δ-FUNCTION OF AN OPERATOR: A WHITE NOISE APPROACH

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ABSTRACT. Let \((E) \subset (L^2) \subset (E)^*\) be the canonical framework of white noise analysis over the Gel’fand triple \(S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S^*(\mathbb{R})\) and \(\mathcal{L} \equiv \mathcal{L}((E),(E)^*)\) be the space of continuous linear operators from \((E)\) to \((E)^*\). Let \(Q\) be a self-adjoint operator in \((L^2)\) with spectral representation \(Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda)\). In this paper, it is proved that under appropriate conditions upon \(Q\), there exists a unique linear mapping \(Z : S^*(\mathbb{R}) \mapsto \mathcal{L}\) such that \(Z(f) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)\) for each \(f \in S(\mathbb{R})\). The mapping is then naturally used to define \(\delta(Q)\) as \(Z(\delta)\), where \(\delta\) is the Dirac δ-function. Finally, properties of the mapping \(Z\) are investigated and several results are obtained.

1. Introduction

Let \(\delta\) be the Dirac δ-function, which is a Schwartz generalized function, and \(Q\) an observable, i.e., a self-adjoint operator in a Hilbert space. Then \(\delta(Q)\), called the δ-function of \(Q\), is of physical significance (cf. [1]). However, from the mathematical point of view, it is a very singular object. What is the mathematical meaning of \(\delta(Q)\) which is both reasonable and rigorous? In [1], the authors gave an interpretation in the context of Hilbert space theory.

On the other hand, white noise analysis initiated by Hida [2], which is essentially an infinite-dimensional analogue of Schwartz generalized function theory, has been considerably developed and successfully applied to many fields including stochastic analysis and quantum physics (see, e.g., [2, 3, 4, 5], [7, 8], [12] and references cited therein). The mathematical framework of the theory is the Gel’fand triple \((E) \subset (L^2) \subset (E)^*\)

\[\delta(Q)\] is defined as \(Z(\delta)\), where \(\delta\) is the Dirac δ-function. Finally, properties of the mapping \(Z\) are investigated and several results are obtained.

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we recall some necessary notions, notation and facts in white noise analysis. In Section 3, we first prove that for a self-adjoint operator $Q$ in $(L^2)$ with spectral representation $Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda)$, under appropriate conditions upon $Q$ there exists a unique linear mapping $Z : S^*(\mathbb{R}) \rightarrow \mathcal{L}$ such that $Z(f) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$ for each $f \in S(\mathbb{R})$. We then naturally use the mapping $Z$ to define $\delta(Q)$ as $Z(\delta)$. Finally, we show that the mapping $Z : S^*(\mathbb{R}) \rightarrow \mathcal{L}$ is continuous and positivity-preserving.

2. Framework of white noise analysis

In this section we briefly recall some notions, notation and facts in white noise analysis. For details see [2], [5], [7] and [9].

We first fix some general notation. Throughout the paper, $\mathbb{R}$ and $\mathbb{C}$ stand for the real line and complex plane, respectively. For any real locally convex space $V$, we denote by $V_\mathbb{C}$ the complexification of $V$. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $V^* \times V$; then the canonical bilinear forms on $V_\mathbb{C}^* \times V_\mathbb{C}$ and $(V_\mathbb{C}^*)^* \times V_\mathbb{C}^*$ are still denoted by $\langle \cdot, \cdot \rangle$. Similarly, if $V$ is a real Hilbert space with norm $|\cdot|$, then the norms of $V_\mathbb{C}$ and $V_\mathbb{C}^*$ are also denoted by the same symbol $|\cdot|$. Now let $H \equiv L^2(\mathbb{R}, dt; \mathbb{R})$ be the Hilbert space of real-valued square integrable functions on $\mathbb{R}$ with norm $|\cdot|_0$ and inner product $\langle \cdot, \cdot \rangle$. Let $A = 1 + t^2 - d^2/dt^2$ be the harmonic oscillator. Then $A$ has a self-adjoint extension in $H$, which is still denoted by $A$.

