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COUNTABLY COMPACT GROUPS FROM A SELECTIVE ULTRAFILTER

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Abstract. We prove that the existence of a selective ultrafilter on ω implies the existence of a countably compact group without non-trivial convergent sequences all of whose powers are countably compact. Hence, by using a selective ultrafilter on ω, it is possible to construct two countably compact groups without non-trivial convergent sequences whose product is not countably compact.

1. Introduction

The first example of a countably compact group without non-trivial convergent sequences was constructed, assuming CH, by A. Hajnal and I. Juhász [7]. A second example was discovered by E. K. van Douwen [3] under the assumption of MA, and one of the most recent examples lies in [10]. All known examples of such a topological group use some form of MA. A similar situation holds in the problem of the existence, in ZFC, of two countably compact groups whose product is not countably compact (see, for instance, [3], [8], [9] and [10]). In this paper, we will construct two countably compact groups without non-trivial convergent sequences whose product is not countably compact from a selective ultrafilter. We also construct a countably compact group without non-trivial convergent sequences all of whose powers are countably compact from a selective ultrafilter on ω.

We shall use standard notation. If $\{x_\xi : \xi < \mathfrak{c}\} \subseteq \{0,1\}^{\mathfrak{c}}$ and $F \in [\mathfrak{c}]^{<\omega}$, then $x_F = \sum_{\xi \in F} x_\xi$. The *type* of a point $p \in \beta(\omega) \setminus \omega = \omega^*$ is denoted by $T(p) = \{q \in \omega^* : \exists \text{ a bijection } f : \omega \to \omega(\overline{f}(p) = q)\}$, where $\overline{f} : \beta(\omega) \to \beta(\omega)$ denotes the Stone-Čech extension of f. An ultrafilter $p \in \omega^*$ is called *selective* if for every $f : \omega \to \omega$ there is $A \in p$ such that $f|_A$ is either constant or one-to-one (the reader may find other combinatorial statements equivalent to selectivity in the book [2]).

The following concept has been very useful in the construction of countably compact spaces with certain properties.

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Definition 1.1 (A. R. Bernstein [1])**.** Let $p \in \omega^*$, and let $(x_n)_{n<\omega}$ be a sequence in a space X. We say that x is a p-limit point of $(x_n)_{n<\omega}$, we write $x = p\text{-}\lim_{n\to\omega} x_n$, if for every neighborhood V of x, $\{n < \omega : x_n \in V\} \in p$.

It is not difficult to prove that a space X is countably compact iff every sequence of points in X has a p-limit point in X, for some $p \in \omega^*$. The following class of spaces was introduced by A. R. Bernstein [1].

Definition 1.2. Let $p \in \omega^*$. A space X is said to be p-compact if for every sequence $(x_n)_{n<\omega}$ of points of X there is $x \in X$ such that $x = p\text{-}\lim_{n\to\omega} x_n$.

We know that p-compactness is preserved under arbitrary products, for each $p \in \omega^*$. Hence, we can find countably compact spaces that are not p-compact for any $p \in \omega^*$ (see [5]). J. Ginsburg and V. Saks [6] showed that all powers of a space X are countably compact iff there is $p \in \omega^*$ such that X is p-compact.

For $p \in \omega^*$, we shall use the properties of the ultrapower $([\mathfrak{c}]^{<\omega})^\omega/p$ considered as a vector space over the field $\{0, 1\}$ with the symmetric difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$ as addition. For $p \in \omega^*$, an element of the ultrapower $([\mathfrak{c}]^{<\omega})^\omega/p$ will be denoted by $[f]_p$, where $f : \omega \to [\mathfrak{c}]^{<\omega}$ is a function. For $F \in [\mathfrak{c}]^{<\omega}$, the constant function whose domain is ω and takes only the value F will be denoted by $\vec{F}$. If $\alpha < \mathfrak{c}$ is an ordinal, then $\vec{\{\alpha\}}$ will be denoted by $\vec{\alpha}$.

2. The examples

Our group G will be generated by a linearly independent subset of $\{0, 1\}^{\mathfrak{c}}$.

