PROBABILISTIC ASPECTS
OF AL-SALAM–CHIHARA POLYNOMIALS

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Abstract. We solve the connection coefficient problem between the Al-Salam–Chihara polynomials and the \(q\)-Hermite polynomials, and we use the resulting identity to answer a question from probability theory. We also derive the distribution of some Al-Salam–Chihara polynomials, and compute determinants of related Hankel matrices.

1. Introduction and main identity

The aim of the paper is to point out the connection of Al-Salam–Chihara polynomials with a regression problem in probability, and to use it to give a new simple derivation of their density. Our approach exploits identity (1.8) below, which connects the Al-Salam–Chihara polynomials to the continuous \(q\)-Hermite polynomials. This connection is more direct and elementary but less general than the technique of attachment exploited in [B196, Section 2]. We also compute determinants of Hankel matrices with entries that are linear combinations of the \(q\)-Hermite polynomials.

The Al-Salam–Chihara polynomials were introduced in [ASC76], and their weight function was found in [AI84]. We are interested in the renormalized Al-Salam–Chihara polynomials \(\{p_n(x|q,a,b)\}\), which are defined by the following three-term recurrence relation:

\[
p_{n+1}(x) = (x - aq^n)p_n(x) - (1 - bq^{n-1}) [n]_q p_{n-1}(x) \quad (n \geq 0),
\]

with the usual initial conditions \(p_{-1} = 0\), \(p_0 = 1\). Here, we use the standard notation

\[
[n]_q = 1 + q + \cdots + q^{n-1},
\]

\[
[n]_q! = [1]_q[2]_q \cdots [n]_q,
\]

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! [k]_q!},
\]

with the usual conventions \([0]_q = 0\), \([0]_q! = 1\).

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For $|q| < 1$, their generating function
\[
f(t, x|q, a, b) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^n} p_n(x|q, a, b)
\]
is given by
\[
f(t, x|q, a, b) = \prod_{k=0}^{\infty} \frac{1 - (1 - q)atq^k + (1 - q)bt^2q^{2k}}{1 - (1 - q)xqt^k + (1 - q)t^2q^{2k}};
\]
compare [AI84] (3.6) and (3.10).

The corresponding (renormalized) continuous $q$-Hermite polynomials $H_n(x|q) = p_n(x|q, 0, 0)$ satisfy the three-term recurrence relation
\[
H_{n+1}(x) = xH_n(x) - [n]_q H_{n-1}(x).
\]
For $|q| < 1$ their generating function $\phi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^n} H_n(x|q)$ is
\[
\phi(t, x|q) = \prod_{k=0}^{\infty} \left(1 - (1 - q)xqt^k + (1 - q)t^2q^{2k}\right)^{-1}.
\]

Of course, these are well-known special cases of (1.1) and (1.2); see [ISV87, (2.11) and (2.12)], which we state here for further reference.

We will also use polynomials $\{B_n(x|q)\}$ defined by the three-term recurrence relation
\[
B_{n+1}(x) = -q^n xB_n(x) + q^{n-1}[n]_q B_{n-1}(x) \quad (n \geq 0)
\]
with the usual initial conditions $B_{-1} = 0, B_0 = 1$. These polynomials are related to the $q$-Hermite polynomials by
\[
B_n(x|q) = \begin{cases} 
q^n n^{(n-2)/2} H_n(i \sqrt{q} x^{1/2}_q) & \text{if } q > 0, \\
(-1)^n n^{(n-1)/2} [q]^{n(n-2)/2} H_n(- \sqrt{q} x^{1/2}_q) & \text{if } q < 0
\end{cases}
\]
and have been studied in [Ask89], [LM94]. Their generating function $\psi(t, x|q) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q^n} B_n(x|q)$ is given by
\[
\psi(t, x|q) = \prod_{k=0}^{\infty} \left(1 - (1 - q)xqt^k + (1 - q)t^2q^{2k}\right).
\]

We now point out the mutual relationship between the Al-Salam–Chihara polynomials $\{p_n(x|q, a, b)\}$ and the polynomials $\{H_n(x|q)\}_{n \geq 0}$ and $\{B_n(x|q)\}_{n \geq 0}$.

