

δ -FUNCTION OF AN OPERATOR: A WHITE NOISE APPROACH

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ABSTRACT. Let $(E) \subset (L^2) \subset (E)^*$ be the canonical framework of white noise analysis over the Gel'fand triple $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$ and $\mathcal{L} \equiv \mathcal{L}[(E), (E)^*]$ be the space of continuous linear operators from (E) to $(E)^*$. Let Q be a self-adjoint operator in (L^2) with spectral representation $Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda)$. In this paper, it is proved that under appropriate conditions upon Q , there exists a unique linear mapping $Z : S^*(\mathbb{R}) \rightarrow \mathcal{L}$ such that $Z(f) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$ for each $f \in S(\mathbb{R})$. The mapping is then naturally used to define $\delta(Q)$ as $Z(\delta)$, where δ is the Dirac δ -function. Finally, properties of the mapping Z are investigated and several results are obtained.

1. INTRODUCTION

Let δ be the Dirac δ -function, which is a Schwartz generalized function, and Q an observable, i.e., a self-adjoint operator in a Hilbert space. Then $\delta(Q)$, called the δ -function of Q , is of physical significance (cf. [1]). However, from the mathematical point of view, it is a very singular object. What is the mathematical meaning of $\delta(Q)$ which is both reasonable and rigorous? In [1], the authors gave an interpretation in the context of Hilbert space theory.

On the other hand, white noise analysis initiated by Hida [2], which is essentially an infinite-dimensional analogue of Schwartz generalized function theory, has been considerably developed and successfully applied to many fields including stochastic analysis and quantum physics (see, e.g., [2, 3, 4, 5], [7, 8], [12] and references cited therein). The mathematical framework of the theory is the Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*$$

over $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})$, where (E) (resp. $(E)^*$) is known as Hida testing (resp. generalized) functional space. Let $\mathcal{L} \equiv \mathcal{L}[(E), (E)^*]$ be the space of continuous linear operators from (E) to $(E)^*$. Elements of \mathcal{L} are usually called generalized operators, which are significant generalizations of bounded operators on the Hilbert space (L^2) .

The main purpose of the present paper is to define $\delta(Q)$ reasonably and rigorously in the context of white noise analysis. The paper is organized as follows. In Section 2

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we recall some necessary notions, notation and facts in white noise analysis. In Section 3, we first prove that for a self-adjoint operator Q in (L^2) with spectral representation $Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda)$, under appropriate conditions upon Q there exists a unique linear mapping $Z : S^*(\mathbb{R}) \mapsto \mathcal{L}$ such that $Z(f) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$ for each $f \in S(\mathbb{R})$. We then naturally use the mapping Z to define $\delta(Q)$ as $Z(\delta)$. Finally, we show that the mapping $Z : S^*(\mathbb{R}) \mapsto \mathcal{L}$ is continuous and positivity-preserving.

2. FRAMEWORK OF WHITE NOISE ANALYSIS

In this section we briefly recall some notions, notation and facts in white noise analysis. For details see [2], [5], [7] and [9].

We first fix some general notation. Throughout the paper, \mathbb{R} and \mathbb{C} stand for the real line and complex plane, respectively. For any real locally convex space V , we denote by $V_{\mathbb{C}}$ the complexification of V . Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $V^* \times V$; then the canonical bilinear forms on $V_{\mathbb{C}}^* \times V_{\mathbb{C}}$ and $(V_{\mathbb{C}}^{\otimes n})^* \times V_{\mathbb{C}}^{\otimes n}$ are still denoted by $\langle \cdot, \cdot \rangle$. Similarly, if V is a real Hilbert space with norm $|\cdot|$, then the norms of $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^{\otimes n}$ are also denoted by the same symbol $|\cdot|$.

Now let $H \equiv L^2(\mathbb{R}, dt; \mathbb{R})$ be the Hilbert space of real-valued square integrable functions on \mathbb{R} with norm $|\cdot|_0$ and inner product $\langle \cdot, \cdot \rangle$. Let $A = 1 + t^2 - d^2/dt^2$ be the harmonic oscillator. Then A has a self-adjoint extension in H , which is still denoted by A .