For every selective ultrafilter $p \in \omega^*$, it is evident that

$$([\mathfrak{c}]^{<\omega})^\omega/p = \{[f]_p : f \in ([\mathfrak{c}]^{<\omega})^\omega \text{ is one-to-one }\} \cup \{[\vec{F}]_{\mathrm{p}} : \mathrm{F} \in [\mathfrak{c}]^{<\omega}\}.$$

Lemma 2.1. *Let $p \in \omega^*$ be selective. Then, there exists a family of one-to-one functions $\{f_\xi : \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ such that:*

1) $\bigcup_{n<\omega} f_\xi(n) \subseteq max\{\omega, \xi\}$, *for every* $\xi < \mathfrak{c}$.
2) $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ *is a base for* $([\mathfrak{c}]^{<\omega})^\omega/p$.
3) *For every one-to-one function $g \in ([\mathfrak{c}]^{<\omega})^\omega$, there are distinct $\zeta_0, \zeta_1 < \mathfrak{c}$ and two increasing sequences of positive integers $(n^0_k)_{k<\omega}$ and $(n^1_k)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n^i_k)$, for every $k < \omega$ and $i \in \{0, 1\}$.*

Proof. Let $\{g_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all one-to-one functions of $([\mathfrak{c}]^{<\omega})^\omega$ in such a way that each element is listed two times, and $\bigcup_{n<\omega} g_\xi(n) \subseteq max\{\omega, \xi\}$, for every $\xi < \mathfrak{c}$. We proceed by transfinite induction. Let $\alpha < \mathfrak{c}$ and suppose that, for each $\xi < \alpha$, we have defined a one-to-one function $f_\xi : \omega \to [\mathfrak{c}]^{<\omega}$ such that:

i) For every $m < \omega$ there is $n < \omega$ such that $f_\xi(m) = g_\xi(n)$, for every $\xi < \alpha$.
ii) $\{[f_\zeta]_p : \zeta < \xi\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, for every $\xi < \alpha$.
iii) If $\{[f_\zeta]_p : \zeta < \xi\} \cup \{[g_\xi]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, then $f_\xi = g_\xi$, for every $\xi < \alpha$.

If $\{[f_\xi]_p : \xi < \alpha\} \cup \{[g_\alpha]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, then we define $f_\alpha = g_\alpha$. Let us assume that $\{[f_\xi]_p : \xi < \alpha\} \cup \{[g_\alpha]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is not linearly independent. Now, let $\{A_\mu : \mu < \mathfrak{c}\}$ be an almost disjoint family of infinite subsets of ω. For each $\mu < \mathfrak{c}$, let $h_\mu : \omega \to A_\mu$ be a bijection. Then, we define $h_{\alpha,\mu} : \omega \to [\mathfrak{c}]^{<\omega}$ by $h_{\alpha,\mu}(n) = g_\alpha(h_\mu(n))$, for each $n < \omega$. It is evident that $\{n < \omega : h_{\alpha,\mu}(n) = h_{\alpha,\nu}(n)\}$ is finite for $\mu < \nu < \mathfrak{c}$. Hence, $\{[h_{\alpha,\mu}]_p : \mu < \mathfrak{c}\}$

are pairwise distinct. So, we can find $\mu_\alpha < \mathfrak{c}$ such that $[h_{\alpha,\mu_\alpha}]_p \notin \langle \{[f_\zeta]_p : \zeta < \xi\} \cup \{[\vec{\beta}]_p : \beta < max\{\omega, \alpha\}\}\rangle$. Put $f_\alpha = h_{\alpha,\mu_\alpha}$. Clearly, conditions $i)$ and $iii)$ are satisfied. We know that $\{[f_\xi]_p : \xi \leq \alpha\} \cup \{[\vec{\beta}]_p : \beta < max\{\omega, \alpha\}\}$ is linearly independent. Since $\bigcup_{n<\omega} f_\xi(n) \subseteq max\{\omega, \alpha\}$, for every $\xi \leq \alpha$, we also have that $\{[f_\xi]_p : \xi \leq \alpha\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent. This shows that condition $ii)$ holds. We claim that the family $\{f_\xi : \xi < \mathfrak{c}\}$ satisfies all the conditions. Indeed, by the construction, $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is a base for $([\mathfrak{c}]^{<\omega})^\omega/p$. Let us prove that condition 3) is satisfied. For this, take $\zeta_0, \zeta_1 < \mathfrak{c}$ so that $\xi_0 < \xi_1$ and $g = g_{\zeta_0} = g_{\zeta_1}$. From condition $i)$ we can find two increasing sequences of positive integers $(n_k^0)_{k<\omega}$ and $(n_k^1)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n_k^i)$, for every $k < \omega$ and $i \in \{0, 1\}$. □

In what follows, we fix a family $\{f_\xi : \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ satisfying the three properties stated in Lemma 2.1, and enumerate $[\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$ as $\{F_\alpha : \alpha < \mathfrak{c}\}$.

