A $q$-SAMPLING THEOREM RELATED TO THE $q$-HANKEL TRANSFORM

L. D. ABREU

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ABSTRACT. A $q$-version of the sampling theorem is derived using the $q$-Hankel transform introduced by Koornwinder and Swarttouw. The sampling points are the zeros of the third Jackson $q$-Bessel function.

1. Introduction

The classical sampling theorem asserts that every function $f$ in the Paley-Wiener space defined by

$$PW = \left\{ f \in L^2 (\mathbb{R}) : f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{ixt} u(t) dt, u \in L^2 (-\pi, \pi) \right\}$$

can be represented by the interpolation series

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x - n)}{\pi (x - n)}.$$

Hardy’s proof of this fact [4] used properties from the kernel of the Fourier transform. Relying on properties of the Hankel transform kernel, Higgins [5] used the theory of reproducing kernels to obtain a sampling theorem where the sampling points are the zeros of the Bessel function. In this note, a $q$-Bessel analogue of the sampling theorem is derived by considering the kernel of the $q$-Hankel transform, $H^\nu_q$, introduced by Koornwinder and Swarttouw [8],

$$(H^\nu_q f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J^{(3)}_\nu (xt; q^2) f(t) dq t$$

where $J^{(3)}_\nu$ denotes the third Jackson $q$-Bessel function defined by the power series

$$(1.1) \quad J^{(3)}_\nu (x; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} x^{\nu} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)} x^n}{(q^{\nu+1}; q)_{\nu}(q; q)_n} x^{2n}$$

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with $0 < q < 1$, $(a; q)_n = (1 - a) (1 - aq) \ldots (1 - aq^{n-1})$ and $(a; q)_\infty = \lim_{n \to \infty} (a; q)_n$.

We are using the definition of the $q$-integral. The $q$-integral in the interval $(0, 1)$ is defined as

\begin{equation}
\int_0^1 f(t) \, dq = (1 - q) \sum_{n=0}^{\infty} f(q^n) q^n
\end{equation}

and in the interval $(0, \infty)$ as

\begin{equation}
\int_0^{\infty} f(t) \, dq = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n) q^n.
\end{equation}

The sampling points will turn out to be $q_{j\nu}(q^2)$, where $j_{\nu}(q^2)$ is the $n^{th}$ zero of $J^{(3)}_{\nu}(x; q^2)$. In [2] it was proved that $j_{\nu}(q^2) = q^{-n+\epsilon_n}, 0 < \epsilon_n < 1$. This shows how big is the spacing between the sampling points.

2. Preliminaries on reproducing kernels

Let $H$ be a class of complex-valued functions, defined in a set $X \subset \mathbb{C}$, such that $X$ is a Hilbert space with the norm of $L^2(X, \mu)$. $g(s, x)$ is a reproducing kernel to $H$ if

i) $g(t, x) \in H$ for every $x \in X$;

ii) $f(x) = \langle f(t), g(t, x) \rangle$ for every $f \in H, x \in X$.

The next result lists the properties of Hilbert spaces with reproducing kernel that will be used in the remainder. Properties (a), (c) and (d) are proved in [5]. Property (b) is a well-known property of the reproducing kernels, of primary importance, because it relates two different kinds of convergence. A proof of (b) can be found in [10], together with an introduction to the general theory.

**Proposition 1.** In the Hilbert space $L^2[(a, b), \mu]$, an operator is defined by

\[
Ku = \langle K(x, t), u(t) \rangle_{L^2[(a, b), \mu]}
\]

The following properties hold:

(a) If $K^{-1}$ is bounded, the range of $K$, denoted by $N$, is a Hilbert space with reproducing kernel.

(b) If the sequence $\{f_n\}$ converges strongly to $f$ in the norm of $H$, with reproducing kernel $g$, then $\{f_n\}$ converges pointwise in $X$ to $f$. The convergence is uniform in every set of $X$ where $g(x, x)$ is bounded.

(c) If $K$ is an isometry, then $g(s, x) = \langle K(s, t), K(x, t) \rangle_{L^2[(a, b), \mu]}$.

(d) Let $\{f_n\}$ be a complete orthogonal sequence in $H$ and $(x_n)$ such that $f_n(x_m) = \delta_{nm}$. Then

\[
f_n(t) = \frac{g(t, x_n)}{g(x_n, x_n)}
\]

We will suppose $N \subset L^2(X, \mu)$, and this implies $K$ bounded. $K^{-1}$ is a transformation of $N$ over $L^2[(a, b), \mu]$, also bounded.