For each integer $p$, let $E_p$ be the completion of Dom $A^p$ with respect to the Hilbertian norm $|\cdot|_p = |A^p \cdot|_0$. Then $E_p$ and $E_{-p}$ can be regarded as each other’s dual if we identify $H$ with its dual. Let $E$ be the projective limit of $\{E_p | p \geq 0\}$ and $E^*$ the topological dual of $E$. Then $E$ is a nuclear space and $E^*$ is the inductive limit of $\{E_{-p} | p \geq 0\}$. Hence we have a real Gel’fand triple $E \subset H \subset E^*$. It is known (cf. [2]) that $E$ and $E^*$ coincide with Schwartz rapidly decreasing function space $S(\mathbb{R})$ and generalized function space $S^*(\mathbb{R})$, respectively. We denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear form on $E^* \times E$, which is consistent with the inner product of $H$.

Let $\mu$ be the standard Gaussian measure on $E^*$, i.e., its characteristic function is

$$\int_{E^*} e^{i\langle x, f \rangle} \mu(dx) = e^{-\frac{1}{2}|f|_0^2}, \quad f \in E.$$ (2.1)

The measure space $(E^*, \mu)$ is known as white noise. Let $(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$ be the Hilbert space of complex-valued $\mu$-square integrable functionals on $E^*$ with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|_0$. Then, by the well-known Wiener-Itô-Segal isomorphism theorem, for each $\varphi \in (L^2)$ there exists a unique sequence $(f_n)_{n=0}^\infty$ with $f_n \in H_\mathbb{C}^\otimes n$ such that $\varphi = \sum_{n=0}^\infty I_n(f_n)$ in norm $\|\cdot\|_0$ and

$$\|\varphi\|_0^2 = \sum_{n=0}^\infty n! |f_n|^2$$ (2.2)

where $I_n(f_n)$ denotes the multiple Wiener integral of order $n$ with kernel $f_n$.

Note that the harmonic oscillator $A$ also has a self-adjoint extension in $H_\mathbb{C}$, which is still denoted by $A$. Let $\Gamma(A)$ be the second quantization operator of $A$ defined by

$$\Gamma(A)\varphi = \sum_{n=0}^\infty I_n(A^\otimes n f_n)$$ (2.3)
where \( \varphi = \sum_{n=0}^{\infty} I_n(f_n) \). Then \( \Gamma(A) \) is a positive self-adjoint operator with Hilbert-Schmidt inverse in \( (L^2) \).

Similarly, for each integer \( p \), let \( (E_p) \) be the completion of \( \text{Dom} \Gamma(A)^p \) with respect to the Hilbertian norm \( \| \cdot \|_p = \| \Gamma(A)^p \cdot \|_0 \). Then \( (E_p) \) becomes a complex Hilbert space. In particular, \( (E_0) = (L^2) \). Let \( (E) \) be the projective limit of \( \{ (E_p) \mid p \geq 0 \} \) and \( (E)^* \) the inductive limit of \( \{ (E_{-p}) \mid p \geq 0 \} \). Then \( (E) \) and \( (E)^* \) can be regarded as each other’s dual. Moreover, \( (E) \) is a nuclear space and we come to a complex Gelfand triple

\[
(E) \subset (L^2) \subset (E)^* ,
\]

which is known as the canonical framework of white noise analysis. Elements of \( (E) \) (resp. \( (E)^* \)) are called Hida testing (resp. generalized) functionals. In the following, we denote by \( \langle \cdot, \cdot \rangle \) the canonical bilinear form on \( (E)^* \times (E) \).

For \( \xi \in E_C \), the exponential functional \( \phi_\xi \) associated with \( \xi \) is defined as

\[
\phi_\xi(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle / 2} = \sum_{n=0}^{\infty} \langle x^\otimes n, \frac{1}{n!} \xi^\otimes n \rangle, \quad x \in (E)^*.
\]

It is known that the set \( \{ \phi_\xi \mid \xi \in E_C \} \) is total in the Hilbert space \( (E_p) \) for each integer \( p \). Hence \( \text{Span}\{ \phi_\xi \mid \xi \in E_C \} \) is a dense subspace of \( (E) \).

Continuous linear operators from \( (E) \) to \( (E)^* \) are usually called generalized operators. The space of all generalized operators is denoted by \( \mathcal{L} \equiv \mathcal{L}((E), (E)^*) \). For \( X \in \mathcal{L} \), its symbol \( \tilde{X} \) is defined as

\[
\tilde{X}(\xi, \eta) = \langle X \phi_\xi, \phi_\eta \rangle, \quad \xi, \eta \in E_C.
\]

The next lemma (cf. [8] and [9]) will be used later.

