

A q -SAMPLING THEOREM RELATED TO THE q -HANKEL TRANSFORM

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ABSTRACT. A q -version of the sampling theorem is derived using the q -Hankel transform introduced by Koornwinder and Swarttouw. The sampling points are the zeros of the third Jackson q -Bessel function.

1. INTRODUCTION

The classical sampling theorem asserts that every function f in the Paley-Wiener space defined by

$$PW = \left\{ f \in L^2(\mathbf{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t) dt, u \in L^2(-\pi, \pi) \right\}$$

can be represented by the interpolation series

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

Hardy's proof of this fact [4] used properties from the kernel of the Fourier transform. Relying on properties of the Hankel transform kernel, Higgins [5] used the theory of reproducing kernels to obtain a sampling theorem where the sampling points are the zeros of the Bessel function. In this note, a q -Bessel analogue of the sampling theorem is derived by considering the kernel of the q -Hankel transform, H_q^ν , introduced by Koornwinder and Swarttouw [8],

$$(H_q^\nu f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2) f(t) d_q t$$

where $J_\nu^{(3)}$ denotes the third Jackson q -Bessel function defined by the power series

$$(1.1) \quad J_\nu^{(3)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1}; q)_n (q; q)_n} x^{2n}$$

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with $0 < q < 1$, $(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$ and $(a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n$. We are using the definition of the q -integral. The q -integral in the interval $(0, 1)$ is defined as

$$(1.2) \quad \int_0^1 f(t) d_q t = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n$$

and in the interval $(0, \infty)$ as

$$(1.3) \quad \int_0^{\infty} f(t) d_q t = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n.$$

The sampling points will turn out to be $qj_{n\nu}(q^2)$, where $j_{n\nu}(q^2)$ is the n^{th} zero of $J_\nu^{(3)}(x; q^2)$. In [2] it was proved that $j_{n\nu}(q^2) = q^{-n+\epsilon_n}$, $0 < \epsilon_n < 1$. This shows how big is the spacing between the sampling points.

2. PRELIMINARIES ON REPRODUCING KERNELS

Let H be a class of complex-valued functions, defined in a set $X \subset \mathbf{C}$, such that X is a Hilbert space with the norm of $L^2(X, \mu)$. $g(s, x)$ is a *reproducing kernel* to H if

- i) $g(t, x) \in H$ for every $x \in X$;
- ii) $f(x) = \langle f(t), g(t, x) \rangle$ for every $f \in H$, $x \in X$.

The next result lists the properties of Hilbert spaces with reproducing kernel that will be used in the remainder. Properties (a), (c) and (d) are proved in [5]. Property (b) is a well-known property of the reproducing kernels, of primary importance, because it relates two different kinds of convergence. A proof of (b) can be found in [10], together with an introduction to the general theory.

Proposition 1. *In the Hilbert space $L^2[(a, b), \mu]$, an operator is defined by*

$$Ku = \langle K(x, t), u(t) \rangle_{L^2[(a, b), \mu]}.$$

The following properties hold:

(a) *If K^{-1} is bounded, the range of K , denoted by N , is a Hilbert space with reproducing kernel.*

(b) *If the sequence $\{f_n\}$ converges strongly to f in the norm of H , with reproducing kernel g , then $\{f_n\}$ converges pointwise in X to f . The convergence is uniform in every set of X where $g(x, x)$ is bounded.*

(c) *If K is an isometry, then $g(s, x) = \langle K(s, t), K(x, t) \rangle_{L^2[(a, b), \mu]}$.*

(d) *Let $\{f_n\}$ be a complete orthogonal sequence in H and (x_n) such that $f_n(x_m) = \delta_{nm}$. Then*

$$f_n(t) = \frac{g(t, x_n)}{g(x_n, x_n)}.$$

We will suppose $N \subset L^2(X, \mu)$, and this implies K bounded. K^{-1} is a transformation of N over $L^2[(a, b), \mu]$, also bounded.

3. A q -SAMPLING THEOREM

We introduce a q -Bessel version of the Paley-Wiener space, and call it PW_q^ν :

$$(3.1) \quad PW_q^\nu = \left\{ f \in L_q^2(0, \infty) : f(x) = \int_0^1 (tx)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2) u(t) d_q t, u \in L_q^2(0, 1) \right\}.$$

The notation $L_q^2(0, 1)$ stands for the Hilbert space associated to the measure of the q -integral in $(0, 1)$. In [8] the following inversion formula was proved:

$$(3.2) \quad f(t) = \int_0^\infty (xt)^{\frac{1}{2}} (H_q^\nu f)(x) J_\nu^{(3)}(xt; q^2) d_q x = (H_q^\nu (H_q^\nu f))(t).$$

Let $f \in L_q^2(0, \infty)$ such that $(H_q^\nu f)(q^{-n}) = 0, n = 1, 2, \dots$. Then $f \in PW_q^\nu$. To see this use the formula (3.2) and compare (1.2) and (1.3) to write f as an element of PW_q^ν .