Lemma 2.2. *Let $p \in \omega^*$ be selective. Suppose that for every $\alpha < \mathfrak{c}$ we have a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0, 1\}$ such that*

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, *for every $\xi < \mathfrak{c}$; and*
ii) $\Phi_\alpha(F_\alpha) = 1$.

For $\xi < \mathfrak{c}$, we define $x_\xi \in \{0, 1\}^\mathfrak{c}$ by $x_\xi(\alpha) = \Phi_\alpha(\{\xi\})$, for every $\alpha < \mathfrak{c}$. Then, the set $X = \{x_\xi : \xi < \mathfrak{c}\}$ is linearly independent in $\{0, 1\}^\mathfrak{c}$ and $G = \langle X\rangle$ is a p-compact group without non-trivial convergent sequences.

Proof. Let $\{\xi_0, ..., \xi_k\} \in [\mathfrak{c}]^{<\omega}$. Choose $\alpha < \mathfrak{c}$ such that $F_\alpha = \{\xi_0, ..., \xi_k\}$. Then, by $ii)$,

$$(x_{\xi_0} + ... + x_{\xi_k})(\alpha) = \Phi_\alpha(\{\xi_0\}) + ... + \Phi_\alpha(\{\xi_k\}) = \Phi_\alpha(F_\alpha) = 1.$$

This shows that $\{x_\xi : \xi < \mathfrak{c}\}$ is linearly independent in $\{0, 1\}^\mathfrak{c}$. Now we will show that G is p-compact. Before proving this, notice from clause $i)$ that

$$x_\xi = p\text{-}\lim_{n\to\omega} \sum_{\mu\in f_\xi(n)} x_\mu = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)},$$

for every $\xi < \mathfrak{c}$. Let $(a_n)_{n<\omega}$ be a sequence in G. Choose $g \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $a_n = x_{g(n)}$, for every $n < \omega$. Since p is selective, there is $A \in p$ such that $g|_A$ is either constant or one-to-one. If $g|_A$ is constant, then there is $F \in [\mathfrak{c}]^{<\omega}$ such that $\{n < \omega : x_{g(n)} = x_F\} \in p$ and so $x_F = p\text{-}\lim_{n\to\omega} x_{g(n)}$. Let us assume that there is a one-to-one function $h \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $h|_A = g|_A$. Since $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is a base for $([\mathfrak{c}]^{<\omega})^\omega/p$, there are $\xi_0, ..., \xi_k < \mathfrak{c}$ and $E \in [\mathfrak{c}]^{<\omega}$ such that $[h]_p = (\Delta_{i\leq k}[f_{\xi_i}]_p)\Delta(\Delta_{\mu\in E}[\vec{\mu}]_p)$. Hence, we can find $B \in p$ such that $B \subseteq A$ and $h(n) = (\Delta_{i\leq k} f_{\xi_i}(n))\Delta E$, for every $n \in B$. It then follows that $x_{h(n)} = \sum_{i\leq k} x_{f_{\xi_i}(n)} + x_E$, for all $n \in B$. So,

$$\sum_{i\leq k} x_{\xi_i} + x_E = p\text{-}\lim_{n\to\omega} x_{h(n)}.$$

This shows that G is p-compact. Let $(y_n)_{n<\omega}$ be a non-trivial sequence in G, and assume that there is a one-to-one function $g \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $y_n = x_{g(n)}$. By clause 3) of Lemma 2.1, there are distinct $\zeta_0, \zeta_1 < \mathfrak{c}$ and two increasing sequences of positive integers $(n_k^0)_{k<\omega}$ and $(n_k^1)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n_k^i)$, for every $k < \omega$ and $i \in \{0, 1\}$. Since $x_{\zeta_i} = p\text{-}\lim_{k\to\omega} x_{f_{\zeta_i}(k)} = p\text{-}\lim_{k\to\omega} x_{g(n_k^i)} = p\text{-}\lim_{k\to\omega} y_{n_k^i}$, for each $i \in \{0,1\}$, we must have that x_{ζ_0} and x_{ζ_1} are both cluster points of $\{y_n : n < \omega\}$. □

