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A GENERAL REARRANGEMENT INEQUALITY

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ABSTRACT. We prove a general rearrangement inequality for multiple integrals, using polarization. We introduce a special class of kernels for which the product inequality holds, and then we prove that it also holds when the product is replaced by a so-called function AL_m.

1. INTRODUCTION

Let $f : \Omega \to \mathbb{R}_+$ be a nonnegative measurable function defined on a measure space (Ω, μ), which satisfies the condition

$$\mu(\{f > t\}) < \infty, \qquad \forall\, t > 0 \tag{1.1}$$

where we write

$$\{f > t\} := \{x \in \Omega : f(x) > t\}.$$

Its *distribution function* λ_f is defined to be

$$\lambda_f(t) = \mu(\{f > t\}), \qquad t \in [0, \infty).$$

Two functions f and g are said to be *equimeasurable* if they have the same distribution function and we write $f \sim g$. Functions which are equimeasurable are also said to be *rearrangements* of each other.

Let X denote either $\mathbb{R}^n$, $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, or the n-dimensional hyperbolic space $\mathbb{H}^n$. We equip X with its geodesic distance d and canonical measure μ, and we fix an origin $\mathbf{o}$ in X. For $\mathbb{R}^n$, d is the Euclidean distance, μ is the Lebesgue measure, and $\mathbf{o}$ is $(0, \dots, 0)$. For $\mathbb{S}^n$, d is the great circle distance on the sphere, μ is the surface area measure and $\mathbf{o}$ is $(1, 0, \dots, 0)$. We take as model for $\mathbb{H}^n$ the open unit ball $\{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the distance element $ds = \frac{2}{1-|x|^2}|dx|$; μ is the volume measure associated to ds, and the origin $\mathbf{o}$ is $(0, \dots, 0)$.

Given $f : X \to \mathbb{R}_+$ satisfying (1.1) we define its *symmetric decreasing rearrangement* $f^\sharp : X \to \mathbb{R}_+$ as follows :

$$f^\sharp(x) = \inf\{t : \lambda_f(t) \le \mu(B(d(x)))\},$$

where $B(d(x))$ is the ball centered at $\mathbf{o}$ and with radius $d(x) = d(x, \mathbf{o})$. It follows easily that $f^\sharp$ is constant on each sphere centered at $\mathbf{o}$ and decreases as $d(x)$ increases. Moreover, $f^\sharp \sim f$. See, e.g., [3].

We define now a very simple rearrangement, called *polarization* (see [2, 3, 6, 7]).

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Let $\mathcal{H}(\mathbb{R}^n)$ be the collection of all $(n-1)$-dimensional affine subspaces H of $\mathbb{R}^n$ such that $\mathbf{o} \notin H$, $\mathcal{H}(\mathbb{S}^n)$ the collection of all intersections of $\mathbb{S}^n$ with hyperplanes through the origin in $\mathbb{R}^{n+1}$ which do not contain $\mathbf{o}$ and $\mathcal{H}(\mathbb{H}^n)$ the collection of all images under the group of hyperbolic motions of the hyperbolic $n-1$ plane $\{x \in \mathbb{R}^n : |x| < 1,\, x_n = 0\} \subset \mathbb{H}^n$ which do not contain the origin $\mathbf{o}$. For $H \in \mathcal{H}(X)$, let $^-$denote the reflection in H (i.e. $\bar{x}$ is the reflection of x in H), and let H^+ and H^- be the two components of $X \setminus H$, such that $\mathbf{o} \in H^+$.

For $f : X \to \mathbb{R}$, we define a function $f^H : X \to \mathbb{R}$ by $f^H(x) = f(x)$ if $x \in H$, and

$$f^H(x) = \begin{cases} \max(f(x), f(\bar{x})) & \text{if } x \in H^+, \\ \min(f(x), f(\bar{x})) & \text{if } x \in H^-. \end{cases} \tag{1.2}$$

We explore, in what follows, some of the rich history of rearrangement inequalities. The most basic rearrangement inequality is due to Hardy and Littlewood [13]:

$$\int_X f(x)g(x)\, d\mu(x) \le \int_X f^\sharp(x) g^\sharp(x)\, d\mu(x),$$

with $f, g : X \to \mathbb{R}_+$ satisfying (1.1). One can show that the inequality also holds for polarizations.