Now, in the language of the preceding section, consider $X = (0, \infty), (a, b) = (0, 1)$ and the kernel $K(x, t) = (xt)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2)$. The corresponding operator K is

$$(Ku)(x) = \langle K(x, t), u(t) \rangle_{L_q^2(0,1)} = \int_0^1 (xt)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2) u(t) d_q t.$$

By (3.2), H_q^ν is a self-inverse operator and consequently, an isometry. Thus, K is also an isometry. The range of K, N , is the set of functions $f \in L_q^2(0, \infty)$ such that $f = Ku$ for some $u \in L_q^2(0, 1)$. By (3.1), $N = PW_q^\nu$. In the next lemma, the reproducing kernel of the space PW_q^ν is evaluated.

Lemma 1. *The set PW_q^ν is a Hilbert space with reproducing kernel given by*

$$(3.3) \quad g(s, x) = (1-q) q^\nu \frac{(xs)^{\frac{1}{2}} [xJ_{\nu+1}^{(3)}(x; q^2) J_\nu^{(3)}(sq^{-1}; q^2) - sJ_{\nu+1}^{(3)}(s; q^2) J_\nu^{(3)}(xq^{-1}; q^2)]}{x^2 - s^2}.$$

Proof. By Proposition 1 (a), PW_q^ν is a space with reproducing kernel $g(s, x)$. From Proposition 1 (c), since K is an isometry,

$$g(s, x) = \langle K(s, t), K(x, t) \rangle_{L_q^2(0,1)} = \int_0^1 t (xs)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2) J_\nu^{(3)}(st; q^2) d_q t.$$

In [7], the following formula was proved:

$$(3.4) \quad (a^2 - b^2) \int_0^z t J_\nu^{(3)}(aqt; q^2) J_\nu^{(3)}(bqt; q^2) d_q t \\ = (1-q) q^{\nu-1} z \left[aJ_{\nu+1}^{(3)}(aqz; q^2) J_\nu^{(3)}(bz; q^2) - bJ_{\nu+1}^{(3)}(bqz; q^2) J_\nu^{(3)}(az; q^2) \right].$$

Setting $z = 1, a = xq^{-1}$ and $b = sq^{-1}$ in (3.4), (3.3) follows. □

The q -sampling theorem can now be stated and proved.

Theorem 1. *If $f \in PW_q^\nu$, then f has the unique representation*

$$(3.5) \quad f(x) = \sum_{n=1}^\infty f(qj_{n\nu}(q^2)) \frac{2(xqj_{n\nu}(q^2))^{\frac{1}{2}} J_\nu^{(3)}(x; q^2)}{\frac{d}{dx} [J_\nu^{(3)}(x; q^2)]_{x=qj_{n\nu}(q^2)} (x^2 - q^2 j_{n\nu}^2(q^2))}$$

where $(j_{n\nu}(q^2))$ denotes the sequence of positive zeros of $J_\nu^{(3)}(x; q^2)$. The series converges uniformly in compact subsets of $(0, \infty)$.

Proof. Consider the sequence $\{f_n(x)\}$ defined by

$$f_n(x) = (xqj_{n\nu}(q^2))^{\frac{1}{2}} J_\nu^{(3)}(qxj_{n\nu}(q^2); q^2).$$

It was proved in [1] that $\{f_n(x)\}$ is a complete orthogonal sequence in $L_q^2(0, 1)$. Taking into account that K is an isometry, the sequence $(Kf_n)(x)$ is also orthogonal and complete in PW_q^ν . Now set

$$F_n(x) = \frac{(Kf_n)(x)}{(Kf_n)(qj_{n\nu}(q^2))}.$$

The orthogonality of $\{f_n(x)\}$ implies

$$(3.6) \quad F_n(qj_{m\nu}(q^2)) = \delta_{nm}.$$

Proposition 1 (d) allows us to write

$$F_n(x) = \frac{g(x, qj_{n\nu}(q^2))}{g(qj_{n\nu}(q^2), qj_{n\nu}(q^2))}.$$

Substituting in (3.3) yields

$$F_n(x) = \frac{2(xqj_{n\nu}(q^2))^{\frac{1}{2}} J_\nu^{(3)}(x; q^2)}{\frac{d}{dx} [J_\nu^{(3)}(x; q^2)]_{x=qj_{n\nu}(q^2)} (x^2 - q^2 j_{n\nu}^2(q^2))}.$$

