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TRANSITIVE FAMILIES OF PROJECTIONS IN FACTORS OF TYPE II_1

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ABSTRACT. We introduce a notion of transitive family of subspaces relative to a type II_1 factor, and hence a notion of transitive family of projections in such a factor. We show that whenever $\mathcal{M}$ is a factor of type II_1 and $\mathcal{M}$ is generated by two self-adjoint elements, then $\mathcal{M} \otimes M_2(\mathbb{C})$ contains a transitive family of 5 projections. Finally, we exhibit a free transitive family of 12 projections that generate a factor of type II_1.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{M} \subseteq B(H)$ be a factor of type II_1. If S is a non-empty set, we say that a family of norm-closed subspaces $\{\mathcal{H}_i\}_{i\in S}$ of $\mathcal{H}$ is transitive relative to $\mathcal{M}$ if for each $i \in S$, the projection P_i of $\mathcal{H}$ onto $\mathcal{H}_i$ lies in $\mathcal{M}$ and only the scalar operators leave all of the $\mathcal{H}_i$ invariant. In this case, we also say that the family $\{P_i\}_{i\in S}$ is a transitive family of projections relative to $\mathcal{M}$. When $\dim(\mathcal{H}) \geq 3$, a transitive family cannot contain only two nontrivial projections P and Q, since in this case $(P-Q)^2$ commutes with both P and Q, and therefore leaves the ranges of both invariant. In this paper we first prove that if $\mathcal{M}$ is a type II_1 factor and is generated by two self-adjoint elements, then there is a transitive family of five projections relative to $\mathcal{M} \otimes M_2(\mathbb{C})$. This leads us to the question of whether or not there is a transitive family of three or four projections relative to some factor of type II_1? To shed light on this question we consider free families of projections. A family $\{P_i\}_{i=1}^n$ of projections in a factor of type II_1 is free if each P_i has trace $\frac{1}{2}$ and the P_i are free with respect to the trace (in the sense of Voiculescu, see [7] and [1]). We shall exhibit a free transitive family of twelve projections.

In $\mathcal{B}(\mathcal{H})$, a family of norm-closed subspaces is transitive if the only bounded operators on $\mathcal{H}$ that leave every subspace in the family invariant are scalars. Transitive families of subspaces were first considered by Paul Halmos in his 1970 paper, "Ten problems in Hilbert space" [3]. In this paper Halmos studied medial subspace lattices, which are families of subspaces that contain $\{0\}$, $\mathcal{H}$, and at least two nontrivial subspaces of $\mathcal{H}$, with the additional property that any pair of nontrivial subspaces K_1, K_2 in the lattice are topologically complementary (that is, $K_1 \cap K_2 = \{0\}$

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and $\overline{span}\{K_1, K_2\} = \mathcal{H}$). Halmos constructed a finite-dimensional example of a transitive medial subspace lattice having five nontrivial elements and raised the question of how small a transitive medial subspace lattice could be. In 1971, Harrison, Radjavi and Rosenthal found that, in a separable, infinite-dimensional Hilbert space, there is a transitive medial subspace lattice having four nontrivial elements [4]. It has become apparent since that the construction of a medial subspace lattice having three elements is a difficult problem. In fact, even finding a transitive family of three nontrivial norm-closed subspaces is hard. Lambrou and Longstaff have shown that in finite (≥ 3) dimensional $\mathcal{H}$, the smallest possible cardinality of a transitive family of subspaces is four [6]. Hadwin, Longstaff and Rosenthal have (when $\dim \mathcal{H}$ is infinite) found a transitive family of two norm-closed subspaces and a linear manifold and have shown that the existence of a three-element transitive family of norm-closed subspaces would follow from the existence of two dense operator ranges in $\mathcal{H}$ such that the only bounded operators leaving both of the ranges invariant are scalars [2].

We note that the questions considered in this paper are closely related to the generator question of von Neumann algebras, which asks if every von Neumann algebra acting on a separable Hilbert space is generated by two self-adjoint elements. The last example in this note shows that free families of projections that generate factors can be transitive.