For each integer p , let E_p be the completion of $\text{Dom } A^p$ with respect to the Hilbertian norm $|\cdot|_p = |A^p \cdot|_0$. Then E_p and E_{-p} can be regarded as each other's dual if we identify H with its dual. Let E be the projective limit of $\{E_p \mid p \geq 0\}$ and E^* the topological dual of E . Then E is a nuclear space and E^* is the inductive limit of $\{E_{-p} \mid p \geq 0\}$. Hence we have a real Gel'fand triple $E \subset H \subset E^*$. It is known (cf. [2]) that E and E^* coincide with Schwartz rapidly decreasing function space $S(\mathbb{R})$ and generalized function space $S^*(\mathbb{R})$, respectively. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $E^* \times E$, which is consistent with the inner product of H .

Let μ be the standard Gaussian measure on E^* , i.e., its characteristic function is

$$(2.1) \quad \int_{E^*} e^{i\langle x, f \rangle} \mu(dx) = e^{-\frac{1}{2}|f|_0^2}, \quad f \in E.$$

The measure space (E^*, μ) is known as *white noise*. Let $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ be the Hilbert space of complex-valued μ -square integrable functionals on E^* with the inner product $(\langle \cdot, \cdot \rangle)$ and norm $\|\cdot\|_0$. Then, by the well-known Wiener-Itô-Segal isomorphism theorem, for each $\varphi \in (L^2)$ there exists a unique sequence $(f_n)_{n=0}^{\infty}$ with $f_n \in H_{\mathbb{C}}^{\otimes n}$ such that $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ in norm $\|\cdot\|_0$ and

$$(2.2) \quad \|\varphi\|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2$$

where $I_n(f_n)$ denotes the multiple Wiener integral of order n with kernel f_n .

Note that the harmonic oscillator A also has a self-adjoint extension in $H_{\mathbb{C}}$, which is still denoted by A . Let $\Gamma(A)$ be the second quantization operator of A defined by

$$(2.3) \quad \Gamma(A)\varphi = \sum_{n=0}^{\infty} I_n(A^{\otimes n} f_n)$$

where $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$. Then $\Gamma(A)$ is a positive self-adjoint operator with Hilbert-Schmidt inverse in (L^2) .

Similarly, for each integer p , let (E_p) be the completion of $\text{Dom } \Gamma(A)^p$ with respect to the Hilbertian norm $\|\cdot\|_p = \|\Gamma(A)^p \cdot\|_0$. Then (E_p) becomes a complex Hilbert space. In particular, $(E_0) = (L^2)$. Let (E) be the projective limit of $\{(E_p) \mid p \geq 0\}$ and $(E)^*$ the inductive limit of $\{(E_{-p}) \mid p \geq 0\}$. Then (E) and $(E)^*$ can be regarded as each other's dual. Moreover, (E) is a nuclear space and we come to a complex Gel'fand triple

$$(E) \subset (L^2) \subset (E)^*,$$

which is known as the *canonical framework of white noise analysis*. Elements of (E) (resp. $(E)^*$) are called *Hida testing* (resp. *generalized*) *functionals*. In the following, we denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$.

For $\xi \in E_{\mathbb{C}}$, the exponential functional ϕ_{ξ} associated with ξ is defined as

$$(2.4) \quad \phi_{\xi}(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle / 2} = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \frac{1}{n!} \xi^{\otimes n} \right\rangle, \quad x \in E^*.$$

It is known that the set $\{\phi_{\xi} \mid \xi \in E_{\mathbb{C}}\}$ is total in the Hilbert space (E_p) for each integer p . Hence $\text{Span}\{\phi_{\xi} \mid \xi \in E_{\mathbb{C}}\}$ is a dense subspace of (E) .

Continuous linear operators from (E) to $(E)^*$ are usually called *generalized operators*. The space of all generalized operators is denoted by $\mathcal{L} \equiv \mathcal{L}[(E), (E)^*]$. For $X \in \mathcal{L}$, its *symbol* \widehat{X} is defined as

$$(2.5) \quad \widehat{X}(\xi, \eta) = \langle\langle X\phi_{\xi}, \phi_{\eta} \rangle\rangle, \quad \xi, \eta \in E_{\mathbb{C}}.$$

The next lemma (cf. [8] and [9]) will be used later.