We will prove inequalities from my thesis [10], of the form

$$\int_X \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m)\, dx \le \int_X \Psi(f_1^\sharp(x_1), \dots, f_m^\sharp(x_m)) K(x_1, \dots, x_m)\, dx,$$

where $dx = d\mu(x_1) \dots d\mu(x_m)$, which generalize the Hardy-Littlewood inequality. See also [11].

The theory of rearrangement inequalities is well-developed in $\mathbb{R}^n$. The following inequality, known as Riesz-Sobolev, was proved for $n = 1$ by F. Riesz in 1930 and for $n \ge 1$ by S. L. Sobolev in 1938 :

$$\int_{\mathbb{R}^{2n}} f(x)g(y)h(x-y)\, dxdy \le \int_{\mathbb{R}^{2n}} f^\sharp(x) g^\sharp(y) h^\sharp(x-y)\, dxdy.$$

The functions f, g, and h are nonnegative.

A general rearrangement inequality involving nonnegative functions is due to H. J. Brascamp, E. H. Lieb, and J. M. Luttinger [5]:

$$\int_{(\mathbb{R}^n)^p} \prod_{j=1}^k f_j\Big(\sum_{m=1}^p a_{jm} x_m\Big)\, dx \le \int_{(\mathbb{R}^n)^p} \prod_{j=1}^k f_j^\sharp\Big(\sum_{m=1}^p a_{jm} x_m\Big)\, dx \tag{1.3}$$

where a_{jm} are constants, $dx = dx_1 \dots dx_p$, and each $x_i \in \mathbb{R}^n$. When $p = 2$, $k = 3$ and a_{jm} are suitably chosen, (1.3) implies the Riesz-Sobolev inequality.

The proof of (1.3) uses the Brunn-Minkowski inequality [15], which states that if K and L are convex bodies (compact convex sets with nonempty interiors) in $\mathbb{R}^k$ and $0 < \lambda < 1$, then

$$V((1-\lambda)K + \lambda L)^{1/k} \ge (1-\lambda)V(K)^{1/k} + \lambda V(L)^{1/k}.$$

V denotes the k-dimensional volume and $+$ is the vector sum of K and L. Further generalizations of (1.3) were given by M. Christ [9] and R. E. Pfiefer [16].

The proofs of the rearrangement inequalities on $\mathbb{R}^n$ require the product structure $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$, as well as the Brunn-Minkowski inequality. These tools are not available on $\mathbb{S}^n$ or $\mathbb{H}^n$, and for this reason complete analogues of (1.3) for $\mathbb{S}^n$ and $\mathbb{H}^n$ are not known. To get rearrangement inequalities on X, polarization proved to be an effective tool. A. Baernstein and B. A. Taylor [2] used it to derive inequalities for $\mathbb{S}^n$, and W. Beckner [4] noticed that it can also be used for $\mathbb{H}^n$.

Baernstein and Taylor showed that the symmetric decreasing rearrangement of a function can be approximated through a sequence of polarizations [2]. The polarization is easier to handle and if an inequality holds for polarizations in all hyperplanes in $\mathcal{H}(X)$, then in most cases one can apply a similar approximation argument to deduce the inequality for the symmetric decreasing rearrangement. We state here the Baernstein-Taylor inequality:

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f(x)g(y)K(d(x,y))\,d\mu(x)\,d\mu(y) \leq \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} f^\sharp(x)g^\sharp(y)K(d(x,y))\,d\mu(x)\,d\mu(y),$$

for $K : \mathbb{R}_+ \to \mathbb{R}_+$ a decreasing function. This inequality was proved for f^H and g^H in place of $f^\sharp$ and $g^\sharp$, for every $H \in \mathcal{H}(X)$, and then they used the approximation argument to deduce it for $f^\sharp$ and $g^\sharp$.

We introduce now a special class of functions of particular interest for our result. For $\underline{x}^1 = (x_1^1, \ldots x_m^1)$, $\underline{x}^2 = (x_1^2, \ldots, x_m^2) \in \mathbb{R}_+^m$ we write $\underline{x}^1 \leq \underline{x}^2$ if $x_i^1 \leq x_i^2$, for every $i = 1, \ldots, m$.