**Theorem 1.** For all $a, c, q \in \mathbb{C}$, $c \neq 0$, and $n \geq 1$ we have
\[
p_n(x|q, a, b) = \sum_{k=1}^{n} \binom{n}{k}_q \frac{a^{n-k}}{c^k} B_{n-k}(\frac{a}{c}|q) \left( H_k(x|q) - c^k H_k(\frac{a}{c}|q) \right),
\]
where $b = c^2$.

**Proof.** From the recurrence relations (1.1), (1.3), and (1.5), it is clear that $p_n(x|q, a, b), H_n(x|q),$ and $B_n(x|q)$ are given by polynomial expressions in the variable $q$. The $q$-binomial coefficient $\binom{n}{k}_q$ is also a polynomial in $q$. Therefore, we see that identity (1.5) is equivalent to a polynomial identity in variable $q \in \mathbb{C}$.
Hence it is enough to prove that (1.8) holds true for all \(|q| < 1\). When \(|q| < 1\), inspecting (1.2), (1.4), and (1.7) we notice that for \(b = c^2\) we have
\[
(1.9) \quad f(t, x|q, a, b) = \psi(tc, a/c|q)\phi(t, x|q)
\]
and
\[
(1.10) \quad \psi(t, x|q)\phi(t, x|q) = 1.
\]
Therefore,
\[
f(t, x|q, a, b) = 1 + \psi(tc, a/c|q) (\phi(t, x|q) - \phi(tc, a/c|q)),
\]
which is valid for all small enough \(|t|\). Comparing the coefficients at \(t^n\) for \(n \geq 1\) and taking into account that \(H_k(x|q) - c^kH_k(a/c|q) = 0\) for \(k = 0\), we get (1.8). \(\square\)

Remark 1. One could split (1.8) into the following two separate identities, which are implied by (1.9) and (1.10) respectively:
\[
(1.11) \quad \forall n \geq 0 : \quad p_n(x|q, a, c^2) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] c^{-k}B_{n-k}(a/c|q)H_k(x|q),
\]
\[
(1.12) \quad \forall n \geq 1 : \quad \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] B_{n-k}(x|q)H_k(x|q) = 0.
\]
Formula (1.11) is a renormalized inverse of formula [IRS99, (4.7)], which expresses the \(q\)-Hermite polynomials as linear combinations of Al-Salam–Chihara polynomials. Formula (1.12) resembles [Car56, (2.28)], which considers \(q\)-Hermite polynomials of the form \(h_n(x|q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k\), paired with \(b_n(x|q) = h_n(x|1/q)\).

2. Probabilistic aspects

Quadratic regression questions in the paper [Bry01] lead to the problem of determining all probability distributions \(\mu\) which are defined indirectly by the relationships
\[
(2.1) \quad \int H_n(x|q)\mu(dx) = \rho^nH_n(y|q), \quad n = 1, 2, \ldots,
\]
where \(y, \rho, q \in \mathbb{R}\) are fixed parameters, and \(\{H_n\}_{n \geq 0}\) is the family of the \(q\)-Hermite polynomials.

Our next result shows that this problem can be solved using the Al-Salam–Chihara polynomials.

Theorem 2. If \(\mu = \mu(dx|\rho, y)\) satisfies (2.1), then its orthogonal polynomials are Al-Salam–Chihara polynomials \(\{p_n(x|q, a, b)\}\) with \(a = \rho y, b = \rho^2\).

Proof. Recall that \(H_n(x|q) = p_n(x|q, 0, 0)\). Thus if \(\rho = 0\), then (2.1) implies that \(\int p_n(x|q, a, b)\mu(dx) = 0\) for all \(n = 1, 2, \ldots\). Suppose now that \(\rho \neq 0\). Combining (1.8) with (2.1) we get
\[
\int p_n(x|q, a, b)\mu(dx) = \sum_{k=1}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \rho^{-k}B_{n-k}(y|q) \int \left( H_k(x|q) - \rho^kH_k(y|q) \right)\mu(dx) = 0
\]
for all \(n = 1, 2, \ldots\). Since \(\{p_n\}\) satisfy a three-step recurrence, this implies \(\int p_n(x)p_n(x)\mu(dx) = 0\) for all \(0 \leq k < n\). \(\square\)
Next we answer an unresolved case from [Bry01].