Lemma 2.1. *Let $\{X_n\}_{n \geq 1} \subset \mathcal{L}$ be such that*

- (1) $\forall \xi, \eta \in E_{\mathbb{C}}$, *the sequence $\{\widehat{X}_n(\xi, \eta)\}_{n \geq 1}$ is convergent in \mathbb{C} ,*
- (2) *there exist constants $a, k, p \geq 0$ such that*

$$(2.6) \quad \sup_{n \geq 1} |\widehat{X}_n(\xi, \eta)| \leq a \exp\{k(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi, \eta \in E_{\mathbb{C}}.$$

Then there exists a unique $X \in \mathcal{L}$ such that $X_n \rightarrow X$ in \mathcal{L} .

3. δ-FUNCTION OF AN OPERATOR

We first make some necessary assumptions. Let $\mathcal{B}(\mathbb{R})$ be the Borel σ -field of the real line \mathbb{R} and $\mathcal{P}[(L^2)]$ the set of projections in (L^2) .

Let Q be a given self-adjoint operator in (L^2) with spectral representation

$$(3.1) \quad Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda),$$

where $P_Q : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{P}[(L^2)]$ is the spectral measure of Q (cf. [10]). It is well known that for a Borel measurable function f on \mathbb{R} , $f(Q) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$ makes sense as a densely defined operator in (L^2) . Moreover, $f(Q)$ is a bounded operator in (L^2) if f is a bounded Borel measurable function (see [10] for details).

For each $\xi, \eta \in E_{\mathbb{C}}$, define $\nu_{\xi, \eta}^Q : \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{C}$ as

$$(3.2) \quad \nu_{\xi, \eta}^Q(S) = \langle\langle P_Q(S)\phi_{\xi}, \phi_{\eta} \rangle\rangle, \quad S \in \mathcal{B}(\mathbb{R}).$$

Obviously $\nu_{\xi,\eta}^Q$ is a complex-valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Throughout the section, we make the following hypothesis.

Hypothesis. For each $\xi, \eta \in E_{\mathbb{C}}$, there exists a function $\rho_{\xi,\eta}^Q \in E_{\mathbb{C}}$ such that

$$(3.3) \quad \nu_{\xi,\eta}^Q(S) = \int_S \rho_{\xi,\eta}^Q(\lambda) d\lambda, \quad S \in \mathcal{B}(\mathbb{R}).$$

We call the function $\rho_{\xi,\eta}^Q$ the spectral density of the operator Q associated with ξ, η .

Proposition 3.1. *The spectral density $\rho_{\xi,\eta}^Q$ is positive definite in the sense that for each $n \geq 1$ and any $z_i \in \mathbb{C}, \xi_i \in E_{\mathbb{C}}, i = 1, 2, \dots, n$,*

$$(3.4) \quad \sum_{i,j=1}^n z_i \bar{z}_j \rho_{\xi_i, \bar{\xi}_j}^Q \geq 0 \quad \text{as a function on } \mathbb{R}.$$

Proof. Let $\varphi = \sum_{i=1}^n z_i \phi_{\xi_i}$. Then for each $S \in \mathcal{B}(\mathbb{R})$, we have

$$\begin{aligned} \int_S \sum_{i,j=1}^n z_i \bar{z}_j \rho_{\xi_i, \bar{\xi}_j}^Q(\lambda) d\lambda &= \sum_{i,j=1}^n z_i \bar{z}_j \nu_{\xi_i, \bar{\xi}_j}^Q(S) \\ &= \sum_{i,j=1}^n z_i \bar{z}_j \langle P_Q(S) \phi_{\xi_i}, \phi_{\bar{\xi}_j} \rangle \\ &= \langle P_Q(S) \varphi, \bar{\varphi} \rangle \\ &= \|P_Q(S) \varphi\|_0^2 \\ &\geq 0 \end{aligned}$$

where $\|\cdot\|_0$ denotes the norm of (L^2) . Hence $\sum_{i,j=1}^n z_i \bar{z}_j \rho_{\xi_i, \bar{\xi}_j}^Q \geq 0$ as a function on \mathbb{R} . □