Definition 1.1. Let $\Psi : \mathbb{R}_+^m \to \mathbb{R}_+$ be a continuous function such that

$$\Psi_{|\{x_i=0\}} = 0, \text{for every } i = 1, \ldots, m.$$

We say that $\Psi \in AL_m(\mathbb{R}_+^m)$ if, for every $\underline{x}^1$, $\underline{x}^2 \in \mathbb{R}_+^m$, such that $\underline{x}^1 \leq \underline{x}^2$, the following condition holds:

$$\sum_{i_1,\ldots,i_m \in \{1,2\}} (-1)^{i_1+\ldots+i_m} \Psi(x_1^{i_1}, \ldots, x_m^{i_m}) \geq 0. \tag{1.4}$$

If $\Psi \in C^m(\mathbb{R}_+^m)$, then (1.4) is equivalent to $\frac{\partial^m \Psi}{\partial x_1 \ldots \partial x_m} \geq 0$.

Consider now a family of m positive random variables $X_1, \ldots, X_m$ on some probability space (Ω, P), and define

$$\Psi(x_1, \ldots, x_m) = P(X_1 \leq x_1, \ldots, X_m \leq x_m),$$

their joint distribution function. Then Ψ satisfies (1.4).

One can show that $\Psi \in AL_m(\mathbb{R}_+^m)$ implies Ψ is increasing in each argument, when the rest are fixed. This can be seen, for example, by taking $\underline{x}^1 = (x_1, 0, \ldots, 0)$ and $\underline{x}^2 = (y_1, x_2, \ldots, x_m)$ with $\underline{x}^1 \leq \underline{x}^2$, and using (1.4) and the fact that Ψ vanishes on the boundary.

We say that a continuous function $\Psi : \mathbb{R}_+^m \to \mathbb{R}_+$ is in $AL_2(\mathbb{R}_+^m)$ if for every fixed $m-2$ variables Ψ is in $AL_2(\mathbb{R}_+^2)$ in the remaining two arguments [6]. This class of functions was studied by Almgren and Lieb [1].

Brock [6] proved an extension of the Hardy-Littlewood inequality: If $\Psi \in AL_2(\mathbb{R}_+^m)$, then

$$\int_X \Psi(f_1(x), \ldots, f_m(x))\,d\mu(x) \leq \int_X \Psi(f_1^\sharp(x), \ldots, f_m^\sharp(x))\,d\mu(x), \tag{1.5}$$

for every $f_1, \ldots, f_m : X \to \mathbb{R}_+$ satisfying (1.1). As before, X is any of the spaces $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$.

In the same spirit, Morpurgo [14] proved the following inequality: For $K_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$, $1 \le i < j \le m$, decreasing functions and $\Psi_0 : \mathbb{R} \to \mathbb{R}$ a continuous convex function:

$$\int_{X^m} \prod_{i<j} K_{ij}(d(x_i, x_j))\Psi_0(\sum_{l=1}^{m} f_l(x_l))\, dx \le \int_{X^m} \prod_{i<j} K_{ij}(d(x_i, x_j))\Psi_0(\sum_{l=1}^{m} f_l^\sharp(x_l))\, dx, \tag{1.6}$$

where $dx = dx_1 \dots dx_m$. A similar result was proved by Burchard and Schmuckenschläger, [8]. Both Morpurgo's result (1.6) and Brock's result (1.5) were first proved for polarization, then obtained for the symmetric decreasing rearrangement by the approximation argument. Also, Morpurgo's inequality contains the Baernstein-Taylor inequality as a special case.

Our result gives a larger class of kernels K for which the polarization inequality holds in Morpurgo's theorem, when $\Psi_0(\sum_{l=1}^m f_l(x_l))$ is replaced by $f_1(x_1)\cdots f_m(x_m)$ and hence by $\Psi(f_1(x_1), \dots, f_m(x_m))$, where $\Psi \in AL_m(\mathbb{R}^m_+)$. We begin with a discrete result.

2. A discrete rearrangement inequality

Let $Q = \{1, 2\}$ be a set with two points, and let $K_0 : Q^m \to \mathbb{R}_+$ be a nonnegative function. We think of Q^m as the vertices of a cube in $\mathbb{R}^m$. Take $k \in \{1, \dots, m\}$ and $S \subset \{1, \dots, m\}$, with $|S| = k$. We write $S = \{i_1, \dots, i_k\}$, with $i_1 < \dots < i_k$. For every S and $y = (y_1, \dots, y_k) \in Q^k$, let

$$A(y, S) = \{x \in Q^m : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}.$$

We can think of $A(y, S)$ as an $(m - k)$-dimensional subcube of Q^m.