**Lemma 2.1.** Let \( \{ X_n \}_{n \geq 1} \subset \mathcal{L} \) be such that

1. \( \forall \xi, \eta \in E_C \), the sequence \( \{ \tilde{X}_n(\xi, \eta) \}_{n \geq 1} \) is convergent in \( \mathbb{C} \),
2. there exist constants \( a, k, p \geq 0 \) such that

\[
\sup_{n \geq 1} |\tilde{X}_n(\xi, \eta)| \leq a \exp\{k(|\xi|^2_p + |\eta|^2_p)\}, \quad \xi, \eta \in E_C.
\]

Then there exists a unique \( X \in \mathcal{L} \) such that \( X_n \longrightarrow X \) in \( \mathcal{L} \).

3. \( \delta \)-FUNCTION OF AN OPERATOR

We first make some necessary assumptions. Let \( \mathcal{B}(\mathbb{R}) \) be the Borel \( \sigma \)-field of the real line \( \mathbb{R} \) and \( \mathcal{P}([L^2]) \) the set of projections in \( (L^2) \).

Let \( Q \) be a given self-adjoint operator in \( (L^2) \) with spectral representation

\[
Q = \int_{\mathbb{R}} \lambda P_Q(d\lambda),
\]

where \( P_Q : \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{P}([L^2]) \) is the spectral measure of \( Q \) (cf. [10]). It is well known that for a Borel measurable function \( f \) on \( \mathbb{R} \), \( f(Q) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda) \) makes sense as a densely defined operator in \( (L^2) \). Moreover, \( f(Q) \) is a bounded operator in \( (L^2) \) if \( f \) is a bounded Borel measurable function (see [10] for details).

For each \( \xi, \eta \in E_C \), define \( \nu_{\xi, \eta}^Q : \mathcal{B}(\mathbb{R}) \longrightarrow \mathbb{C} \) as

\[
\nu_{\xi, \eta}^Q(S) = \langle P_Q(S) \phi_\xi, \phi_\eta \rangle, \quad S \in \mathcal{B}(\mathbb{R}).
\]
Obviously $\nu_{\xi, \eta}^Q$ is a complex-valued measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Throughout the section, we make the following hypothesis.

**Hypothesis.** For each $\xi, \eta \in E_C$, there exists a function $\rho_{\xi, \eta}^Q \in E_C$ such that

\[(3.3) \quad \nu_{\xi, \eta}^Q(S) = \int_S \rho_{\xi, \eta}^Q(\lambda) d\lambda, \quad S \in \mathcal{B}(\mathbb{R}).\]

We call the function $\rho_{\xi, \eta}^Q$ the spectral density of the operator $Q$ associated with $\xi, \eta$.

**Proposition 3.1.** The spectral density $\rho_{\xi, \eta}^Q$ is positive definite in the sense that for each $n \geq 1$ and any $z_i \in \mathbb{C}, \xi_i \in E_C, i = 1, 2, \cdots, n$,

\[(3.4) \quad \sum_{i,j=1}^n z_i \overline{z_j} \rho_{\xi_i, \xi_j}^Q \geq 0 \quad \text{as a function on } \mathbb{R}.\]

**Proof.** Let $\varphi = \sum_{i=1}^n z_i \phi_{\xi_i}$. Then for each $S \in \mathcal{B}(\mathbb{R})$, we have

\[
\int_S \sum_{i,j=1}^n z_i \overline{z_j} \rho_{\xi_i, \xi_j}^Q(\lambda) d\lambda = \sum_{i,j=1}^n z_i \overline{z_j} \nu_{\xi_i, \xi_j}^Q(S)
\]

\[
= \sum_{i,j=1}^n z_i \overline{z_j} \left\langle \left\langle P_Q(S) \phi_{\xi_i}, \phi_{\xi_j} \right\rangle \right\rangle
\]

\[
= \left\langle \left\langle P_Q(S) \varphi, \varphi \right\rangle \right\rangle \quad \text{as a function on } \mathbb{R}.
\]

where $\| \cdot \|_0$ denotes the norm of $(L^2)$. Hence $\sum_{i,j=1}^n z_i \overline{z_j} \rho_{\xi_i, \xi_j}^Q \geq 0$ as a function on $\mathbb{R}$. \(\square\)

**Proposition 3.2.** Let $\text{Dom} Q^n$ be the domain of $Q^n$, where $n \geq 0$. Then $\{ \phi_\xi \mid \xi \in E_C \} \subset \text{Dom} Q^n$.