Proposition 3.2. *Let $\text{Dom } Q^n$ be the domain of Q^n , where $n \geq 0$. Then $\{\phi_{\xi} \mid \xi \in E_{\mathbb{C}}\} \subset \text{Dom } Q^n$.*

Proof. Let $\xi \in E_{\mathbb{C}}$. By Proposition 3.1 and the Hypothesis, we have

$$0 \leq \int_{\mathbb{R}} \lambda^{2n} \rho_{\xi, \bar{\xi}}^Q(\lambda) d\lambda < +\infty.$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} |\lambda^n|^2 (P_Q(d\lambda) \phi_{\xi}, \phi_{\xi}) &= \int_{\mathbb{R}} \lambda^{2n} \langle P_Q(d\lambda) \phi_{\xi}, \bar{\phi}_{\xi} \rangle \\ &= \int_{\mathbb{R}} \lambda^{2n} \nu_{\xi, \bar{\xi}}^Q(d\lambda) \\ &= \int_{\mathbb{R}} \lambda^{2n} \rho_{\xi, \bar{\xi}}^Q(\lambda) d\lambda \\ &< +\infty, \end{aligned}$$

which implies that $\phi_{\xi} \in \text{Dom } Q^n$. □

The above propositions show useful properties of the operator Q . Now we use them to define Schwartz generalized functions of Q .

Theorem 3.3. Assume that the spectral density $\rho_{\xi,\eta}^Q$ satisfies that for each $q \geq 0$ there exist constants $a, k, p \geq 0$ such that

$$(3.5) \quad |\rho_{\xi,\eta}^Q|_q \leq k \exp\{a(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi, \eta \in E_{\mathbb{C}}.$$

Then for each Schwartz generalized function $\omega \in E^* = S^*(\mathbb{R})$, there exists a unique generalized operator $X_{\omega}^Q \in \mathcal{L}$ such that

$$(3.6) \quad \widehat{X_{\omega}^Q}(\xi, \eta) = \langle \omega, \rho_{\xi,\eta}^Q \rangle, \quad \xi, \eta \in E_{\mathbb{C}}.$$

Proof. Obviously (3.6) implies the uniqueness of X_{ω}^Q . Now we prove the existence. Let $\omega \in E^*$. Then there is $q \geq 0$ such that $\omega \in E_{-q}$. Since E is dense in E_{-q} , we can take a sequence $\{f_n\}_{n \geq 1} \subset E$ such that $f_n \rightarrow \omega$ in the norm $|\cdot|_{-q}$. For each $n \geq 1$, $f_n(Q) = \int_{\mathbb{R}} f_n(\lambda) P_Q(d\lambda)$ is a bounded linear operator on (L^2) since f_n is a bounded function. Hence $f_n(Q) \in \mathcal{L}$ for all $n \geq 1$.

We assert that the sequence $\{f_n(Q)\}$ satisfies the two conditions of Lemma 2.1. In fact, for each $\xi, \eta \in E_{\mathbb{C}}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{f_n(Q)}(\xi, \eta) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(\lambda) \nu_{\xi,\eta}^Q(d\lambda) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n(\lambda) \rho_{\xi,\eta}^Q(\lambda) d\lambda \\ &= \lim_{n \rightarrow \infty} \langle f_n, \rho_{\xi,\eta}^Q \rangle \\ &= \langle \omega, \rho_{\xi,\eta}^Q \rangle. \end{aligned}$$

On the other hand, by the assumption we have

$$\begin{aligned} |\widehat{f_n(Q)}(\xi, \eta)| &= |\langle f_n, \rho_{\xi,\eta}^Q \rangle| \\ &\leq |f_n|_q |\rho_{\xi,\eta}^Q|_q \\ &\leq \alpha k \exp\{a(|\xi|_p^2 + |\eta|_p^2)\} \end{aligned}$$

$\forall \xi, \eta \in E_{\mathbb{C}}$, where $\alpha = \sup_{n \geq 1} |f_n|_{-q} < \infty$ since $\{f_n\}_{n \geq 1}$ is convergent in the norm $|\cdot|_{-q}$.