Let $1_k = (1, \dots, 1) \in Q^k$. Then $A(1_k, S)$ is the set of points x in Q^m for which $x_{i_1} = \dots = x_{i_k} = 1$.

We define

$$K_0(y, S) = \sum_{x \in A(y,S)} K_0(x), \quad y \in Q^k. \tag{2.1}$$

Definition 2.1. A function $K_0 : Q^m \to \mathbb{R}_+$ is called a "good" cube kernel if

$$K_0(y, S) \le K_0(1_k, S), \tag{2.2}$$

for every $y \in Q^k$, every $S \subset \{1, \dots, m\}$ with $|S| = k$, and every $k \in \{1, \dots, m\}$.

Let us consider some particular cases. For $k = m$, we have $A(y, S) = \{y\}$ and $S = \{1, \dots, m\}$. Then (2.2) says that

$$K_0(y) \le K_0(1_m), \qquad \forall\, y \in Q^m.$$

For $k = m - 1$ and $S = \{1, \dots, m\} \setminus \{j\}$, (2.2) asserts that

$$K_0(y_1, \dots, y_{j-1}, 1, y_{j+1}, \dots, y_m) + K_0(y_1, \dots, y_{j-1}, 2, \dots, y_m) \le K_0(1, 1, \dots, 1) + K_0(1, 1, \dots, 2, \dots, 1), \quad \forall\, y \in Q^{m-1}. \tag{2.3}$$

One may picture the left-hand side of (2.3) as the sum of K_0 over an edge of the m-cube Q^m, and then (2.3) says that among all parallel edges, the maximal edge sum is achieved when the edge contains $(1, \dots, 1)$.

Likewise, for every S, with $|S| = k$, $A(y, S)$ represents the sum of K_0 over the vertices of an $m-k$ subcube of Q^m and (2.2) requires that among all parallel $m-k$ subcubes, the maximal sum is attained when the subcube contains $(1, \dots, 1)$.

For $g : Q \to \mathbb{R}_+$ we define g^*, its decreasing rearrangement, as follows:

$$g^*(1) = \max(g(1), g(2)) \qquad \text{and} \qquad g^*(2) = \min(g(1), g(2)).$$

The following lemma is immediate from the definition of K_0.

Lemma 2.2. *Let $K_0 : Q^m \to \mathbb{R}_+$ be a nonnegative function. Then, the following inequality is true:*

$$\int_{Q^m} g_1(\varepsilon_1) \cdots g_m(\varepsilon_m) K_0(\varepsilon_1, \dots, \varepsilon_m) d\varepsilon_1 \dots d\varepsilon_m \le \int_{Q^m} g_1^*(\varepsilon_1) \cdots g_m^*(\varepsilon_m) K_0(\varepsilon_1, \dots, \varepsilon_m) d\varepsilon_1 \dots d\varepsilon_m \tag{2.4}$$

for every $g_1, \dots, g_m : Q \to \mathbb{R}_+$, if and only if K_0 is a good cube kernel.

The proof is achieved by considering each g_i to be the characteristic function of a point in Q. This result implies the following:

Proposition 2.3. *Let $f_i : Q \to \mathbb{R}_+$ be m nonnegative functions, let $\Psi \in AL_m(\mathbb{R}_+^m)$ and let $K_0 : Q^m \to \mathbb{R}_+$ be a good cube kernel. We define*

$$I[g_1, \dots, g_m] = \int_{Q^m} \Psi(g_1(\varepsilon_1), \dots, g_m(\varepsilon_m)) K_0(\varepsilon_1, \dots, \varepsilon_m)\, d\varepsilon_1 \dots d\varepsilon_m.$$

Sometimes we write $I(g_1, \dots, g_m, \Psi, K)$ if we want to emphasize Ψ and K. Then

$$I[g_1, \dots, g_m] \le I[g_1^*, \dots, g_m^*].$$

Conversely, if for some $K_0 : Q^m \to \mathbb{R}_+$ we assume that the inequality holds for every $\Psi \in AL_m(\mathbb{R}_+^m)$ and all choices of g_i, then K_0 is a good cube kernel.

