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ON THE EVALUATION OF SALIÉ SUMS

ÁRPÁD TÓTH

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Abstract. The Salié sum $S(m,n;c)$ can be evaluated as the product of a Gauss sum and an exponential sum involving square roots of $mn \bmod c$. We give a new proof of this fact that can simultaneously handle a twisted version of these sums that arise in the theory of half-integral weight modular forms.

The exponential sum

$$K(m,n;c)=\sum_{a\overline{a}\equiv 1(c)} \varepsilon_a \left(\frac{c}{a}\right) \mathrm{e}((ma+n\overline{a})/c)$$

arises in the theory of modular forms of half-integral weight. $K(m,n;c)$ is only defined when $4|c$, and then $\varepsilon_a = 1$ or i according to whether $a\equiv 1$ or $3 \bmod 4$, $\left(-\right)$ is the extension of the Legendre-Jacobi symbol as in [5], and $\mathrm{e}(z)=\exp(2\pi i z)$. In Iwaniec's celebrated estimates for the Fourier coefficients of said forms (see [2]), an essential role is played by the identity

$$K(D,1;c)=\frac{G(1,0;c)}{2}\sum_{x^2\equiv D\ (c)} \mathrm{e}(2x/c) \tag{1}$$

valid when $8|c$ and the analogous formula

$$S(D,1;c)=\sum_{a\overline{a}\equiv 1\ (c)} \left(\frac{a}{c}\right) \mathrm{e}((Da+\overline{a})/c) = G(1,0;c)\sum_{x^2\equiv D(c)} \mathrm{e}(2x/c), \tag{2}$$

which is valid whenever c is odd. In the formula above $G(a,b;c)$ is the Gauss sum

$$G(a,b;c)=\sum_{x(c)} \mathrm{e}((ax^2+bx)/c).$$

When the modulus is prime this identity was first proved by Salié [3]; see also [6]. Iwaniec derived a formula for the general modulus by pasting the local results together using quadratic reciprocity, while Sarnak gave a "global" proof of (2) that works when $(2D,c)=1$; see [4].

The purpose of this note is to give a simple argument that leads to (1) when c is even and to (2) when c is odd, without any assumption on D. It is based on a very simple idea: we start by the sum

$$A=\sum_{x^2\equiv D(c)} \mathrm{e}(2x/c)$$

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and sieve out the support of the sum by

$$A = \frac{1}{c}\sum_{x(c)} \mathrm{e}(2x/c) \sum_{a(c)} \mathrm{e}(a(x^2 - D)/c).$$

Interchanging the two sums, we are led to

$$A = \frac{1}{c}\sum_{a(c)} G(a, 2; c)\mathrm{e}(-aD/c) = \sum_{d|c} A_d$$

where

$$A_d = \frac{1}{c} \sum_{\substack{a(c) \\ (a,c)=d}} G(a, 2; c)\mathrm{e}(-aD/c).$$

It is well known that $G(a, b; c) = dG(a/d, b/d; c/d)$ if $d = (a, c)|b$ and is zero otherwise. This shows that $A_d = 0$ for all $d > 1$, when c is odd or $8|c$, because we even have $G(a', 1; c') = 0$ if $4|c'$.

Now, when $(a, c) = 1$,

$$G(a, 2; c) = \mathrm{e}(-\overline{a}/c)G(a, 0; c),$$

and so

$$A = A_1 = \frac{G(1, 0; c)}{c} \sum_{\substack{a(c) \\ (a,c)=1}} \frac{G(a, 0; c)}{G(1, 0; c)}\mathrm{e}(-\overline{a}/c)\mathrm{e}(-aD/c).$$

This proves (1) and (2), since

$$G(a, 0; c)/G(1, 0; c) = \begin{cases} \left(\dfrac{a}{c}\right) & \text{if } c \text{ is odd,} \\ \\ \left(\dfrac{c}{a}\right) \varepsilon_a^{-1} & \text{if } 4|c. \end{cases} \tag{3}$$

Since the explicit form of the θ-multiplier arises from $G(a, 0; c)/G(1, 0; c)$, see e.g. [4], identity (3) can be eliminated from the argument. In this respect, note that in [1], using the Davenport-Hasse relation, Duke evaluated generalized Salié sums that bear the same relation to metaplectic forms on GL_n as $K(m, n; c)$ to half-integral weight forms. Although the method presented here does not seem to be applicable in a direct fashion, this fact still suggests that Duke's theorem can be generalized to non-prime arguments, without any recourse to the explicit evaluation of generalized Gauss sums.

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Departament of Analysis, Eötvös Lórand University, Pázmány Péter Sétány 1/c, H-1117 Budapest, Hungary

E-mail address: `toth@cs.elte.hu`