**Corollary 1.** Fix \( q > 1, y \in \mathbb{R} \). Let \( \mathcal{R}_q = \{1, 1/q, 1/q^2, \ldots, 1/q^n, \ldots, 0\} \).

(i) If \( \rho^2 \notin \mathcal{R}_q \), then (2.1) has no probabilistic solution \( \mu \).

(ii) If \( \rho^2 \in \mathcal{R}_q \) is non-zero, then the probabilistic solution of (2.1) exists, and is a discrete measure supported on \( 1 + \log_q(1/\rho^2) \) points.

**Proof.** Suppose that \( \mu \) is positive and solves (2.1). Therefore its monic orthogonal polynomials satisfy the three-term recurrence relation

\[
 p_{n+1}(x) = (x - \rho y q^n)p_n(x) - (1 - \rho^2 q^{n-1})p_{n-1}(x).
\]

For a positive non-degenerate measure \( \mu_y(dx) \), and \( n \geq 1 \) we have

\[
 \int p_n^2(x) \mu_y(dx) = (1 - \rho^2 q^{n-1}) \int p_{n-1}^2(x) \mu_y(dx).
\]

If \( \rho^2 \notin \mathcal{R}_q \), then \( 1 - \rho^2 q^{n-1} \neq 0 \) for all \( n \). Since \( \int p_n^2(x) \mu_y(dx) > 0 \), this shows that \( \int p_n^2(x) \mu_y(dx) > 0 \) for all \( n \geq 0 \). But then the coefficients \( 1 - \rho^2 q^{n-1} \) must be non-negative for all \( n \), which is false. This proves (i).

To conclude the proof it remains to notice that if \( \rho^2 = 1/q^m \), then from (2.2) and (the proof of) Favard’s theorem, see [Fre71, Theorem II.1.5], it follows that the solution of (2.1) is given by a measure supported on the roots of the polynomial \( p_{m+1} \). Indeed, (2.2) implies that the polynomial \( p_{m+2} \) is divisible by \( p_{m+1} \). Therefore, \( p_{m+1} \) is the common factor of all polynomials \( \{p_k : k \geq m + 1\} \). It is also known, see [Fre71 Theorem I.2.2], that \( p_{m+1} \) has exactly \( m + 1 \) distinct real roots \( x_1, \ldots, x_{m+1} \). Thus, any measure \( \mu(dx) = \sum \lambda_j \delta_{x_j} \), supported on the roots of the polynomial \( p_{m+1} \) satisfies \( \int p_{m+1+k} \mu(dx) = 0 \). Solving the remaining \( m + 1 \) equations \( \int p_k \mu(dx) = 1 \), and \( \int p_k(x) \mu(dx) = 0 \), \( k = 1, 2, \ldots, m \) for \( \lambda_j \), we get a measure that solves (2.1). This measure is non-negative since the coefficients at the third term in the recurrence (2.2) are non-negative for \( n = 1, \ldots, m \); see [Fre71 page 58].

From Theorem 2 it follows that if the solution of (2.1) exists, then it is given by the distribution of the Al-Salam–Chihara polynomials. The distribution of the Al-Salam–Chihara polynomials is derived in [A184 Chapter 3]. However, in [Bry01 Proposition 8.1] we found the solution of (2.1) that relies solely on the facts about the \( q \)-Hermite polynomials. We repeat the latter argument here, and then use it to re-derive the distribution of the corresponding Al-Salam–Chihara polynomials.

**Corollary 2.** If \( \rho, q, y \in \mathbb{R} \) are such that \( |\rho| < 1, |q| < 1, \) and \( y^2(1-q) < 4 \), then the probabilistic solution of (2.1) is given by the absolutely continuous measure \( \mu \) with the density on \( x^2 < 4/(1-q) \) given by

\[
 f_H(x) = \frac{\sqrt{1-q}}{2\pi \sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \frac{1 - \rho^2 q^{k+1} + (1 + q^k) \left((1 + q^k)^2 - (1 - q)x^2q^k\right)}{(1 - \rho^2 q^{2k+2})^2 - (1 - q)q^{k(1 + q^k)}x(1 - q)x^2(1 - \rho^2 q^{2k})q^{k+1}}.
\]