Proof. For every $a > 0$ we define the set

$$C(a) = \mathbb{R}_+^m \setminus \{(x_1, \dots, x_m) : x_i > a, \forall\, i = 1, \dots, m\}.$$

We assume first that $\Psi \in C^m$ (Ψ is m times continuously differentiable) and that it vanishes on $C(a)$, for some a. We already have the inequality for the case when Ψ is the product function, $\Psi(y_1, \dots, y_m) = y_1 \cdots y_m$. So, we need to express Ψ in terms of a product. To condense notation, let

$$\Phi(s) = \frac{\partial^m \Psi}{\partial s_1 \dots \partial s_m}(s),$$

$d\varepsilon = d\varepsilon_1 \dots d\varepsilon_m$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $ds = ds_1 \dots ds_m$, $s = (s_1, \dots, s_m)$. Then $\Phi \ge 0$ since $\Psi \in AL_m$.

Since all the boundary terms are zero, we can write

$$\begin{aligned}\Psi(y_1, \dots, y_m) &= \int_0^{y_1} \int_0^{y_2} \dots \int_0^{y_m} \Phi(s)\, ds_1 \dots ds_m \\ &= \int_0^{\infty} \dots \int_0^{\infty} \Phi(s) \chi_{\{s_1 < y_1\}}(s_1) \cdots \chi_{\{s_m < y_m\}}(s_m)\, ds.\end{aligned}$$

Setting $y_i = g_i(\varepsilon_i)$, multiplying by $K_0(\varepsilon)$ and integrating, we obtain

$$I[g_1, \dots, g_m] = \int_{Q^m} \int_{(0,\infty)^m} \Phi(s) \chi_{\{g_1 > s_1\}}(\varepsilon_1) \cdots \chi_{\{g_m > s_m\}}(\varepsilon_m) K_0(\varepsilon)\, ds\, d\varepsilon.$$

Now reverse the order of integration. For fixed $s_1,\ldots, s_m$, positive,

$$\chi^*_{\{g_i>s_i\}} = \chi_{\{g_i^*>s_i\}}, \quad \text{as functions on } Q,$$

and we can apply Lemma 2.2 to show that

$$\begin{aligned}&\int_{Q^m} \chi_{\{g_1>s_1\}}(\varepsilon_1)\cdots\chi_{\{g_m>s_m\}}(\varepsilon_m)K_0(\varepsilon)\,d\varepsilon\\ \le&\int_{Q^m} \chi_{\{g_1^*>s_1\}}(\varepsilon_1)\cdots\chi_{\{g_m^*>s_m\}}(\varepsilon_m)K_0(\varepsilon)\,d\varepsilon.\end{aligned}$$

It follows that $I(g_1,\ldots,g_m,\Psi,K_0) \le I(g_1^*,\ldots,g_m^*,\Psi,K_0)$ when $\Psi\in C^m$ and Ψ vanishes on $C(a)$.

If Ψ vanishes on $C(a)$ and is not smooth, then we can approximate it by smooth functions by convolving it with C_0^∞ functions. First, we extend Ψ to $\mathbb{R}^m$ by defining it to be zero outside $\mathbb{R}^m_+$, its domain. The new function Ψ is continuous on $\mathbb{R}^m$.

Let $G:\mathbb{R}^m\to\mathbb{R}_+$ be a C_0^∞ function such that $\int_{\mathbb{R}^m} G = 1$. Define $G_\delta(x) = \delta^{-m}G(x/\delta)$ and

$$\Psi_\delta(x) = (\Psi * G_\delta)(x) = \int_{\mathbb{R}^m}\Psi(y)G_\delta(x-y)\,dy.$$

Then Ψ_δ is smooth and in $AL_m(\mathbb{R}^m_+)$, and since $\Psi=0$ on $C(a)$, Ψ_δ is zero on $C(a_\delta)$, when δ is small enough, for some $a_\delta>0$. Thus we can apply our inequality for Ψ_δ to get

$$I(g_1,\ldots,g_m,\Psi_\delta,K_0)\le I(g_1^*,\ldots,g_m^*,\Psi_\delta,K_0).$$

Since the Ψ_δ are uniformly close to Ψ on the discrete set $f_1(Q)\times\ldots\times f_m(Q)$, we let $\delta\to 0$ to get

$$I(g_1,\ldots,g_m,\Psi,K_0)\le I(g_1^*,\ldots,g_m^*,\Psi,K_0),$$

for $\Psi\in AL_m(\mathbb{R}^m_+)$ and $\Psi=0$ on $C(a)$.