**Proof.** Let $\xi \in E_C$. By Proposition 3.1 and the Hypothesis, we have

\[
0 \leq \int_{\mathbb{R}} \lambda^{2n} \rho_{\xi, \xi}^Q(\lambda) d\lambda < +\infty.
\]

Hence

\[
\int_{\mathbb{R}} |\lambda|^2 \left\langle \left\langle P_Q(d\lambda) \phi_\xi, \phi_\xi \right\rangle \right\rangle = \int_{\mathbb{R}} \lambda^{2n} \left\langle \left\langle P_Q(d\lambda) \phi_\xi, \phi_\xi \right\rangle \right\rangle
\]

\[
= \int_{\mathbb{R}} \lambda^{2n} \rho_{\xi, \xi}^Q(d\lambda)
\]

\[
< +\infty,
\]

which implies that $\phi_\xi \in \text{Dom} Q^n$. \(\square\)

The above propositions show useful properties of the operator $Q$. Now we use them to define Schwartz generalized functions of $Q$. 

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Theorem 3.3. Assume that the spectral density $\rho_{\xi,\eta}^Q$ satisfies that for each $q \geq 0$ there exist constants $a$, $k$, $p \geq 0$ such that
\[ |\rho_{\xi,\eta}^Q|_q \leq k \exp\{a(\xi^2_p + \eta^2_p)\}, \quad \xi, \eta \in E_C. \]
Then for each Schwartz generalized function $\omega \in E^* = S^*(\mathbb{R})$, there exists a unique generalized operator $X_Q^\omega \in \mathcal{L}$ such that
\[ \hat{X}_Q^\omega(\xi,\eta) = \langle \omega, \rho_{\xi,\eta}^Q \rangle, \quad \xi, \eta \in E_C. \]

Proof. Obviously (3.6) implies the uniqueness of $X_Q^\omega$. Now we prove the existence. Let $\omega \in E^*$. Then there is $q \geq 0$ such that $\omega \in E_{-q}$. Since $E$ is dense in $E_{-q}$, we can take a sequence $\{f_n\}_{n \geq 1} \subset E$ such that $f_n \to \omega$ in the norm $|\cdot|_{-q}$. For each $n \geq 1$, $f_n(Q) = \int_{\mathbb{R}} f_n(\lambda) P_Q(d\lambda)$ is a bounded linear operator on $(L^2)$ since $f_n$ is a bounded function. Hence $f_n(Q) \in \mathcal{L}$ for all $n \geq 1$.

We assert that the sequence $\{f_n(Q)\}$ satisfies the two conditions of Lemma 2.1. In fact, for each $\xi, \eta \in E_C$, we have
\[
\lim_{n \to \infty} f_n(Q)(\xi,\eta) = \lim_{n \to \infty} \int_{\mathbb{R}} f_n(\lambda) \nu_{\xi,\eta}^Q(d\lambda) = \lim_{n \to \infty} \int_{\mathbb{R}} f_n(\lambda) \rho_{\xi,\eta}^Q(\lambda) d\lambda = \lim_{n \to \infty} \langle f_n, \rho_{\xi,\eta}^Q \rangle = \langle \omega, \rho_{\xi,\eta}^Q \rangle.
\]

On the other hand, by the assumption we have
\[
|f_n(Q)(\xi,\eta)| = |\langle f_n, \rho_{\xi,\eta}^Q \rangle| \\
\leq |f_n|_q |\rho_{\xi,\eta}^Q|_q \\
\leq \alpha k \exp\{a(\xi^2_p + \eta^2_p)\}
\]
\[\forall \xi, \eta \in E_C, \text{ where } \alpha = \sup_{n \geq 1} |f_n|_{-q} < \infty \text{ since } \{f_n\}_{n \geq 1} \text{ is convergent in the norm } |\cdot|_{-q}.
\]

By Lemma 2.1 there exists a generalized operator, denoted by $X_Q^\omega$, such that
\[ f_n(Q) \to X_Q^\omega \text{ in } \mathcal{L}, \text{ which implies } \lim_{n \to \infty} f_n(Q)(\xi,\eta) = X_Q^\omega(\xi,\eta), \quad \forall \xi, \eta \in E_C,
\]
which implies (3.6).

