ON THE EVALUATION OF SALIÉ SUMS

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Abstract. The Salié sum $S(m, n; c)$ can be evaluated as the product of a Gauss sum and an exponential sum involving square roots of $mn$ mod $c$. We give a new proof of this fact that can simultaneously handle a twisted version of these sums that arise in the theory of half-integral weight modular forms.

The exponential sum

$$K(m, n; c) = \sum_{a \equiv 1 (c)} \varepsilon_a \left(\frac{c}{a}\right) e((ma + n\overline{a})/c)$$

arises in the theory of modular forms of half-integral weight. $K(m, n; c)$ is only defined when $4|c$, and then $\varepsilon_a = 1$ or $i$ according to whether $a \equiv 1$ or $3 \text{ mod } 4$, $\left(\frac{-}{a}\right)$ is the extension of the Legendre-Jacobi symbol as in [5], and $e(z) = \exp(2\pi i z)$. In Iwaniec’s celebrated estimates for the Fourier coefficients of said forms (see [2]), an essential role is played by the identity

$$K(D, 1; c) = \frac{G(1, 0; c)}{2} \sum_{x^2 \equiv D (c)} e(2x/c)$$

valid when $8|c$ and the analogous formula

$$S(D, 1; c) = \sum_{a \equiv 1 (c)} \left(\frac{c}{a}\right) e((Da + \overline{a})/c) = G(1, 0; c) \sum_{x^2 \equiv D(c)} e(2x/c),$$

which is valid whenever $c$ is odd. In the formula above $G(a, b; c)$ is the Gauss sum

$$G(a, b; c) = \sum_{x(c)} e((ax^2 + bx)/c).$$

When the modulus is prime this identity was first proved by Salié [3]; see also [6]. Iwaniec derived a formula for the general modulus by pasting the local results together using quadratic reciprocity, while Sarnak gave a “global” proof of (2) that works when $(2D, c) = 1$; see [4].

The purpose of this note is to give a simple argument that leads to (1) when $c$ is even and to (2) when $c$ is odd, without any assumption on $D$. It is based on a very simple idea: we start by the sum

$$A = \sum_{x^2 \equiv D(c)} e(2x/c)$$
and sieve out the support of the sum by
\[ A = \frac{1}{c} \sum_{x \equiv (c)} e(2x/c) \sum_{a \equiv (c)} e(a(x^2 - D)/c). \]
Interchanging the two sums, we are led to
\[ A = \frac{1}{c} \sum_{a \equiv (c)} G(a, 2; c) e(-aD/c) = \sum_{d | c} A_d \]
where
\[ A_d = \frac{1}{c} \sum_{a \equiv (c) \pmod{(a,c)=d}} G(a, 2; c) e(-aD/c). \]

It is well known that \( G(a, b; c) = dG(a/d, b/d; c/d) \) if \( d = (a, c)|b \) and is zero otherwise. This shows that \( A_d = 0 \) for all \( d > 1 \), when \( c \) is odd or \( 8 \nmid c \), because we even have \( G(a', 1; c') = 0 \) if \( 4 | c' \).

Now, when \( (a, c) = 1 \),
\[ G(a, 2; c) = e(-\frac{a}{c}) G(a, 0; c), \]
and so
\[ A = A_1 = \frac{G(1, 0; c)}{c} \sum_{a \equiv (c) \pmod{(a,c)=1}} \frac{G(a, 0; c)}{G(1, 0; c)} e(-\frac{a}{c}) e(-aD/c). \]

This proves (1) and (2), since
\[
G(a, 0; c)/G(1, 0; c) = \begin{cases} \left( \frac{a}{c} \right) & \text{if } c \text{ is odd,} \\ \left( \frac{c}{a} \right) \epsilon_a^{-1} & \text{if } 4 | c. \end{cases}
\]

Since the explicit form of the \( \theta \)-multiplier arises from \( G(a, 0; c)/G(1, 0; c) \), see e.g. [3], identity (3) can be eliminated from the argument. In this respect, note that in [1], using the Davenport-Hasse relation, Duke evaluated generalized Salié sums that bear the same relation to metaplectic forms on \( GL_n \) as \( K(m, n; c) \) to half-integral weight forms. Although the method presented here does not seem to be applicable in a direct fashion, this fact still suggests that Duke’s theorem can be generalized to non-prime arguments, without any recourse to the explicit evaluation of generalized Gauss sums.

References


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