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2. Main results

For basic information about von Neumann algebras, we refer the reader to [5].

Definition 1. Let $\mathcal{H}$ be a complex, separable Hilbert space, let I denote the identity in $\mathcal{B}(\mathcal{H})$, and let $I \in \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1. Let S be a nonempty set. A family $\{P_i\}_{i \in S}$ of projections in $\mathcal{B}(\mathcal{H})$ is transitive relative to $\mathcal{M}$ if each P_i is in $\mathcal{M}$ and the only operators $T \in \mathcal{M}$ that satisfy $(I - P_i)TP_i = 0$ for all $i \in S$ are scalars.

Remark 1. When there is no danger of confusing which factor we are considering, we say that a family of projections $\{P_i\}_{i \in S} \subseteq \mathcal{M}$ is transitive when $\{P_i\}_{i \in S}$ is transitive relative to $\mathcal{M}$.

Proposition 1. *Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 such that $\mathcal{M}$ is generated, as a von Neumann algebra, by three projections P_1, P_2 and P_3. Then the family $\{P_1, I - P_1, P_2, I - P_2, P_3, I - P_3\}$ is transitive relative to $\mathcal{M}$.*

Proof. If $T \in \mathcal{M}$ leaves the ranges of each of these projections invariant, then $TP_i - P_iT = P_iTP_i + (I - P_i)TP_i - P_iTP_i - P_iT(I - P_i) = 0$ for $i = 1, 2, 3$. It follows that $T \in \mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. □

We now extend an idea of Halmos [3].

Proposition 2. *Let $\mathcal{H}$ be a complex, separable Hilbert space, and let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor of type II_1 such that $\mathcal{M}$ is generated, as a von Neumann algebra, by two self-adjoint elements A, B. There is a transitive family of* 5 *projections relative to $\mathcal{M} \otimes M_2(\mathbb{C})$.*

Proof. We realize $\mathcal{M}\otimes M_2(\mathbb{C})$ as $M_2(\mathcal{M})$ acting on $\mathcal{H}\oplus\mathcal{H}$. Let I denote the identity in $\mathcal{M}$, and I_2 the identity in $M_2(\mathcal{M})$. Now each of the projections

$$P_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & \frac{1}{2}I \end{bmatrix}$$

lies in $M_2(\mathcal{M})$. If an operator $T \in M_2(\mathcal{M})$ leaves the ranges of each of these three projections invariant, then T must have the form

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}$$

with $T_1 \in \mathcal{M}$. We consider the matrix

$$A_1 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}A & \frac{1}{2}A \end{bmatrix} \in M_2(\mathcal{M}).$$

Letting $\lambda = ||A_1A_1^*||^{-1}$, we note that there is the following equality of range projections $R(\lambda A_1A_1^*) = R(A_1A_1^*) = R(A_1)$. Now notice that $\lambda A_1A_1^*$ is a positive element of norm 1, and therefore $0 \le \lambda A_1A_1^* \le I_2$, and by Lemma 5.15 in the first volume of [5], the sequence $\{(\lambda A_1A_1^*)^{\frac{1}{n}}\}$ converges in the strong-operator topology to $R(\lambda A_1A_1^*)$ and therefore $R(A_1) \in M_2(\mathcal{M})$ since $M_2(\mathcal{M})$ is a von Neumann algebra. Note that the range of the operator A_1 is a closed subspace of $\mathcal{H}\oplus\mathcal{H}$ since it is the graph of the bounded operator A. Now if T leaves the ranges of P_1, P_2, P_3, and A_1 invariant, then for any $x \in \mathcal{H}$ it must be that

$$\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}\begin{bmatrix} x \\ Ax \end{bmatrix} = \begin{bmatrix} T_1x \\ T_1Ax \end{bmatrix} = \begin{bmatrix} T_1x \\ AT_1x \end{bmatrix}.$$

It follows that $T_1A = AT_1$.