Proposition 3.4. Let $\rho_{\xi,\eta}^Q$ be as in Theorem 3.3 and $f \in E = S(\mathbb{R})$. Then
\[ X_Q^f = f(Q) \quad \text{(as generalized operators)}\]
where $f(Q) = \int_{\mathbb{R}} f(\lambda) P_Q(d\lambda)$, which is well known as the $f$-function of $Q$.

Proof. $f(Q)$ is a bounded operator on $(L^2)$, which means $f(Q) \in \mathcal{L}$. For each $\xi, \eta \in E_C$, by a straightforward computation, we find that
\[ f(Q)(\xi,\eta) = \hat{X}_Q^f(\xi,\eta), \]
which implies (3.7).
Motivated by Proposition 3.4, we now give the definition of Schwartz generalized functions of the operator \( Q \) as follows.

**Definition 3.1.** Let \( \rho_{\xi,\eta}^Q \) be as in Theorem 3.3. For a Schwartz generalized function \( \omega \in E^* \), we define
\[
(3.8) \quad \omega(Q) = X^Q_{\omega}
\]
and call it the \( \omega \)-function of \( Q \).

**Remark 3.1.** Let \( \delta \) be the Dirac \( \delta \)-function. Then \( \delta \in E^* \). Hence, under the above conditions upon \( Q \) and \( \rho_{\xi,\eta}^Q \), \( \delta(Q) \) makes sense as a generalized operator.

In the following, we investigate properties of the Schwartz generalized functions of \( Q \) defined above.

**Theorem 3.5.** Let \( \rho_{\xi,\eta}^Q \) be as in Theorem 3.3 and \( n \geq 0 \). Let \( \omega_n \in E^* \) be defined by
\[
(3.9) \quad \langle \omega_n, f \rangle = \int_\mathbb{R} \lambda^n f(\lambda) \, d\lambda, \quad f \in E.
\]
Then
\[
(3.10) \quad \omega_n(Q)\varphi = Q^n\varphi, \quad \varphi \in D
\]
where \( D \equiv \text{Span}\{ \phi_\xi \mid \xi \in E_\mathbb{C} \} \) is the linear subspace of \( E \) spanned by \( \{ \phi_\xi \mid \xi \in E_\mathbb{C} \} \).

**Proof.** Firstly, by Proposition 3.2, we see that \( D \subset \text{Dom} Q^n \). On the other hand, for each \( \xi, \eta \in E_\mathbb{C} \), we have
\[
\langle Q^n\phi_\xi, \phi_\eta \rangle = \left( \int_\mathbb{R} \lambda^n P_Q(d\lambda)\phi_\xi, \phi_\eta \right),
\]
\[
= \int_\mathbb{R} \lambda^n \langle P_Q(d\lambda)\phi_\xi, \phi_\eta \rangle,
\]
\[
= \int_\mathbb{R} \lambda^n \rho_{\xi,\eta}^Q(\lambda) \, d\lambda,
\]
\[
= \langle \omega_n, \rho_{\xi,\eta}^Q \rangle = \langle \omega_n(Q)\phi_\xi, \phi_\eta \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) means the inner product of \( (L^2) \). Hence (3.10) follows. \( \square \)

**Remark 3.2.** Let \( \rho_{\xi,\eta}^Q \) be as in Theorem 3.3. Then from Theorem 3.5 we see that \( Q \varphi = \omega_1(Q) \varphi, \quad \varphi \in D \).

Note that \( D \) is not only a dense subspace of \( E \) but also a dense subspace of \( (L^2) \). Hence \( Q \) itself can be viewed as a generalized operator.