$F_n(x)$ is an orthonormal complete sequence in N . Thus, every $f \in PW_q^\nu$ has a unique series expansion in the form

$$(3.7) \quad f(x) = \sum_{n=1}^{\infty} a_n F_n(x)$$

where a_n are the Fourier coefficients of f in $\{F_n(x)\}$. The series in (3.7) is convergent in the norm of $L_q^2(0, 1)$ and also in the norm of PW_q^ν . The real-valued function $g(x, x)$ is continuous, thus bounded in every compact subset of $(0, \infty)$. It follows from Proposition 1 (b) that (3.7) converges uniformly in compact subsets of $(0, \infty)$. Finally, setting $x = qj_{n\nu}(q^2)$ in (3.7), (3.6) implies $f(qj_{m\nu}(q^2)) = a_m$ and thus, (3.7) can be written in the form (3.5). \square

4. APPLICATION

The following formula is a consequence of the product representation for the classical Bessel function:

$$(4.1) \quad \frac{\frac{d}{dx} J_\nu(x)}{J_\nu(x)} = 2x \sum_{n=1}^{\infty} \frac{1}{j_{n\nu}^2 - x^2} + \frac{\nu}{x}.$$

Using the recurrence $x \frac{d}{dx} J_\nu(x) - \nu J_\nu(x) = -x J_{\nu+1}(x)$, (4.1) becomes

$$(4.2) \quad \frac{J_{\nu+1}(x)}{J_\nu(x)} = -2x \sum_{n=1}^{\infty} \frac{1}{j_{n\nu}^2 - x^2}$$

where $j_{n\nu}$ stands for the zeros of $J_\nu(x)$. In the case of the q -analogues of the Bessel function, this analysis cannot be done, for there are no formulas to establish a simple relation between a q -Bessel function and its derivative. While the q -analogue of (4.1) is very simple to derive from the Hadamard factorization theorem or using residues, the q -analogue of (4.2) is harder to obtain. In [6], Ismail studied the second Jackson q -Bessel function, $J_\nu^{(2)}(x; q)$, and found such a q -analogue using the orthogonality

measure of the modified q -Lommel polynomials associated to $J_\nu^{(2)}(x; q)$. Kvitsinsky [9] found a recurrence relation for the coefficients h_n in the identity

$$(4.3) \quad \frac{J_{\nu+1}^{(3)}(x; q)}{J_\nu^{(3)}(x; q)} = \sum_{n=1}^{\infty} h_n x^{2n-1}.$$

In this section an explicit formula for the coefficients h_n will be obtained as a special case of the expansion of a particular function as a sampling series. Preliminary to this expansion, a q -integral formula connecting two q -Bessel functions of different orders is established.

Lemma 2. *For $y > 0$, $\nu > -\frac{1}{2}$ and $x \in \mathbf{R}$, the following relation holds:*

$$(4.4) \quad \frac{(q; q)_\infty}{(q^y; q)_\infty} x^{-y} J_{\nu+y}^{(3)}(x; q) = \int_0^1 t^{\frac{y}{2}} \frac{(tq; q)_\infty}{(tq^y; q)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}}; q) d_q t.$$

Proof. The q -analogues of the gamma and beta functions will be critical in the proof. According to [3, 1.10], the q -gamma function, $\Gamma_q(x)$, is defined by

$$(4.5) \quad \Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}$$

and the q -beta function, $\beta_q(x, y)$, by

$$(4.6) \quad \beta_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

The q -beta function has the q -integral representation

$$(4.7) \quad \beta_q(x, y) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^y; q)_\infty} d_q t, \quad \operatorname{Re}(x) > 0, y \neq 0, -1, -2, \dots$$

Using the series representation (1.1) and the q -integral representation (4.7), it is easy to see that, if $\nu > -\frac{1}{2}$ and $y > 0$,

$$(4.8) \quad \begin{aligned} & \int_0^1 t^{\frac{y}{2}} \frac{(tq; q)_\infty}{(tq^y; q)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}}; q) d_q t \\ &= x^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\frac{k(k+1)}{2}}}{(q; q)_k (q^{\nu+1}; q)_k} x^{2k} \beta_q(k + \nu + 1, y). \end{aligned}$$