Lemma 2.3. *Let $p \in \omega^*$ be selective. For every $E_0 \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, there are $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subseteq [\mathfrak{c}]^{<\omega}$ such that*

1) *$\omega \subseteq \bigcup_{i<\omega} E_i$;*
2) *$E_i \cup [\bigcup_{\xi \in E_i} f_\xi(b_i)] \subseteq E_{i+1}$, for every $i < \omega$; and*
3) *$\{f_\xi(b_i) : \xi \in E_i\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent, for every $i < \omega$.*

Proof. Put $F_0 = E_0$, and let $F_{n+1} = n \cup F_n \cup [\bigcup_{\xi \in F_n} \bigcup_{m \leq n} f_\xi(m)]$, for every $1 \leq n < \omega$. Since $\{[f_\xi]_p : \xi \in S\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, we have that

$$A_n = \{k < \omega : \{f_\xi(k) : \xi \in F_n\} \cup \{\{\mu\} : \mu \in F_n\} \text{ is linearly independent}\} \in p,$$

for every $n < \omega$. By the selectivity of p, we can find $A = \{a_n : n < \omega\} \in p$ such that $m < a_m < a_n$ and $a_n \in A_n$, for every $m < n < \omega$. Let us define a coloring P_0 and P_1 on $[\omega]^2$ as $\{a,b\} \in P_0$ iff $a < b$, $a, b \in A$, $a = a_m$, $b = a_n$ and $a_m < n$, and $\{a,b\} \in P_1$ otherwise. Since p is selective, there is $B \in p$ such that $B \subseteq A$ and either $[B]^2 \subseteq P_0$ or $[B]^2 \subseteq P_1$. Choose $I \in [\omega]^\omega$ such that $B = \{a_n : n \in I\}$. Let $\{i_k : k < \omega\}$ be the enumeration of I in increasing order. Suppose that $[B]^2 \subseteq P_1$. Since $\{a_{i_0}, a_{i_k}\} \in P_1$, then $a_{i_0} \geq i_k$, for every $1 \leq k < \omega$, but this is a contradiction. Therefore, $[B]^2 \subseteq P_0$. Hence, we have that $i_k < a_{i_k} < i_{k+1}$, for every $k < \omega$. By using this, we obtain that

$$F_{i_k} \cup [\bigcup_{\xi \in F_{i_k}} f_\xi(a_{i_k})] \subseteq F_{i_{k+1}} \cup [\bigcup_{\xi \in F_{i_k}} \bigcup_{m < i_{k+1}} f_\xi(m)] \subseteq F_{i_{k+1}},$$

for all $k < \omega$. Notice that, for every $k < \omega$,

$$\{f_\xi(a_{i_k}) : \xi \in F_{i_k}\} \cup \{\{\mu\} : \mu \in F_{i_k}\}$$

is linearly independent, since $a_{i_k} \in A_{i_k}$. Then, for every $1 \leq k < \omega$, we define $E_k = F_{i_k}$ and, for every $k < \omega$, we put $b_k = a_{i_k}$. It is evident that 2) and 3) are satisfied. We remark that $E_0 \subseteq F_{i_0}$ and

$$\omega \subseteq \bigcup_{n<\omega} F_n = \bigcup_{k<\omega} F_{i_k} = \bigcup_{k<\omega} E_k,$$

and $B = \{b_k : k < \omega\} \in p$. □

Example 2.4. If $p \in \omega^*$ is selective, then there is a p-compact subgroup of size $\mathfrak{c}$ without non-trivial convergent sequences.

Proof. According to Lemma 2.2, it suffices to construct, for each $\alpha < \mathfrak{c}$, a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0,1\}$ such that

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, for every $\xi < \mathfrak{c}$; and
ii) $\Phi_\alpha(F_\alpha) = 1$.

Fix $\alpha < \mathfrak{c}$. By applying Lemma 2.3 to $E_0 = F_\alpha$, we get $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subseteq [\mathfrak{c}]^{<\omega}$ such that

1) $\omega \subseteq \bigcup_{i<\omega} E_i =: E$;
2) $E_i \cup [\bigcup_{\xi \in E_i} f_\xi(b_i)] \subseteq E_{i+1}$, for every $i < \omega$; and
3) $\{f_\xi(b_i) : \xi \in E_i\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent, for every $i < \omega$.