**Theorem 3.6.** Let \( \rho_{\xi,\eta}^Q \) be as in Theorem 3.3. Define a mapping \( Z : E^* \rightarrow \mathcal{L} \) as follows:
\[
(3.12) \quad Z(\omega) = \omega(Q), \quad \omega \in E^*.
\]
Then \( Z : E^* \rightarrow \mathcal{L} \) is a continuous linear mapping.
Proof. $Z$ is obviously linear. We now prove its continuity. Let $\{\omega^{(k)}\}_{k \geq 1} \subset E^*$ and $\omega \in E^*$ be such that $\omega^{(k)} \longrightarrow \omega$ in $E^*$. Then there exists some $q \geq 0$ such that $\omega, \omega^{(k)} \in E_{-q}$, $k \geq 1$ and

$$\omega(k) \longrightarrow \omega \quad \text{(in the norm } | \cdot |_{-q}).$$

With an argument similar to that in the proof of Theorem 3.3 we can get a generalized operator $X$ such that

$$Z(\omega^{(k)}) = \omega^{(k)}(Q) \longrightarrow X \quad \text{(in } L).$$

On the other hand, we have

$$\hat{X}(\xi, \eta) = \lim_{k \to \infty} Z(\omega^{(k)})(\xi, \eta)$$

$$= \lim_{k \to \infty} \langle \omega^{(k)}, \rho^Q_{\xi, \eta} \rangle$$

$$= \langle \omega, \rho^Q_{\xi, \eta} \rangle$$

$$= \langle \omega(Q)(\xi, \eta) \rangle$$

$$= \langle \hat{Z}(\omega)(\xi, \eta) \rangle,$$

$\forall \xi, \eta \in E_C$, which implies $X = Z(\omega)$. Hence $Z(\omega^{(k)}) \longrightarrow Z(\omega)$ in $L$. □

**Theorem 3.7.** Let $\rho^Q_{\xi, \eta}$ be as in Theorem 3.3 and $Z : E^* \longrightarrow L$ as in Theorem 3.6. Then $Z$ is positivity-preserving in the sense that

$$\langle \langle Z(\omega) \varphi, \varphi \rangle \rangle \geq 0, \quad \varphi \in (E)$$

whenever $\omega \in E^*$ and $\omega \geq 0$.

Proof. Let $\omega \in E^*$ with $\omega \geq 0$. To prove (3.13), we only need to show that for each $n \geq 1$ and any $z_i \in \mathbb{C}, \xi_i \in E_C, i = 1, 2, \ldots, n$,

$$\langle \langle Z(\omega) \sum_{i=1}^{n} z_i \phi_{\xi_i}, \sum_{i=1}^{n} z_i \phi_{\xi_i} \rangle \rangle \geq 0.$$

In fact, we have

$$\langle \langle Z(\omega) \sum_{i=1}^{n} z_i \phi_{\xi_i}, \sum_{i=1}^{n} z_i \phi_{\xi_i} \rangle \rangle = \sum_{i,j=1}^{n} z_i \mathbf{\overline{z}}_j \langle Z(\omega) \phi_{\xi_i}, \phi_{\xi_j} \rangle$$

$$= \sum_{i,j=1}^{n} z_i \mathbf{\overline{z}}_j \langle \omega, \rho^Q_{\xi_i, \xi_j} \rangle$$

$$= \langle \omega, \sum_{i,j=1}^{n} z_i \mathbf{\overline{z}}_j \rho^Q_{\xi_i, \xi_j} \rangle$$

$$\geq 0,$$

where, by Proposition 3.1, $\sum_{i,j=1}^{n} z_i \mathbf{\overline{z}}_j \rho^Q_{\xi_i, \xi_j} \geq 0$ as a function on $\mathbb{R}$. □

By Theorem 3.7 we immediately come to the following proposition.

**Proposition 3.8.** $\delta(Q)$ is positive, i.e., $\langle \langle \delta(Q) \varphi, \varphi \rangle \rangle \geq 0, \forall \varphi \in (E)$.
Remark 3.3. The physical meaning of the fact that $\delta(Q)$ is positive can be interpreted as follows. From the physical point of view, the self-adjoint operator $Q$ stands for an observable. Naturally, as a generalized operator, $\delta(Q)$ can be viewed as an observable associated with the observable $Q$. Hence the positivity property of $\delta(Q)$ implies that it is a positive observable.

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