Similarly, we consider

$$B_1 = \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}B & \frac{1}{2}B \end{bmatrix} \in M_2(\mathcal{M})$$

and see that $R(B_1) \in M_2(\mathcal{M})$, and if T leaves the ranges of P_1, P_2, P_3 and B_1 invariant, then $T_1B = BT_1$. Hence T_1 commutes with both generators of the factor $\mathcal{M}$, and hence $T_1 \in \mathcal{M}\cap\mathcal{M}' = \mathbb{C}I$. Thus the family of projections

$$\{P_1, P_2, P_3, R(A_1), R(B_1)\} \subseteq M_2(\mathcal{M})$$

is transitive. □

Corollary 1. *There is a transitive family of* 5 *projections relative to the hyperfinite* II_1 *factor* $\mathcal{R}$.

Proof. It is well known that $\mathcal{R} \simeq \mathcal{R}\otimes M_2(\mathbb{C})$ and that $\mathcal{R}$ is generated by two self-adjoint elements. From the above proposition, $\mathcal{R}\otimes M_2(\mathbb{C})$ contains a transitive family of five projections. It follows that $\mathcal{R}$ contains a transitive family of five projections. □

We now exhibit a free transitive family of projections.

Let $\{G_i\}_{i=1}^n$ be groups, and let e_i be the identity of G_i for $i = 1, 2, ..., n$. Let $\mathop{*}\limits_{i=1}^{n} G_i$ denote the group free product of the G_i, and let e denote its identity element. Recall that elements in $\mathop{*}\limits_{i=1}^{n} G_i$ are given, in reduced form, by elements in the set

$\{e\} \cup \bigcup_{k\in\mathbb{N}} \{g_{i_1}g_{i_2}g_{i_3}...g_{i_k}\colon\ i_j \in \{1,2,...,n\};\ i_j \neq i_{j+1}$ for $j \in \{1,2,...,(n-1)\};$ $g_{i_j} \in G_{i_j}\backslash\{e_{i_j}\}\}$.

Let G denote the group free product $\underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * ... * \mathbb{Z}_2}_{12 \text{ times}}$, and let $\mathbb{A} = \{a_1, a_2, a_3, ..., a_{12}\}$ denote the canonical set of generators of G. This group is I.C.C.; therefore the group von Neumann algebra $\mathcal{L}_G$ acting on $l_2(G)$ is a factor of type II_1. Recall that each element in $\mathcal{L}_G$ has the form L_x, where $x \in l_2(G)$ and the action on a function $y \in l_2(G)$ is defined by $(L_x y)(g) = \sum_{h\in G} x(gh^{-1})y(h)$. With $g \in G$, let x_g be the function in $l_2(G)$ that takes the value 1 on g and 0 on every other group element. To avoid excessive use of subscripts, we everywhere write L_g in place of L_{x_g}. Since $L_{a_i}^2 = I$ for each $i \in \{1,2,3,...,12\}$, it is evident that $P_i = \frac{I+L_{a_i}}{2}$ is a projection in $\mathcal{L}_G$, and the family $\{P_1, P_2, ..., P_{12}\}$ is free with respect to the trace on $\mathcal{L}_G$.

Theorem 1. *In $\mathcal{L}_G$, the family $\{P_1, P_2, P_3, ..., P_{12}\}$ is transitive.*

Proof. Suppose that $L_f \in \mathcal{L}_G$ is a solution to the system $(I - P_i)TP_i = 0$ $(i = 1,2,3,...,12)$, and therefore

$$(I - L_{a_i})L_f(I + L_{a_i}) = 0 \qquad (i = 1,2,3,...,12).$$

Let both sides of this equation act on x_e to obtain

$$L_f x_e = L_{a_i}L_f x_e - L_f L_{a_i} x_e + L_{a_i} L_f L_{a_i} x_e \ (i = 1,2,3,...,12).$$