3. A $q$-sampling theorem

We introduce a $q$-Bessel version of the Paley-Wiener space, and call it $PW_q^\nu$:

\begin{equation}
PW_q^\nu = \left\{ f \in L^2_q(0, \infty) : f(x) = \int_0^1 (tx)^{\frac{\nu}{2}} J^{(3)}_{\nu}(xt; q^2) u(t) \, dt, u \in L^2_q(0, 1) \right\}.
\end{equation}
The notation $L^2_q(0, 1)$ stands for the Hilbert space associated to the measure of the $q$-integral in $(0, 1)$. In [8] the following inversion formula was proved:

$$f(t) = \int_0^\infty (xt)^{\frac{1}{2}} \left( H^q f \right)(x) J^{(3)}_{\nu}(xt; q^2) \, dq \, x = \left( H^q \left( H^q f \right) \right)(t).$$

Let $f \in L^2_q(0, \infty)$ such that $(H^q f)(q^n) = 0$, $n = 1, 2, \ldots$. Then $f \in PW^q$.

Now, in the language of the preceding section, consider $X = (0, \infty)$, $(a, b) = (0, 1)$ and the kernel $K(x, t) = (xt)^{\frac{1}{2}} J^{(3)}_{\nu}(xt; q^2)$. The corresponding operator $K$ is

$$(K u)(x) = \langle K(x, t), u(t) \rangle_{L^2_q(0, 1)} = \int_0^1 (xt)^{\frac{1}{2}} J^{(3)}_{\nu}(xt; q^2) u(t) \, dq \, t.$$

By (3.2), $H^q$ is a self-inverse operator and consequently, an isometry. Thus, $K$ is also an isometry. The range of $K$, $N$, is the set of functions $f \in L^2_q(0, \infty)$ such that $f = Ku$ for some $u \in L^2(0, 1)$. By (3.3), $N = PW^q$. In the next lemma, the reproducing kernel of the space $PW^q$ is evaluated.

**Lemma 1.** The set $PW^q$ is a Hilbert space with reproducing kernel given by

$$g(s, x) = (1 - q) q^{sx} \sqrt{x} \left[ x J^{(3)}_{\nu+1}(x; q^2) J^{(3)}_{\nu}(sq^{-1}; q^2) - s J^{(3)}_{\nu+1}(s; q^2) J^{(3)}_{\nu}(xq^{-1}; q^2) \right].$$

**Proof.** By Proposition 1(a), $PW^q$ is a space with reproducing kernel $g(s, x)$. From Proposition 1(c), since $K$ is an isometry,

$$g(s, x) = \langle K(s, t), K(x, t) \rangle_{L^2_q(0, 1)} = \int_0^1 t(x, s) \frac{1}{2} J^{(3)}_{\nu}(xt; q^2) J^{(3)}_{\nu}(st; q^2) \, dq \, t.$$

In [7], the following formula was proved:

$$(a^2 - b^2) \int_0^z t J^{(3)}_{\nu}(aq; q^2) J^{(3)}_{\nu}(bt; q^2) \, dq \, t = (1 - q) q^{z^{-1}} \left[ a J^{(3)}_{\nu+1}(az; q^2) J^{(3)}_{\nu}(bz; q^2) - b J^{(3)}_{\nu+1}(bqz; q^2) J^{(3)}_{\nu}(az; q^2) \right].$$

Setting $z = 1$, $a = xq^{-1}$ and $b = sq^{-1}$ in (3.4), (3.3) follows. \qed

The $q$-sampling theorem can now be stated and proved.

**Theorem 1.** If $f \in PW^q$, then $f$ has the unique representation

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\frac{d}{dx} J^{(3)}_{\nu}(x; q^2)_{x=qj_{\nu}(q^2)} (x^2 - q^2 j^2_{\nu}(q^2))} \left( x q j_{\nu}(q^2) \right)^{\frac{1}{2}} J^{(3)}_{\nu}(q x j_{\nu}(q^2); q^2).$$

where $(j_{\nu}(q^2))$ denotes the sequence of positive zeros of $J^{(3)}_{\nu}(x; q^2)$. The series converges uniformly in compact subsets of $(0, \infty)$.