Now, suppose that for $i < \omega$, we have defined Φ_α on $[E_i]^{<\omega}$ so that $\Phi_\alpha(F_\alpha) = 1$ and $\Phi_\alpha(f_\xi(b_i)) = \Phi_\alpha(\{\mu\})$, for every $\xi, \mu \in E_i$. Since $\{f_\xi(b_{i+1}) : \xi \in E_{i+1}\} \cup \{\{\mu\} : \mu \in E_{i+1}\}$ is linearly independent, and $E_i \cup \bigcup_{\xi \in E_i} f_\xi(b_i) \subseteq E_{i+1}$, we may extend $\Phi_\alpha : [E_i]^{<\omega} \to \{0,1\}$ to a homomorphism from $[E_{i+1}]^{<\omega}$ to $\{0,1\}$ in such a way that $\Phi_\alpha(f_\xi(b_{i+1})) = \Phi_\alpha(\{\xi\})$, for every $\xi \in E_{i+1}$. Thus, we have defined Φ_α on $[E]^{<\omega}$. Observe that $\Phi_\alpha(f_\xi(b_i)) = \Phi_\alpha(\{\xi\})$, for every $\xi \in E_i$ and $i < \omega$. Hence, $\{n < \omega : \Phi_\alpha(f_\xi(n)) = \Phi_\alpha(\{\xi\})\} \in p$, for every $\xi \in E$. Our next task is to extend Φ_α to $[\mathfrak{c}]^{<\omega}$. We will do this by transfinite induction on $\mathfrak{c} \setminus E$. Let $\gamma \in \mathfrak{c} \setminus E$ and suppose that Φ_α has been defined on $[E \cup \gamma]^{<\omega}$. Since $f_\gamma(n) \subseteq \gamma$, for every $n < \omega$, $\Phi_\alpha(\{\mu\})$ has been defined for each $\mu < \gamma$ and $\{\{\gamma\}\} \cup \{\{\mu\} : \mu < \gamma\}$ is linearly independent, Φ_α can be extended to $[E \cup (\gamma + 1)]^{<\omega}$ in such a way that

$$\Phi_\alpha(\{\gamma\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\gamma(n)).$$

It is evident that Φ_α satisfies the required properties. □

The following example follows from E. K. van Douwen's construction [3, 6.1] applied to Example 2.4.

Example 2.5. If there is a selective ultrafilter on ω, then there are two countably compact groups without non-trivial convergent sequences whose product is not countably compact.

3. One more example

In this section, we will improve a little bit Example 2.5.

For $p \in \omega^*$, we say that a space is *almost p-compact* if for every sequence $(x_n)_{n<\omega}$ in X there is a function $\sigma : \omega \to \omega$ such that $\overline{\sigma}(p) \in \omega^*$ and $\overline{\sigma}(p)\text{-}\lim_{n\to\omega} x_n \in X$ (this concept was introduced in [4]). It is evident that every p-compact space is almost p-compact, and every almost p-compact space is countably compact. All these notions are different from each other.

The following lemma is a generalization of Lemma 2.1.

Lemma 3.1. *Let $p \in \omega^*$ be a selective ultrafilter. Then, there exists a family of one-to-one functions $\{f_\xi : \omega \le \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ and pairwise disjoint sets $I_0, I_1, I_2, I_3 \in [\mathfrak{c} \setminus \omega]^{\mathfrak{c}}$ such that:*

- a) $\bigcup_{n<\omega} f_\xi(n) \subseteq \xi$ *for every* $\omega \le \xi < \mathfrak{c}$.
- b) b.0) $\bigcup_{n<\omega} f_\xi(n) \subseteq \omega$, *for every* $\xi \in I_0$.
  - b.1) $\bigcup_{n<\omega} f_\xi(n) \subseteq \omega$, *for every* $\xi \in I_1$.
  - b.2) $\bigcup_{n<\omega} f_\xi(n) \subseteq I_0 \cup I_2$, *for every* $\xi \in I_2$.
  - b.3) $\bigcup_{n<\omega} f_\xi(n) \subseteq I_1 \cup I_3$, *for every* $\xi \in I_3$.
- c) $\{[f_\xi]_p : \omega \le \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ *is linearly independent.*
- d) *For every $j \in \{0,1\}$ and for every one-to-one function $g \in ([\omega]^{<\omega})^\omega$, there exists a bijection $\sigma : \omega \to \omega$ and $\xi \in I_j$ such that $[g \circ \sigma]_p = [f_\xi]_p$.*
- e) *For every $j \in \{0,1\}$, $\{[f_\xi]_p : \xi \in I_{j+2}\} \cup \{[\vec{\beta}]_p : \beta \in I_j \cup I_{j+2}\}$ is a base for $([I_j \cup I_{j+2}]^{<\omega})^\omega/p$.*