**Proof.** The distribution of the \( q \)-Hermite polynomials \( H_n(x|q) \) is supported on \( x^2 < 4/(1-q) \) with the density

\[
 f_H(x) = \frac{\sqrt{1-q}}{2\pi \sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \left((1 + q^k)^2 - (1 - q)x^2q^k\right) \prod_{k=0}^{\infty} (1 - q^{k+1});
\]
Corollary 3. By Theorem 2, the distribution of polynomials converges uniformly and defines the Poisson-Mehler kernel, which is given by
\begin{equation}
g_H(x, y, \rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{|n|!} H_n(x)H_n(y)
\end{equation}

this is the renormalized version of the well-known result; see e.g. [IS88] (2.2), which considers the $q$-Hermite polynomials given by \{(1 - q)^{-n/2}H_n(2x/\sqrt{1 - q})\} instead of our \{H_n(xq)\}.

Since (13) implies that \(\int \frac{H_2^2(x|q)f_H(x)dx}{\overline{2/\sqrt{1-q}}} = |n|\), it follows from (2.4) that
\(\int_{\overline{2/\sqrt{1-q}}} H_n(x|q)g_H(x, y, \rho)f_H(x)dx = \rho^nH_n(y).\)

Corollary 3. If $q, a, b \in \mathbb{R}$ are such that $|q| < 1, 0 < b < 1$, and $a^2(1 - q) < 4b$, then the distribution of the Al-Salam–Chihara polynomials \(\{p_n(x|q, a, b)\}\) is absolutely continuous with the density on $x^2 < 4/(1 - q)$. The density is given by
\[
\frac{\sqrt{1 - q}}{2\pi\sqrt{4 - (1 - q)x^2}} \prod_{k=0}^{\infty} \frac{(1 - bq^k)(1 - q^{k+1})(1 + q^{k+1})(1 + q)2q^k}{(1 - bq^k)^2 - (1 - q)aq^k(1 + bq^k)x + (1 - q)(bq^2 + a^2)q^2k}.
\]

Proof. By Theorem 2 the distribution of polynomials $p_n$ solves (2.1) with $\rho = \sqrt{b}, y = a/\rho$. Thus the formula follows from Corollary 2.

Remark 2. Iterating (2.4) we see that the measure corresponding to the parameter $\rho_1\rho_2$ instead of $\rho$ is given by
\[
\mu(\cdot|\rho_1\rho_2, x) = \int \mu(\cdot|\rho_1, y)\mu(dy|\rho_2, x).
\]

For $|q| < 1, |\rho| < 1$ the density of $\mu$ is given in Corollary 2, hence, after simplifying common factors and substituting $x = 2\xi/\sqrt{1 - q}, y = 2\eta/\sqrt{1 - q}$, $z = 2\zeta/\sqrt{1 - q}$, the relationship (2.6) takes the following form:
\[
\int_{-1}^{1} \prod_{k=0}^{\infty} \frac{(1 - \rho_1^2q^k)(1 - q^{k+1})(1 + q^k)^2 - 4\eta^2q^k}{(1 - \rho_1^2q^k)^2 - 4\eta_1q^k(1 + \rho_1^2q^k)\eta_2 + 4\rho_1^2(\eta^2 + \zeta^2)q^{2k}} \times \prod_{k=0}^{\infty} \frac{1 - \rho_2^2q^k}{(1 - \rho_2^2q^{2k})^2 - 4\rho_2q^k(1 + \rho_2^2q^{2k})\xi_2 + 4\rho_2^2(\eta^2 + \zeta^2)q^{2k}} \frac{d\eta}{2\pi\sqrt{1 - \eta^2}}
\]
3. Determinants of Hankel Matrices

In this section we are interested in calculating the determinants of the Hankel matrices

$$M_n = [m_{i+j}]_{i,j=0,\ldots,n-1},$$

where $m_i = \int x^i \mu(dx)$ are the moments of a certain (perhaps signed) measure $\mu$. It is well known that for positive measures we must have $\det M_n \geq 0$, and that these determinants can be read out from the three-term recurrence for the corresponding monic orthogonal polynomials.