**Proof.** Consider the sequence $\{f_n(x)\}$ defined by

$$f_n(x) = \left( x q j_{\nu}(q^2) \right)^{\frac{1}{2}} J^{(3)}_{\nu}(q x j_{\nu}(q^2); q^2).$$
It was proved in [1] that \( \{f_n (x)\} \) is a complete orthogonal sequence in \( L_q^2 (0, 1) \). Taking into account that \( K \) is an isometry, the sequence \( (K f_n) (x) \) is also orthogonal and complete in \( PW_q^\nu \). Now set

\[
F_n (x) = \frac{(K f_n) (x)}{(K f_n)(qJ_{\nu} (q^2))}.
\]

The orthogonality of \( \{f_n (x)\} \) implies

\[
F_n (qJ_{\nu} (q^2)) = \delta_{nm}.
\]

Proposition [1] (d) allows us to write

\[
F_n (x) = \frac{g (x, qJ_{\nu} (q^2))}{g (qJ_{\nu} (q^2), qJ_{\nu} (q^2))}.
\]

Substituting in (3.3) yields

\[
F_n (x) = \frac{2 (xqJ_{\nu} (q^2))^2 J_{\nu}^{(3)} (x; q^2)}{\frac{dx}{x} J_{\nu}^{(3)} (x; q^2)_{x=qJ_{\nu} (q^2)} ((x^2 - q^2J_{\nu}^2 (q^2))}
\]

\( F_n (x) \) is an orthonormal complete sequence in \( N \). Thus, every \( f \in PW_q^\nu \) has a unique series expansion in the form

\[
f (x) = \sum_{n=1}^{\infty} a_n F_n (x)
\]

where \( a_n \) are the Fourier coefficients of \( f \) in \( \{F_n (x)\} \). The series in (3.7) is convergent in the norm of \( L_q^2 (0, 1) \) and also in the norm of \( PW_q^\nu \). The real-valued function \( g (x, x) \) is continuous, thus bounded in every compact subset of \( (0, \infty) \). It follows from Proposition [1] (b) that (3.7) converges uniformly in compact subsets of \( (0, \infty) \). Finally, setting \( x = qJ_{\nu} (q^2) \) in (3.7), (4.1) implies \( f (qJ_{\nu} (q^2)) = a_m \) and thus, (3.7) can be written in the form (3.5).

\[\square\]

4. Application

The following formula is a consequence of the product representation for the classical Bessel function:

\[
\frac{d}{dx} J_{\nu} (x) = 2x \sum_{n=1}^{\infty} \frac{1}{J_{\nu}^2 (x) - x^2} + \frac{\nu}{x}.
\]

Using the recurrence \( x \frac{d}{dx} J_{\nu} (x) - \nu J_{\nu} (x) = -xJ_{\nu+1} (x) \), (4.1) becomes

\[
\frac{J_{\nu+1} (x)}{J_{\nu} (x)} = -2x \sum_{n=1}^{\infty} \frac{1}{J_{\nu}^2 (x) - x^2}
\]

where \( j_{\nu} \) stands for the zeros of \( J_{\nu} (x) \). In the case of the \( q \)-analogues of the Bessel function, this analysis cannot be done, for there are no formulas to establish a simple relation between a \( q \)-Bessel function and its derivative. While the \( q \)-analogue of (1.1) is very simple to derive from the Hadamard factorization theorem or using residues, the \( q \)-analogue of (4.2) is harder to obtain. In [6], Ismail studied the second Jackson \( q \)-Bessel function, \( J_{\nu}^{(2)} (x; q) \), and found such a \( q \)-analogue using the orthogonality...
measure of the modified q-Lommel polynomials associated to $J^{(3)}_
u(x;q)$. Kvitsinsky [9] found a recurrence relation for the coefficients $h_n$ in the identity

$$
\frac{J^{(3)}_{\nu+1}(x;q)}{J^{(3)}_{\nu}(x;q)} = \sum_{n=1}^{\infty} h_n x^{2n-1}.
$$

(4.3)

In this section an explicit formula for the coefficients $h_n$ will be obtained as a special case of the expansion of a particular function as a sampling series. Preliminary to this expansion, a $q$-integral formula connecting two $q$-Bessel functions of different orders is established.

**Lemma 2.** For $y > 0$, $\nu > -\frac{1}{2}$ and $x \in \mathbb{R}$, the following relation holds:

$$
\frac{(q; q)_\infty}{(q^n; q)_\infty} x^{-y} J^{(3)}_{\nu+y}(x;q) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} J^{(3)}_{\nu}(xt^\frac{1}{2}; q) dq t.
$$

(4.4)

**Proof.** The $q$-analogues of the gamma and beta functions will be critical in the proof. According to [3, 1.10], the $q$-gamma function, $\Gamma_q(x)$, is defined by

$$
\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}
$$

(4.5)

and the $q$-beta function, $\beta_q(x, y)$, by

$$
\beta_q(x, y) = \frac{\Gamma_q(x) \Gamma_q(y)}{\Gamma_q(x+y)}.
$$

(4.6)

The $q$-beta function has the $q$-integral representation

$$
\beta_q(x, y) = \int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} dq t, \quad Re(x) > 0, \quad y \neq 0, -1, -2, \ldots
$$

(4.7)

