^{n_3} < 1/4 \).

Let us first deal with the easy case \( \sup |f'| \geq 2C|t| \sup |f| \). Setting \( \gamma_1 = \max((2C|t|)^{-1}, \rho^{n_3} + 3/4) < 1 \), we have \( \sup |\tilde{L}_\sigma^n f| \leq \sup |f| \leq \frac{1}{2e|t|/\sigma} \sup |f'| \leq \gamma_1 \|f\|_{1,t} \), and, by Lemma 2.2

\[
\frac{|(\tilde{L}_\sigma^n f)|}{|t|} \leq C(\Sigma, \tilde{K}) \frac{|s|}{|t|} \sup \frac{|f|}{|t|} + \frac{\rho^{n_3}}{|t|} \sup \frac{|f'|}{|t|}
\]

\[
\leq \left( \frac{\sqrt{\max |\sigma|^2 + |f|^2}}{2|t|} + \rho^{n_3} \right) \sup \frac{|f'|}{|t|} \leq \gamma_1 \|f\|_{1,t}.
\]

If \( \sup |f'| < 2C|t| \sup |f| \), then the function \( g^2 \) satisfies (2.3) for \( 2C \max(1, \sup |g|) \) for which Lemmas 2.7, 2.8 hold. Note also that a slight modification of the Cauchy-Schwartz argument in the beginning of the proof of Lemma 2.8 yields

\[
|\tilde{L}_\sigma^{mn_3}(g)(x)|^2 \leq K \frac{\lambda_\sigma^{mn_3}}{\lambda_\sigma^{mn_3}} \tilde{L}_0^{mn_3} |g^2|(x),
\]
for some $K$ independent of $mn_3$ and $f$. Next, assume $\sup |f'| < 2C|t|\sup |f|$ and assume $\|f\|_{1,t} = 1$. By the spectral properties of $L_0$ on the space of Lipschitz functions endowed with the norm $\sup |g| + \text{Lip}(g)$ (with $\text{Lip}(g)$ the smallest Lipschitz constant of $g$), there are $R_\sigma < \infty$, $\tau_0^L < 1$ (independent of $f$ and $t$), with:

\[
K \frac{\lambda_{mn_3}}{\lambda_{2mn_3}} \left( \int |\tilde{L}_{s}^{mn_3}(f)|^2 \, dv_0 + R_\sigma (\tau_0^L)^{mn_3} \sup (\text{Lip}) (|\tilde{L}_{s}^{2mn_3}(f)|^2) \right) \\
\leq K \frac{\lambda_{mn_3}}{\lambda_{2mn_3}} \left( \sup |f|^2 \beta_m + R_\sigma (\tau_0^L)^{mn_3} \sup (\text{Lip}) (|\tilde{L}_{s}^{mn_3}(f)|^2) \right)
\]

using Lemma 2.2. (for $n = n_3$ and Cauchy-Schwartz). Lemma 2.7 gives

\[
\sup |\tilde{L}_{s}^{mn_3}(f)|^2 = \sup |f|^2 |v_m|^2 \leq \sup |f|^2u_m^2 \leq \sup |f|^2 \leq 1,
\]

and $\text{Lip}(\tilde{L}_{s}^{mn_3}(f))^2 \leq 2 \sup |f|^2 \cdot \sup |f| |v_m|' \leq 2 \sup |f|^2 2C|t| \leq 4C|t|$, since $\text{Lip}(|v_m|) \leq \text{Lip}(v_m) = \sup |v_m'|$.

In order to find $\max (\beta, (\tau_0^L)^{mn_3} \cdot \frac{\lambda_{mn_3}}{\lambda_{2mn_3}} < \gamma_2^2 < 1$ so that (for all $m$)

\[
K \frac{\lambda_{mn_3}}{\lambda_{2mn_3}} \left( \beta_m + R_\sigma (\tau_0^L)^{mn_3} (1 + 4C|t|) \right) \leq \gamma_2^{2m},
\]

To obtain the derivative, invoke Lemma 2.2 exploiting the bounds just obtained:

\[
\sup \frac{\left | \frac{\partial \tilde{L}_{s}^{mn_3}(f)}{\partial t} \right |} {\left | t \right |} \leq C(\Sigma, \tilde{K}) \left | \frac{\partial |v_m'|'}{\partial t} \right | \sup (\text{Lip}) (|\tilde{L}_{s}^{2mn_3}(f)|^2) + \rho^{mn_3} \left | \frac{\partial \tilde{L}_{s}^{mn_3}(f)}{\partial t} \right |) \\
\leq C(\Sigma, \tilde{K}) \frac{|v_m'|'}{|t|} \gamma_2^{m} + 2C\rho^{mn_3} \sqrt{K} \frac{\lambda_{mn_3}}{\lambda_{2mn_3}} \gamma_2^{m} \leq \gamma_3^{m}.
\]

Take $n_4 = 2n_3$ and large enough $A \geq \tilde{A}$. $oxdot$

References


CNRS, UMR 7586, INSTITUT DE MATHEMATIQUES DE JUSSIEU, 75251 PARIS, FRANCE
*E-mail address:* baladi@math.jussieu.fr

CNRS, GREYC, UNIVERSITÉ DE CAEN, 14032 CAEN, FRANCE
*E-mail address:* brigitte.vallee@info.unicaen.fr

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