We see that

$$\begin{aligned}
(L_f x_e)(g) &= \sum\nolimits_{h\in G} f(gh^{-1})x_e(h) = f(g),\\
(L_{a_i}(L_f x_e))(g) &= (L_{a_i} f)(g) = \sum\nolimits_{h\in G} x_{a_i}(gh^{-1})f(h) = f(a_i g),\\
(L_f(L_{a_i} x_e))(g) &= (L_f x_{a_i})(g) = \sum\nolimits_{h\in G} f(gh^{-1})x_{a_i}(h) = f(g a_i),\\
(L_{a_i}(L_f L_{a_i} x_e))(g) &= (L_{a_i} L_f x_{a_i})(g)\\
&= \sum\nolimits_{h\in G} x_{a_i}(gh^{-1})(\sum\nolimits_{k\in G} f(hk^{-1})x_{a_i}(k)) = f(a_i g a_i).
\end{aligned}$$

From these it follows that for all $g \in G$,

$$f(g) = f(a_i g) - f(g a_i) + f(a_i g a_i) \qquad (i = 1,2,3,...,12).$$

By the triangle inequality, we see that

$$|f(g)| \leq |f(a_i g)| + |f(g a_i)| + |f(a_i g a_i)| \quad (i = 1,2,3,...,12),$$

and by the well-known inequality $(x_1 + ... + x_k)^2 \leq k(x_1^2 + ... + x_k^2)$ for non-negative real k, we see that

$$|f(g)|^2 \leq 3(|f(a_i g)|^2 + |f(g a_i)|^2 + |f(a_i g a_i)|^2) \quad (i = 1,2,3,...,12).$$

With $i, j \in \{1,2,...,12\}$ given, let $S_{ij} = \{g \in G : g$ begins with a_i and ends with a_j in its reduced form in the free product$\}$. With $g \in S_{ij}$, let $\{b_{ij}^1, ..., b_{ij}^{10}\} = \mathbb{A}\backslash\{a_i, a_j\}$. We then have

$$\begin{aligned}
|f(g)|^2 &\leq 3(|f(b_{ij}^1 g)|^2 + |f(g b_{ij}^1)|^2 + |f(b_{ij}^1 g b_{ij}^1)|^2),\\
|f(g)|^2 &\leq 3(|f(b_{ij}^2 g)|^2 + |f(g b_{ij}^2)|^2 + |f(b_{ij}^2 g b_{ij}^2)|^2),\\
&\vdots\\
|f(g)|^2 &\leq 3(|f(b_{ij}^{10} g)|^2 + |f(g b_{ij}^{10})|^2 + |f(b_{ij}^{10} g b_{ij}^{10})|^2).
\end{aligned}$$

Adding these inequalities, we obtain that

$$10|f(g)|^2 \leq 3\sum_{k=1}^{10}(|f(b_{ij}^k g)|^2 + |f(gb_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Summing over $g \in S_{ij}$, we have

$$\sum_{g\in S_{ij}} |f(g)|^2 \leq \frac{3}{10}\sum_{g\in S_{ij}}\sum_{k=1}^{10}(|f(b_{ij}^k g)|^2 + |f(gb_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Suppose that $g_1 \neq g_2$ are elements in S_{ij}. Note that by construction, all elements of the form $b_{ij}^k g_1$, $b_{ij}^k g_1 b_{ij}^l$ or $g_2 b_{ij}^k$ are in reduced form in the free product, for k, $l \in \{1, 2, ..., 10\}$. Consider any $k_1, k_2, k_3, k_4 \in \{1, 2, ..., 10\}$. Since $g_1 \neq g_2$, it follows that $b_{ij}^{k_1} g_1 \neq b_{ij}^{k_2} g_2$, $b_{ij}^{k_1} g_1 b_{ij}^{k_2} \neq b_{ij}^{k_3} g_2 b_{ij}^{k_4}$, $b_{ij}^{k_1} g_1 \neq g_2 b_{ij}^{k_2}$, $b_{ij}^{k_1} g_1 \neq b_{ij}^{k_2} g_2 b_{ij}^{k_3}$, and $g_1 b_{ij}^{k_1} \neq b_{ij}^{k_2} g_2 b_{ij}^{k_3}$. Therefore the right-hand sum above can have no repeated terms, meaning for any $g_0 \in G$, the term $|f(g_0)|^2$ shows up at most once on the right-hand side of the inequality. We sum over all i, j to obtain

$$\sum_{i,j=1}^{12}\sum_{g\in S_{ij}} |f(g)|^2 \leq \frac{3}{10}\sum_{i,j=1}^{12}\sum_{g\in S_{ij}}\sum_{k=1}^{10}(|f(b_{ij}^k g)|^2 + |f(gb_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2).$$