For general $\Psi\in AL_m(\mathbb{R}^m_+)$, let

$$\Psi_k(x) = \Psi(x-(1/k)1_m), \quad \text{with } 1_m=(1,\ldots,1)\in\mathbb{R}^m,\ x\in\mathbb{R}^m_+\setminus C(1/k).$$

Then we can define Ψ_k to be zero on $C(1/k)$. The new Ψ_k's are in $AL_m(\mathbb{R}^m_+)$. Since $\Psi_k\uparrow\Psi$ on R^m_+, and since

$$I(g_1,\ldots,g_m,\Psi_k,K_0)\le I(g_1^*,\ldots,g_m^*,\Psi_k,K_0),$$

we can pass to the limit under the integral sign, and we obtain

$$I(g_1,\ldots,g_m,\Psi,K_0)\le I(g_1^*,\ldots,g_m^*,\Psi,K_0).$$

The converse statement of Proposition 2.3 follows from Lemma 2.2, by taking $\Psi(y_1,\ldots,y_m)=y_1\cdots y_m$. □

3. Main result

We return now to analysis on X. We consider a measurable function $K: X^m\to\mathbb{R}_+$, and take $H\in\mathcal{H}(X)$. For each $(x_1,\ldots,x_m)\in(H^+)^m$, we define $K_0: Q^m\to\mathbb{R}_+$ as follows:

$$K_0(\varepsilon_1,\ldots,\varepsilon_m) := K(y_1,\ldots,y_m), \quad \text{with } y_i = \begin{cases} x_i & \text{if } \varepsilon_i = 1,\\ \bar{x}_i & \text{if } \varepsilon_i=2.\end{cases}$$

As usual, $\bar{x}_i$ is the reflection of x_i in the hyperplane H. We note that K_0 depends on $(x_1,\ldots,x_m)$.

Definition 3.1. We call $K : X^m \to \mathbb{R}_+$ a good polarization kernel if each K_0 is a good cube kernel for almost every $(x_1, \dots, x_m) \in (H^+)^m$ and every $H \in \mathcal{H}(X)$ with $0 \in H^+$.

Here is an equivalent definition of a good polarization kernel, in terms of sums. Consider $k \in \{1, \dots, m\}$ and $S \subset \{1, \dots, m\}$, $S = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$, and $H \in \mathcal{H}(X)$ with $(x_1, \dots, x_m) \in (H^+)^m$ being fixed. For every $y = (y_1, \dots, y_k) \in \{x_{i_1}, \bar{x}_{i_1}\} \times \dots \times \{x_{i_k}, \bar{x}_{i_k}\}$, let

$$A_H(y, S) = \{x \in \{x_1, \bar{x}_1\} \times \dots \times \{x_m, \bar{x}_m\} : x_{i_1} = y_1, \dots, x_{i_k} = y_k\}.$$

We define

$$K(y, S) = \sum_{x \in A_H(y,S)} K(x). \tag{3.1}$$

Let $y^* = (x_{i_1}, \dots, x_{i_k})$. Then we see that our definition is equivalent to the condition $K(y, S) \le K(y^*, S)$, for every $S \subset \{1, \dots, m\}$, $H \in \mathcal{H}(X)$ and a.e. $(x_1, \dots, x_m) \in (H^+)^m$.

Now, consider $\Psi : \mathbb{R}_+^m \to \mathbb{R}_+$ and $f_i : X \to \mathbb{R}_+$, $1 \le i \le m$. Define

$$I(f_1, \dots, f_m, K, \Psi) = \int_{X^m} \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m)\, dx_1 \dots dx_m. \tag{3.2}$$

If we split each integral $\int_X = \int_{H^+} + \int_{H^-}$ and use reflection in the second integral and collect terms, we get in (3.2):

$$I(f_1, \dots, f_m, K, \Psi) = \int_{(H^+)^m} I(g_1, \dots, g_m, \Psi, K_0)\, dx_1 \dots dx_m. \tag{3.3}$$

Here the $g_i : Q \to \mathbb{R}_+$ depend on x_i and are defined as follows:

$$g_i(1) = f_i(x_i), \qquad g_i(2) = f_i(\bar{x}_i), \quad x_i \in X.$$

Theorem 3.2. *Let $X = \mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$, let $f_i : X \to \mathbb{R}_+$ be m nonnegative functions, let $\Psi \in AL_m(\mathbb{R}_+^m)$, and let K be a good polarization kernel. We define*

$$I[f_1, \dots, f_m] = \int_{X^m} \Psi(f_1(x_1), \dots, f_m(x_m)) K(x_1, \dots, x_m)\, dx_1 \dots dx_m.$$