Using the series representation (1.1) and the $q$-integral representation (4.7), it is easy to see that, if $\nu > -\frac{1}{2}$ and $y > 0$,

$$
\int_0^1 t^{x-1} \frac{(tq; q)_\infty}{(tq^n; q)_\infty} J^{(3)}_{\nu}(xt^\frac{1}{2}; q) dq t
$$

$$
= x^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\nu(k+1)+1}}{q^k (q^\nu+1; q)_k} x^{2k} \beta_q(k + \nu + 1, y).
$$

(4.8)

Now use (4.8) and (4.9) to express $\beta_q(k + \nu + 1, y)$ as a quotient of infinite products. Then, some algebraic manipulations using the formula $(a; q)_\infty = (a; q)_n (aq^n; q)_\infty$ allow us to see that the right-hand member of (4.8) is equal to the left-hand member of (4.9).

Before moving to the next theorem, it is convenient to point out that, from the definition (1.3), one can verify the relation:

$$
\int_0^1 f(t^\frac{1}{2}) dq t = (1 + q) \int_0^1 tf(t) dq t.
$$

(4.9)

**Theorem 2.** If $u > \nu > -\frac{1}{2}$, the following identity holds:

$$
x^{\nu-u} \frac{J^{(3)}_u(x; q^2)}{J^{(3)}_{\nu}(x; q^2)} = -2 \sum_{n=1}^{\infty} \frac{d}{dx} \left[ J^{(3)}_{\nu}(x; q^2) \right]_{x=q_j^{nu}(q^2)} \frac{(q^{nu}(q^2))^{\nu-u+1} J^{(3)}_{\nu}(q^{nu}(q^2); q^2)}{x=q_j^{nu}(q^2) (q^{2u} J^{(3)}_{\nu}(q^2) - x^2)}.
$$

(4.10)

Proof. Setting \( y = u - \nu \) in (4.13) and replacing \( q \) by \( q^2 \), the result is, if \( u > \nu \),
\[
\left(\frac{q^2; q^2}{q^{2u-2\nu}; q^2}\right)_\infty x^{u-\nu} J^{(3)}_u(x; q^2) = \int_0^1 t^{\frac{1}{2}} \left(\frac{t^2 q^2; q^2}{t^{2u-2\nu}; q^2}\right)_\infty J^{(3)}_u(x t^{\frac{1}{2}}; q^2) d_q t.
\]
Taking (4.9) into account, this can be rewritten as
\[
\left(\frac{q^2; q^2}{q^{2u-2\nu}; q^2}\right)_\infty x^{u-\nu} J^{(3)}_u(x; q^2) = (1 + q) \int_0^1 t^{\nu + \frac{1}{2}} \left(\frac{t^2 q^2; q^2}{t^{2u-2\nu}; q^2}\right)_\infty J^{(3)}_u(x; q^2) d_q t.
\]

Considering
\[
u(t) = t^{\nu + \frac{1}{2}} \left(1 + q \right) \left(\frac{q^{2u-2\nu}; q^2}{q^2; q^2}\right) \left(\frac{t^{2\nu}; q^2}{t^{2u-2\nu}; q^2}\right)_\infty,
\]
relation (4.11) yields
\[
x^{u-\nu + \frac{1}{2}} J^{(3)}_u(x; q^2) = \int_0^1 (tx)^{\frac{1}{2}} J^{(3)}_u(x t; q^2) u(t) d_q t.
\]
Thus,
\[
f(x) = x^{u-\nu + \frac{1}{2}} J^{(3)}_u(x; q^2) \in PW^\nu.
\]
Now it is possible to apply Theorem 1 to \( f \). The result of this application is (4.10).
\(\square\)

Taking \( u = \nu + 1 \) in (4.10) and replacing \( q^2 \) by \( q \), the result is the analogue of (4.2) previously mentioned:
\[
\frac{J^{(3)}_{\nu+1}(x; q)}{J^{(3)}_\nu(x; q)} = -2x \sum_{n=1}^{\infty} \frac{J^{(3)}_{\nu+1}(q^{2\nu}; q^n; q)_\infty}{J^{(3)}_\nu(x; q)_\infty} \frac{1}{q^{2\nu}_n(q) - x^2}.
\]
Expanding \( 1/(q^{2\nu}_n(q) - x^2) \) in power series of \( x \) and substituting in (4.12), the coefficients \( h_n \) in (4.3) can be seen to be
\[
h_n = \sum_{k=1}^{\infty} \frac{J^{(3)}_{\nu+1}(q^{2\nu}; q^n; q)_\infty}{J^{(3)}_\nu(x; q)_\infty} \left(\frac{1}{q^{2\nu}_n(q)}\right)^{2n}.
\]

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Department of Mathematics, Universidade de Coimbra, Portugal
E-mail address: daniel@mat.uc.pt.