Proof. Let I_0, I_1, I_2 and I_3 be a partition of $\mathfrak{c} \setminus \omega$ in subsets of size $\mathfrak{c}$, and let $\{g_\xi : \omega \le \xi < \mathfrak{c}\}$ be such that:

- i) For each $j \in \{0,1\}$, we have that $\{g_\xi : \xi \in I_j\}$ is an enumeration of all one-to-one functions in $([\omega]^{<\omega})^\omega$.

ii) For each $j \in \{0,1\}$, we have that $\{g_\xi : \xi \in I_{j+2}\}$ is an enumeration of all one-to-one functions in $([I_j \cup I_{j+2}]^{<\omega})^\omega$ in such a way that $\bigcup_{n<\omega} g_\xi(n) \subseteq \xi$, for every $\xi \in I_{j+2}$.

By applying the proof of Lemma 2.1 to $\{g_\xi : \xi \in I_j\}$, for $j \in 2,3$, we get a set of one-to-one functions $\{f_\xi : \xi \in I_j\}$ satisfying a), b.2), b.3), and e). On the other hand, we apply the proof of Lemma 2.1 to $\{g_\xi : \xi \in I_0 \cup I_1\}$ to obtain a family of one-to-one functions $\{f_\xi : \xi \in I_j\}$ satisfying b.0) and b.1). Furthermore, $\{[f_\xi]_p : \xi \in I_0 \cup I_1\} \cup \{[\vec{n}]_p : n < \omega\}$ is linearly independent. Thus, condition c) also holds. Let us see how we get condition d). Following the notation of the proof of Lemma 2.1, at stage $\alpha < \mathfrak{c}$, we choose $\mu_\alpha < \mathfrak{c}$ such that $\{[h_{\alpha,\mu_\alpha}]_p : \alpha < \mathfrak{c}\} \cup \{[f_\xi]_p : \xi < \alpha\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent. We know that $h_{\mu_\alpha} : \omega \to A_{\mu_\alpha}$ is a bijection. Now, pick $B \subseteq A_{\mu_\alpha}$ such that $|B| = |A_{\mu_\alpha} \setminus B| = \omega$ and $h_{\mu_\alpha}^{-1}(B) \in p$. Choose a bijection $\sigma : \omega \to \omega$ for which $\sigma(n) = h_{\mu_\alpha}(n)$ for every $n \in h_{\mu_\alpha}^{-1}(B)$ and $\sigma[\omega \setminus h_{\mu_\alpha}^{-1}(B)] = \omega \setminus B$. So, we define $f_\alpha(n) = g_\alpha(\sigma(n))$, for every $n < \omega$. Then, we have that $f_\alpha = g_\alpha(\sigma(n)) = g_\alpha(h_{\mu_\alpha}(n)) = h_{\alpha,\mu_\alpha}(n)$, for every $n \in h_{\mu_\alpha}^{-1}(B)$. Therefore, $[f_\alpha]_p = [g_\alpha \circ \sigma]_p = [h_{\alpha,\mu_\alpha}]_p$. This shows condition d). □

In the next example, we fix a family $\{f_\xi : \omega \leq \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ and sets $I_0, I_1, I_2, I_3 \in [\mathfrak{c} \setminus \omega]^{\mathfrak{c}}$ satisfying all the properties of Lemma 3.1.

Example 3.2. If there is a selective ultrafilter on ω, then there are two almost p-compact groups whose product is not countably compact.

Proof. By using clause c), Lemma 2.3 and the proof of Example 2.4, we can define, for every $\alpha < \mathfrak{c}$, a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0,1\}$ so that

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, for every $\omega \leq \xi < \mathfrak{c}$; and
ii) $\Phi_\alpha(F_\alpha) = 1$.