Then, the following inequalities hold:

$$I[f_1, \dots, f_m] \le I[f_1^H, \dots, f_m^H] \qquad \forall H \in \mathcal{H}(X) \tag{3.4}$$

and

$$I[f_1, \dots, f_m] \le I[f_1^\sharp, \dots, f_m^\sharp]. \tag{3.5}$$

If $K : X^m \to \mathbb{R}_+$ is such that the polarization inequality (3.4) *holds for every $H \in \mathcal{H}(X)$ and every $\Psi \in AL_m(\mathbb{R}_+^m)$, then K is a good polarization kernel.*

Proof. The polarization inequality in Theorem 3.2 follows from Proposition 2.3 applied to the integrand in (3.3), using the fact that if $g_1, \dots, g_m$ correspond to $f_1, \dots, f_m$ for a certain $(x_1, \dots, x_m)$, then $g_1^*, \dots, g_m^*$ correspond to $f_1^H, \dots, f_m^H$. Also, inequality (3.5) can be deduced using an approximation argument similar to the one presented in [2].

Conversely, to prove that K must be a good polarization kernel if (3.4) always holds, let $\Psi(y_1, \dots, y_m) = y_1 \cdots y_m$. Let us fix $H \in \mathcal{H}(X)$ and consider

$(x_1, \ldots, x_m) \in (H^+)^m$. For $S = \{i_1, \ldots, i_k\}$ and $y = (y_1, \ldots, y_k) \in \{x_{i_1}, \bar{x}_{i_1}\} \times \ldots \times \{x_{i_k}, \bar{x}_{i_k}\}$, define

$$f_{i_1} = (1/c_\varepsilon)\chi_{B(y_1,\varepsilon)}, \quad \ldots, \quad f_{i_k} = (1/c_\varepsilon)\chi_{B(y_k,\varepsilon)}$$

and $f_j = (1/c_\varepsilon)\chi_{B(x_j,\varepsilon)} + (1/c_\varepsilon)\chi_{B(\bar{x}_j,\varepsilon)}$, for $j \in S^c$. Here c_ε represents the measure of any ball of radius ε in X.

For ε sufficiently small,

$$f_{i_1}^H = (1/c_\varepsilon)\chi_{B(x_{i_1},\varepsilon)}, \quad \ldots, \quad f_{i_k}^H = (1/c_\varepsilon)\chi_{B(x_{i_k},\varepsilon)}$$

and $f_j = f_j^H$ for $j \in S^c$.

Without loss of generality, we can assume that K is integrable ($K \in L^1(X^m)$).

Since

$$I(f_1, \ldots, f_m) \le I(f_1^H, \ldots, f_m^H),$$

and by passing to the limit as $\varepsilon \to 0$ we get that LHS $\to K(y, S)$, RHS $\to K(y^*, S)$ for a.e. $(x_1, \ldots, x_m) \in (H^+)^m$ (see [12, p. 93]), it follows that

$$K(y, S) \le K(y^*, S), \quad \text{for a.e. } (x_1, \ldots, x_m) \in (H^+)^m.$$

Since $H \in \mathcal{H}(X)$ was arbitrarily chosen, this proves that K is a good polarization kernel. □

We conclude this section with a few remarks. Firstly, when $m = 2$, $K(x, y) = K_1(d(x, y))$, with K_1 a decreasing function, is a good polarization kernel.

Kernels of the form $K(x_1, \ldots, x_m) = k(|x_1|^{\alpha_1} + |x_2|^{\alpha_2} + \ldots + |x_m|^{\alpha_m})$, where $\alpha_1, \ldots, \alpha_m \ge 0$ are positive real numbers and k is a decreasing function of one variable, are good polarization kernels. Here $|x|$ denotes the distance to the origin, $d(x, \mathbf{o})$. To see this, let $H \in \mathcal{H}(X)$ and $(x_1, \ldots, x_m) \in (H^+)^m$. With the notation in (3.1), $K(x) \le K(z)$ for $x \in A_H(y, S)$ and $z \in A_H(y^*, S)$ such that $x_i = z_i$ for every $i \in S^c$, since any component of z has distance to the origin less than or equal to the distance to the origin of the corresponding component of x and since k is decreasing. There is a one-to-one correspondence between elements in $A_H(y, S)$ and $A_H(y^*, S)$, and thus

$$K(y, S) = \sum_{x \in A_H(y,S)} K(x) \le \sum_{z \in A_H(y^*,S)} K(z) = K(y^*, S),$$

for every $H \in \mathcal{H}(X)$, and $(x_1, \ldots, x_m) \in (H^+)^m$ and y. This proves that K is a good polarization kernel.