Consider first the moments $m_k(y) = \int x^k \mu(dx)$ of the (perhaps signed) measure $\mu = \mu_{y,\rho}$, which solves (2.1) Then $m_k(y)$ are polynomials of degree $k$ in the variable $y$ and can be written as follows. Let $a_{n,2i}$, $i \leq \lfloor n/2 \rfloor$ be the coefficients in the expansion of the monomial $x^n$ into the $q$-Hermite polynomials,

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,2i} H_{n-2i} (x|q), \quad n \geq 0.$$  

Then

$$m_n(y) = \int x^n \, d\mu(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho^{n-2k} a_{n,2k} H_{n-2k} (y|q).$$

Let $S_n$ be the Hankel matrix of moments $m_k(y)$,

$$S_n(y|q, \rho) = \begin{bmatrix} m_0(y) & m_1(y) & \cdots & m_{n-1}(y) \\ m_1(y) & m_2(y) & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ m_{n-1}(y) & & \cdots & m_{2n-2}(y) \end{bmatrix}.$$  

It is well known that $\det S_n$ is the product of the coefficients at the third term of (22), which implies the following.

**Corollary 4.** $\det S_{n+1} / \det S_n = [n]_q! \prod_{i=1}^n (1 - \rho^2 q^{i-1}).$

Our second Hankel matrix has an even simpler form. As indicated in [IS97], [IS02] the $q$-Hermite polynomials can be viewed as moments of a signed measure, $H_n(x|q) = \int u^n \mu(du|x,q)$. It turns out that if $q \neq 0$, the measure $\mu(du|x,q)$ cannot be positive even for a single value of $x$. To see this, consider the following $n \times n$ matrices:

$$M_n(x|q) = \begin{bmatrix} H_0(x|q) & H_1(x|q) & H_2(x|q) & \cdots & H_{n-1}(x|q) \\ H_1(x|q) & H_2(x|q) & H_3(x|q) & & H_n(x|q) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ H_{n-1}(x|q) & H_n(x|q) & & \cdots & H_{2n-2}(x|q) \end{bmatrix}.$$  

The following $q$-generalization of [Kra99 (3.55)] shows that the determinants $\det M_n(x|q)$ are free of the variable $x$ and take negative values.

**Theorem 3.**

$$\frac{\det M_{n+1}}{\det M_n} = (-1)^n q^{n(n-1)/2} [n]_q!.$$  

Proof. Using (1.3), we row-reduce the first column of the matrix. Namely, from the second row of \( M_{n+1} \), we subtract the first one multiplied by \( x \). Similarly, for \( i \geq 3 \), we subtract \( x \) times row \( i - 1 \) and add the \((i-2)\)-th row multiplied by \( [i-1]_q \). Taking (1.3) into account, \( \det M_{n+1}(x|q) \) becomes

\[
\begin{bmatrix}
H_0 & H_1 & H_2 & \cdots & H_{n-1}
0 & (0-[1])H_0 & (0-[2])H_1 & \cdots & (0-[n])H_{n-1}
0 & (1-[2])H_1 & (1-[3])H_2 & \cdots & (1-[n+1])H_{n-1}
\vdots & \ddots & \ddots & \ddots & \ddots
0 & ([n-1]-[n])H_{n-1} & ([n-1]-[n+1])H_n & \cdots & ([n-1]-[2n-1])H_{2n-2}
\end{bmatrix}
\]

Now, we use the fact that for \( m \leq n \) we have \([n]_q - [m]_q = q^n [n-m]_q \). Thus \( \det M_{n+1}(x|q) \) becomes

\[
\det M_{n+1} = (-1)^n q^{\sum_{i=1}^n j} \prod_{j=1}^n [j]_q \det M_n = (-1)^n q^{n(n-1)/2}[n]_q! \det M_n. \quad \square
\]

The formula stated in Corollary 3 was originally discovered through symbolic computations and motivated this paper. We were unable to find a direct algebraic proof along the lines of the proof of Theorem 3 and our search for the explanation of why \( \det S_n(y) \) does not depend on \( y \) led us to Al-Salam–Chihara polynomials and identity (1.8).

The fact that Hankel determinants formed of certain linear combinations of the \( q \)-Hermite polynomials do not depend on the argument of these polynomials as exposed in Theorem 3 and Corollary 3 is striking and unexpected to us. A natural question arises whether other linear combinations have this property.

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