Let $S = \sum_{i,j=1}^{12}\sum_{g\in S_{ij}}\sum_{k=1}^{10}(|f(b_{ij}^k g)|^2 + |f(gb_{ij}^k)|^2 + |f(b_{ij}^k g b_{ij}^k)|^2)$. We now note that there can be repeated terms in S. We shall list the ways that a given term $|f(g_0)|^2$ may be repeated in the sum S. Suppose that $a, b \in \mathbb{A}$ and that g_0 begins with a and ends with b in its reduced form. Each occurrence of the term $|f(g_0)|^2$ in S corresponds to an appearence of $|f(g_0)|^2$ on the right side of an inequality of the form $|f(g')|^2 \leq 3(|f(a_i g')|^2 + |f(g' a_i)|^2 + |f(a_i g' a_i)|^2)$, where $a_i \in \mathbb{A}$ and g' is one of the group elements ag_0, $g_0 b$ or $ag_0 b$. If $a \neq b$, then there can only be two occurrences of $|f(g_0)|^2$ in the sum S, one coming from the inequality

$$|f(ag_0)|^2 \leq 3(|f(g_0)|^2 + |f(ag_0 a)|^2 + |f(g_0 a)|^2),$$

and one from the inequality

$$|f(g_0 b)|^2 \leq 3(|f(bg_0 b)|^2 + |f(g_0)|^2 + |f(bg_0)|^2).$$

If $a = b$, then $|f(g_0)|^2$ may occur three times, once in

$$|f(ag_0)|^2 \leq 3(|f(g_0)|^2 + |f(ag_0 a)|^2 + |f(g_0 a)|^2),$$

again in the inequality

$$|f(g_0 a)|^2 \leq 3(|f(ag_0 a)|^2 + |f(g_0)|^2 + |f(ag_0)|^2),$$

and finally, in the inequality

$$|f(ag_0 a)|^2 \leq 3(|f(g_0 a)|^2 + |f(ag_0)|^2 + |f(g_0)|^2).$$

We therefore note that any term $|f(g_0)|^2$ in S may occur at most three times. We call the number of times the term $|f(g_0)|^2$ appears in the sum S the multiplicity of $|f(g_0)|^2$ in S.

Let script $\mathcal{T}$ denote $\{t : t \text{ is a term in } S\}$. Then, since all terms in S are non-negative, $S = \sum_{t\in\mathcal{T}} n_t t$ where n_t is the multiplicity of the term t in S.

We now have that

$$\begin{aligned}\sum_{g\in G\setminus\{e\}}|f(g)|^2 &= \sum_{i,j=1}^{12}\sum_{g\in S_{ij}}|f(g)|^2\\ &\leq \frac{3}{10}S = \frac{3}{10}\sum\nolimits_{t\in\mathcal{T}} n_t t \leq \frac{9}{10}\sum\nolimits_{t\in\mathcal{T}} t\\ &\leq \frac{9}{10}\sum_{g\in G\setminus\{e\}}|f(g)|^2.\end{aligned}$$

Therefore $\sum_{g\in G\setminus\{e\}}|f(g)|^2$ is necessarily zero and $f(g)=0$ when $g\neq e$. We have now that only $f(e)$ may be nonzero, and hence L_f must be a scalar. It follows that the family $\{P_1,...,P_{12}\}$ is transitive. □

Remark 2. The number of projections in the above theorem may be reduced. We believe that 4 such free projections should form a transitive family, but new techniques may be needed to prove this.

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