By Lemma 2.1, there exists a generalized operator, denoted by X_{ω}^Q , such that $f_n(Q) \rightarrow X_{\omega}^Q$ in \mathcal{L} , which implies

$$\lim_{n \rightarrow \infty} \widehat{f_n(Q)}(\xi, \eta) = \widehat{X_{\omega}^Q}(\xi, \eta), \quad \forall \xi, \eta \in E_{\mathbb{C}},$$

which implies (3.6). □

Proposition 3.4. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3 and $f \in E = S(\mathbb{R})$. Then

$$(3.7) \quad X_f^Q = f(Q) \quad (\text{as generalized operators})$$

where $f(Q) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$, which is well known as the f -function of Q .

Proof. $f(Q)$ is a bounded operator on (L^2) , which means $f(Q) \in \mathcal{L}$. For each $\xi, \eta \in E_{\mathbb{C}}$, by a straightforward computation, we find that

$$\widehat{f(Q)}(\xi, \eta) = \widehat{X_f^Q}(\xi, \eta),$$

which implies (3.7). □

Motivated by Proposition 3.4, we now give the definition of Schwartz generalized functions of the operator Q as follows.

Definition 3.1. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3. For a Schwartz generalized function $\omega \in E^*$, we define

$$(3.8) \quad \omega(Q) = X_\omega^Q$$

and call it the ω -function of Q .

Remark 3.1. Let δ be the Dirac δ -function. Then $\delta \in E^*$. Hence, under the above conditions upon Q and $\rho_{\xi,\eta}^Q$, $\delta(Q)$ makes sense as a generalized operator.

In the following, we investigate properties of the Schwartz generalized functions of Q defined above.

Theorem 3.5. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3 and $n \geq 0$. Let $\omega_n \in E^*$ be defined by

$$(3.9) \quad \langle \omega_n, f \rangle = \int_{\mathbb{R}} \lambda^n f(\lambda) d\lambda, \quad f \in E.$$

Then

$$(3.10) \quad \omega_n(Q)\varphi = Q^n\varphi, \quad \varphi \in \mathcal{D}$$

where $\mathcal{D} \equiv \text{Span}\{\phi_\xi \mid \xi \in E_{\mathbb{C}}\}$ is the linear subspace of (E) spanned by $\{\phi_\xi \mid \xi \in E_{\mathbb{C}}\}$.

Proof. Firstly, by Proposition 3.2, we see that $\mathcal{D} \subset \text{Dom } Q^n$. On the other hand, for each $\xi, \eta \in E_{\mathbb{C}}$, we have

$$\begin{aligned} \langle Q^n \phi_\xi, \phi_\eta \rangle &= \left(\left(\int_{\mathbb{R}} \lambda^n P_Q(d\lambda) \phi_\xi, \overline{\phi_\eta} \right) \right) \\ &= \int_{\mathbb{R}} \lambda^n \langle P_Q(d\lambda) \phi_\xi, \phi_{\bar{\eta}} \rangle \\ &= \int_{\mathbb{R}} \lambda^n \langle P_Q(d\lambda) \phi_\xi, \phi_\eta \rangle \\ &= \int_{\mathbb{R}} \lambda^n \rho_{\xi,\eta}^Q(\lambda) d\lambda \\ &= \langle \omega_n, \rho_{\xi,\eta}^Q \rangle \\ &= \langle \omega_n(Q) \phi_\xi, \phi_\eta \rangle, \end{aligned}$$

where (\cdot, \cdot) means the inner product of (L^2) . Hence (3.10) follows. \square

Remark 3.2. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3. Then from Theorem 3.5 we see that

$$(3.11) \quad Q\varphi = \omega_1(Q)\varphi, \quad \varphi \in \mathcal{D}.$$

Note that \mathcal{D} is not only a dense subspace of (E) but also a dense subspace of (L^2) . Hence Q itself can be viewed as a generalized operator.

Theorem 3.6. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3. Define a mapping $Z : E^* \mapsto \mathcal{L}$ as follows:

$$(3.12) \quad Z(\omega) = \omega(Q), \quad \omega \in E^*.$$

Then $Z : E^* \mapsto \mathcal{L}$ is a continuous linear mapping.