Now use (4.5) and (4.6) to express $\beta_q(k + \nu + 1, y)$ as a quotient of infinite products. Then, some algebraic manipulations using the formula $(a; q)_\infty = (a; q)_n (aq^n; q)_\infty$ allow us to see that the right-hand member of (4.8) is equal to the left-hand member of identity (4.4). \square

Before moving to the next theorem, it is convenient to point out that, from the definition (1.2), one can verify the relation:

$$(4.9) \quad \int_0^1 f(t^{\frac{1}{2}}) d_{q^2} t = (1 + q) \int_0^1 t f(t) d_q t.$$

Theorem 2. *If $u > \nu > -\frac{1}{2}$, the following identity holds:*

$$(4.10) \quad x^{\nu-u} \frac{J_u^{(3)}(x; q^2)}{J_\nu^{(3)}(x; q^2)} = -2 \sum_{n=1}^{\infty} \frac{(qj_{n\nu}(q^2))^{\nu-u+1} J_u^{(3)}(qj_{n\nu}(q^2); q^2)}{\frac{d}{dx} [J_\nu^{(3)}(x; q^2)]_{x=qj_{n\nu}(q^2)} (q^2 j_{n\nu}^2(q^2) - x^2)}.$$

Proof. Setting $y = u - \nu$ in (4.4) and replacing q by q^2 , the result is, if $u > \nu$,

$$\frac{(q^2; q^2)_\infty}{(q^{2u-2\nu}; q^2)_\infty} x^{\nu-u} J_u^{(3)}(x; q^2) = \int_0^1 t^{\frac{\nu}{2}} \frac{(tq^2; q^2)_\infty}{(tq^{2u-2\nu}; q^2)_\infty} J_\nu^{(3)}(xt^{\frac{1}{2}}; q^2) d_{q^2} t.$$

Taking (4.9) into account, this can be rewritten as

$$(4.11) \quad \frac{(q^2; q^2)_\infty}{(q^{2u-2\nu}; q^2)_\infty} x^{\nu-u} J_u^{(3)}(x; q^2) = (1+q) \int_0^1 t^{\nu+1} \frac{(t^2 q^2; q^2)_\infty}{(t^2 q^{2u-2\nu}; q^2)_\infty} J_\nu^{(3)}(xt; q^2) d_{q^2} t.$$

Considering

$$u(t) = t^{\nu+\frac{1}{2}} \frac{(1+q) (q^{2u-2\nu}; q^2)_\infty (t^2 q^2; q^2)_\infty}{(q^2; q^2)_\infty (t^2 q^{2u-2\nu}; q^2)_\infty},$$

relation (4.11) yields

$$x^{\nu-u+\frac{1}{2}} J_u^{(3)}(x; q^2) = \int_0^1 (tx)^{\frac{1}{2}} J_\nu^{(3)}(xt; q^2) u(t) d_{q^2} t.$$

Thus,

$$f(x) = x^{\nu-u+\frac{1}{2}} J_u^{(3)}(x; q^2) \in PW_q^\nu.$$

Now it is possible to apply Theorem 1 to f . The result of this application is (4.10). \square

Taking $u = \nu + 1$ in (4.10) and replacing q^2 by q , the result is the analogue of (4.2) previously mentioned:

$$(4.12) \quad \frac{J_{\nu+1}^{(3)}(x; q)}{J_\nu^{(3)}(x; q)} = -2x \sum_{n=1}^{\infty} \frac{J_{\nu+1}^{(3)}(q^{\frac{1}{2}} j_{n\nu}(q); q)}{\frac{d}{dx} [J_\nu^{(3)}(x; q)]_{x=q^{\frac{1}{2}} j_{n\nu}(q)}} \frac{1}{q j_{n\nu}^2(q) - x^2}.$$

Expanding $1/(j_{n\nu}^2(q) - x^2)$ in power series of x and substituting in (4.12), the coefficients h_n in (4.3) can be seen to be

$$h_n = \sum_{k=1}^{\infty} \frac{J_{\nu+1}^{(3)}(q^{\frac{1}{2}} j_{k\nu}(q); q)}{\frac{d}{dx} [J_\nu^{(3)}(x; q)]_{x=q^{\frac{1}{2}} j_{k\nu}(q)}} \left(\frac{1}{q j_{k\nu}^2(q)} \right)^{2n}.$$

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