Hence, for each $\xi < \mathfrak{c}$ we define $x_\xi(\alpha) = \Phi_\alpha(\{\xi\})$, for every $\alpha < \mathfrak{c}$. Then, we have that $\{x_\xi : \xi < \mathfrak{c}\}$ is a linearly independent subset of $\{0,1\}^{\mathfrak{c}}$ and $x_\xi = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)}$, for every $\omega \leq \xi < \mathfrak{c}$. We put

$E = \langle \{x_n : n < \omega\} \rangle$,
$H_0 = \langle \{x_\xi : \xi \in I_0 \cup I_2\} \rangle$,
$H_1 = \langle \{x_\xi : \xi \in I_1 \cup I_3\} \rangle$,
$G_0 = E + H_0 = \langle \{x_\xi : \xi \in \omega \cup I_0 \cup I_2\} \rangle$ and
$G_1 = E + H_1 = \langle \{x_\xi : \xi \in \omega \cup I_1 \cup I_3\} \rangle$.

It is evident that $G_0 \cap G_1 = E$. Hence, we deduce that $G_0 \times G_1$ is not countably compact. As in Lemma 2.2, both H_0 and H_1 are p-compact groups. We shall show that G_j is almost p-compact, for $j \in \{0,1\}$. For this, fix a sequence $(a_n)_{n<\omega}$ in G_j. Choose two sequences $(e_n)_{n<\omega}$ in E and $(h_n)_{n<\omega}$ in H_j so that $a_n = e_n + h_n$, for every $n < \omega$. By the selectivity of p, there is $A \in p$ such that either $e_n = e$, for all $n \in A$, for some $e \in E$, or the function $n \to e_n$, for $n \in A$, is one-to-one. In the former case, $e + h = p\text{-}\lim_{n\to\omega}(e_n + h_n) \in E + H_j = G_j$, where $h = p\text{-}\lim_{n\to\omega} h_n$. In the latter case, we can find a one-to-one function $g \in ([\omega]^{<\omega})^\omega$ such that $e_n = x_{g(n)}$, for every $n \in A$. According to clause e) of Lemma 3.1, there are a bijection $\sigma : \omega \to \omega$ and $\xi \in I_j$ such that $[g \circ \sigma]_p = [f_\xi]_p$. Pick $B \in p$ so that $B \subseteq A$ and $g(\sigma(n)) = f_\xi(n)$, for every $n \in B$. Hence, $e_{\sigma(n)} = x_{g(\sigma(n))} = x_{f_\xi(n)}$, for every $n \in B$. This implies that

$$p\text{-}\lim_{n\to\omega} e_{\sigma(n)} = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)} = x_\xi \in H_j.$$

So, $x_\xi = \overline{\sigma}(p)\text{-}\lim_{n\to\omega} e_n$ and $q = \overline{\sigma}(p) \in T(p)$. Since H_j is p-compact, it is also q-compact. Thus, $q\text{-}\lim_{n\to\omega} h_n = h \in H_j$. Hence, $q\text{-}\lim_{n\to\omega} a_n = x_\xi + h \in H_j + H_j \subseteq G_j$. Therefore, G_j is almost p-compact. □

Finally, we list some open problems that the authors were unable to solve.

Question 3.3. For an arbitrary $p \in \omega^*$, is there a p-compact group without non-trivial convergent sequences?

Question 3.4. Does the existence of a P-point in ω^* imply the existence of two countably compact groups whose product is not countably compact?

Question 3.5. Does the existence of a selective ultrafilter on ω imply the existence of a countably compact group whose square is not countably compact?

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References

1. A. R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185-193. MR0251697 (40:4924)
2. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974. MR0396267 (53:135)
3. E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), 417 - 427. MR0586725 (82b:22002)
4. S. Garcia-Ferreira, *Quasi M-compact spaces*, Czechoslovak Math. J. **46** (1996), 161 - 177. MR1371698 (97b:54033)
5. L. Gillman and M. Jerison, *Rings of continuous functions*, Lectures Notes in Mathematics No. **27**, Springer-Verlag, 1976. MR0407579 (53:11352)
6. J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), 403-418. MR0380736 (52:1633)
7. A. Hajnal and I. Juhász, *A separable normal topological group need not be Lindelöf*, Gen. Topology Appl. **6** (1976), 199-205. MR0431086 (55:4088)
8. K. P. Hart and J. van Mill, *A countably compact topological group H such that $H \times H$ is not countably compact*, Trans. Amer. Math. Soc. **323** (1991), 811- 821. MR0982236 (91e:54025)
9. A. H. Tomita, *A group under $MA_{countable}$ whose square is countably compact but whose cube is not*, Topology Appl. **91** (1999), 91-104. MR1664516 (2000d:54039)
10. A. H. Tomita, *Countable compactness and finite powers of topological groups without convergent sequences*, submitted.

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