Proof. Z is obviously linear. We now prove its continuity. Let $\{\omega^{(k)}\}_{k \geq 1} \subset E^*$ and $\omega \in E^*$ be such that $\omega^{(k)} \rightarrow \omega$ in E^* . Then there exists some $q \geq 0$ such that $\omega, \omega^{(k)} \in E_{-q}, k \geq 1$ and

$$\omega^{(k)} \rightarrow \omega \quad (\text{in the norm } |\cdot|_{-q}).$$

With an argument similar to that in the proof of Theorem 3.3, we can get a generalized operator X such that

$$Z(\omega^{(k)}) = \omega^{(k)}(Q) \rightarrow X \quad (\text{in } \mathcal{L}).$$

On the other hand, we have

$$\begin{aligned} \widehat{X}(\xi, \eta) &= \lim_{k \rightarrow \infty} \widehat{Z(\omega^{(k)})}(\xi, \eta) \\ &= \lim_{k \rightarrow \infty} \langle \omega^{(k)}, \rho_{\xi, \eta}^Q \rangle \\ &= \langle \omega, \rho_{\xi, \eta}^Q \rangle \\ &= \widehat{\omega(Q)}(\xi, \eta) \\ &= \widehat{Z(\omega)}(\xi, \eta), \end{aligned}$$

$\forall \xi, \eta \in E_{\mathbb{C}}$, which implies $X = Z(\omega)$. Hence $Z(\omega^{(k)}) \rightarrow Z(\omega)$ in \mathcal{L} . □

Theorem 3.7. Let $\rho_{\xi, \eta}^Q$ be as in Theorem 3.3 and $Z : E^* \mapsto \mathcal{L}$ as in Theorem 3.6. Then Z is positivity-preserving in the sense that

$$(3.13) \quad \langle\langle Z(\omega)\varphi, \overline{\varphi} \rangle\rangle \geq 0, \quad \varphi \in (E)$$

whenever $\omega \in E^*$ and $\omega \geq 0$.

Proof. Let $\omega \in E^*$ with $\omega \geq 0$. To prove (3.13), we only need to show that for each $n \geq 1$ and any $z_i \in \mathbb{C}, \xi_i \in E_{\mathbb{C}}, i = 1, 2, \dots, n$,

$$\left\langle\left\langle Z(\omega) \sum_{i=1}^n z_i \phi_{\xi_i}, \overline{\sum_{i=1}^n z_i \phi_{\xi_i}} \right\rangle\right\rangle \geq 0.$$

In fact, we have

$$\begin{aligned} \left\langle\left\langle Z(\omega) \sum_{i=1}^n z_i \phi_{\xi_i}, \overline{\sum_{i=1}^n z_i \phi_{\xi_i}} \right\rangle\right\rangle &= \sum_{i,j=1}^n z_i \overline{z_j} \langle\langle Z(\omega) \phi_{\xi_i}, \phi_{\xi_j} \rangle\rangle \\ &= \sum_{i,j=1}^n z_i \overline{z_j} \langle \omega, \rho_{\xi_i, \xi_j}^Q \rangle \\ &= \left\langle \omega, \sum_{i,j=1}^n z_i \overline{z_j} \rho_{\xi_i, \xi_j}^Q \right\rangle \\ &\geq 0, \end{aligned}$$

where, by Proposition 3.1, $\sum_{i,j=1}^n z_i \overline{z_j} \rho_{\xi_i, \xi_j}^Q \geq 0$ as a function on \mathbb{R} . □

By Theorem 3.7, we immediately come to the following proposition.

Proposition 3.8. $\delta(Q)$ is positive, i.e., $\langle\langle \delta(Q)\varphi, \overline{\varphi} \rangle\rangle \geq 0, \forall \varphi \in (E)$.

Remark 3.3. The physical meaning of the fact that $\delta(Q)$ is positive can be interpreted as follows. From the physical point of view, the self-adjoint operator Q stands for an observable. Naturally, as a generalized operator, $\delta(Q)$ can be viewed as an observable associated with the observable Q . Hence the positivity property of $\delta(Q)$ implies that it is a positive observable.

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