More generally, $K(x_1, \ldots, x_m)$, where K is symmetric decreasing in each argument, is a good polarization kernel.

We show now that Morpurgo's kernels are good polarization kernels. Let $K(x_1, \ldots, x_m) = \prod_{i<j} K_{ij}(d(x_i, x_j))$ where $x_1, \ldots, x_m \in X$, and $K_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are decreasing. We have $I(f_1, \ldots, f_m, \Psi, K) \le I(f_1^H, \ldots, f_m^H, \Psi, K)$ for

$$\Psi(y_1, \ldots, y_m) = y_1 \cdots y_m \in AL_m(\mathbb{R}_+^m)$$

and all nonnegative f_i's and $H \in \mathcal{H}(X)$. This follows from Morpurgo's result (1.6), which was proved using polarization, and by taking $\Psi_0(y) = e^y$ and using the fact that $e^{f^H} = (e^f)^H$. The proof of the necessity part of Theorem 3.2 required only the case when Ψ is the product function. Thus, we can conclude that K is a good polarization kernel.

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References

[1] F. J. Almgren, Jr. and E. H. Lieb. Symmetric decreasing rearrangement is sometimes continuous. *J. Amer. Math. Soc.*, 2(4):683–773, 1989. MR1002633 (90f:49038)

[2] A. Baernstein, II and B. A. Taylor. Spherical rearrangements, subharmonic functions, and *-functions in n-space. *Duke Math. J.*, 43(2):245–268, 1976. MR0402083 (53:5906)

[3] A. Baernstein II. A unified approach to symmetrization. *Partial Differential Equations of Elliptic Type, eds. A.Alvino et al, Symposia Mathematica, Cambridge Univ.Press*, 35:47–91, 1995. MR1297773 (96e:26019)

[4] W. Beckner. Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n. *Proc. Nat. Acad. Sci. U.S.A.*, 89(11):4816–4819, 1992. MR1164616 (93d:26018)

[5] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger. A general rearrangement inequality for multiple integrals. *J. Funct. Anal.*, 17:227–237, 1974. MR0346109 (49:10835)

[6] F. Brock. A general rearrangement inequality à la Hardy-Littlewood. *J. Inequal. Appl.*, 5(4):309–320, 2000. MR1784980 (2001d:26046)

[7] F. Brock and A. Y. Solynin. An approach to symmetrization via polarization. *Trans. Amer. Math. Soc.*, 352(4):1759–1796, 2000. MR1695019 (2001a:26014)

[8] A. Burchard and M. Schmuckenschläger. Comparison theorems for exit times. *Geom. Funct. Anal.*, 11(4):651–692, 2001. MR1866798 (2002h:60168)

[9] M. Christ. Estimates for the k-plane transform. *Indiana Univ. Math. J.*, 33(6):891–910, 1984. MR0763948 (86k:44004)

[10] C. Draghici. *Polarization and rearrangement inequalities for multiple integrals*. Ph.D. thesis, Washington University, 2003.

[11] C. Draghici. Rearrangement inequalities with applications to ratios of heat kernels. *preprint*, 2003.

[12] G. B. Folland. *Real analysis*. Pure and Applied Mathematics. John Wiley & Sons Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication. MR0767633 (86k:28001)

[13] G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition. MR0944909 (89d:26016)

[14] C. Morpurgo. Sharp inequalities for functional integrals and traces of conformally invariant operators. *Duke Math. J.*, 114(3):477–554, 2002. MR1924571 (2004f:58035)

[15] R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 1978. MR0500557 (58:18161)

[16] R. E. Pfiefer. Maximum and minimum sets for some geometric mean values. *J. Theoret. Probab.*, 3(2):169–179, 1990. MR1046328 (91d:60028)

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