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PROBABILISTIC ASPECTS OF AL-SALAM–CHIHARA POLYNOMIALS

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Abstract. We solve the connection coefficient problem between the Al-Salam–Chihara polynomials and the q-Hermite polynomials, and we use the resulting identity to answer a question from probability theory. We also derive the distribution of some Al-Salam–Chihara polynomials, and compute determinants of related Hankel matrices.

1. Introduction and main identity

The aim of the paper is to point out the connection of Al-Salam–Chihara polynomials with a regression problem in probability, and to use it to give a new simple derivation of their density. Our approach exploits identity (1.8) below, which connects the Al-Salam–Chihara polynomials to the continuous q-Hermite polynomials. This connection is more direct and elementary but less general than the technique of attachment exploited in [BI96, Section 2]. We also compute determinants of Hankel matrices with entries that are linear combinations of the q-Hermite polynomials.

The Al-Salam–Chihara polynomials were introduced in [ASC76], and their weight function was found in [AI84]. We are interested in the renormalized Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$, which are defined by the following three-term recurrence relation:

$$p_{n+1}(x) = (x - aq^n)\, p_n(x) - \left(1 - bq^{n-1}\right) [n]_q p_{n-1}(x) \ \ (n \geq 0), \tag{1.1}$$

with the usual initial conditions $p_{-1} = 0$, $p_0 = 1$. Here, we use the standard notation

$$\begin{aligned} [n]_q &= 1 + q + \cdots + q^{n-1}, \\ [n]_q! &= [1]_q[2]_q \ldots [n]_q, \\ \begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[n-k]_q![k]_q!}, \end{aligned}$$

with the usual conventions $[0]_q = 0, [0]_q! = 1$.

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For $|q|<1$, their generating function

$$f(t,x|q,a,b)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}p_n(x|q,a,b)$$

is given by

$$f(t,x|q,a,b)=\prod_{k=0}^{\infty}\frac{1-(1-q)atq^k+(1-q)bt^2q^{2k}}{1-(1-q)xtq^k+(1-q)t^2q^{2k}}; \tag{1.2}$$

compare [AI84, (3.6) and (3.10)].

The corresponding (renormalized) continuous q-Hermite polynomials $H_n(x|q)=p_n(x|q,0,0)$ satisfy the three-term recurrence relation

$$H_{n+1}(x)=xH_n(x)-[n]_qH_{n-1}(x). \tag{1.3}$$

For $|q|<1$ their generating function $\phi(t,x|q)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}H_n(x|q)$ is

$$\phi(t,x|q)=\prod_{k=0}^{\infty}\left(1-(1-q)xtq^k+(1-q)t^2q^{2k}\right)^{-1}. \tag{1.4}$$

Of course, these are well-known special cases of (1.1) and (1.2); see [ISV87, (2.11) and (2.12)], which we state here for further reference.

We will also use polynomials $\{B_n(x|q)\}$ defined by the three-term recurrence relation

$$B_{n+1}(x)=-q^nxB_n(x)+q^{n-1}[n]_qB_{n-1}(x)\ \ (n\geq 0) \tag{1.5}$$

with the usual initial conditions $B_{-1}=0, B_0=1$. These polynomials are related to the q-Hermite polynomials by

$$B_n(x|q)=\begin{cases} i^nq^{n(n-2)/2}H_n(i\sqrt{q}\,x|\frac{1}{q}) & \text{if } q>0,\\ (-1)^{n(n-1)/2}|q|^{n(n-2)/2}H_n(-\sqrt{|q|}\,x|\frac{1}{q}) & \text{if } q<0\end{cases} \tag{1.6}$$

and have been studied in [Ask89], [IM94]. Their generating function $\psi(t,x|q)=\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}B_n(x|q)$ is given by

$$\psi(t,x|q)=\prod_{k=0}^{\infty}\left(1-(1-q)xtq^k+(1-q)t^2q^{2k}\right). \tag{1.7}$$

We now point out the mutual relationship between the Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$ and the polynomials $\{H_n(x|q)\}_{n\geq 0}$ and $\{B_n(x|q)\}_{n\geq 0}$.

Theorem 1. *For all $a,c,q\in\mathbb{C}$, $c\neq 0$, and $n\geq 1$ we have*

$$p_n(x|q,a,b)=\sum_{k=1}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q c^{n-k}B_{n-k}(\frac{a}{c}|q)\left(H_k(x|q)-c^kH_k(\frac{a}{c}|q)\right), \tag{1.8}$$

where $b=c^2$.

Proof. From the recurrence relations (1.1), (1.3), and (1.5), it is clear that $p_n(x|q,a,b)$, $H_n(x|q)$, and $B_n(x|q)$ are given by polynomial expressions in the variable q. The q-binomial coefficient $\begin{bmatrix} n\\ k\end{bmatrix}_q$ is also a polynomial in q. Therefore, we see that identity (1.8) is equivalent to a polynomial identity in variable $q\in\mathbb{C}$.

Hence it is enough to prove that (1.8) holds true for all $|q| < 1$. When $|q| < 1$, inspecting (1.2), (1.4), and (1.7) we notice that for $b = c^2$ we have

$$f(t, x|q, a, b) = \psi(tc, a/c|q)\phi(t, x|q) \tag{1.9}$$

and

$$\psi(t, x|q)\phi(t, x|q) = 1. \tag{1.10}$$

Therefore,

$$f(t, x|q, a, b) = 1 + \psi(ct, a/c|q)\left(\phi(t, x|q) - \phi(ct, a/c|q)\right),$$

which is valid for all small enough $|t|$. Comparing the coefficients at t^n for $n \geq 1$ and taking into account that $H_k(x|q) - c^k H_k(\frac{a}{c}|q) = 0$ for $k = 0$, we get (1.8). $\square$

Remark 1. One could split (1.8) into the following two separate identities, which are implied by (1.9) and (1.10) respectively:

$$\forall n \geq 0: \ p_n(x|q, a, c^2) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q c^{n-k} B_{n-k}(\frac{a}{c}|q) H_k(x|q), \tag{1.11}$$

$$\forall n \geq 1: \ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-k}(x|q) H_k(x|q) = 0. \tag{1.12}$$

Formula (1.11) is a renormalized inverse of formula [IRS99, (4.7)], which expresses the q-Hermite polynomials as linear combinations of Al-Salam–Chihara polynomials. Formula (1.12) resembles [Car56, (2.28)], which considers q-Hermite polynomials of the form $h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$, paired with $b_n(x|q) = h_n(x|1/q)$.

2. Probabilistic aspects

Quadratic regression questions in the paper [Bry01] lead to the problem of determining all probability distributions μ which are defined indirectly by the relationships

$$\int H_n(x|q)\mu(dx) = \rho^n H_n(y|q), \ n = 1, 2, \ldots, \tag{2.1}$$

where $y, \rho, q \in \mathbb{R}$ are fixed parameters, and $\{H_n\}_{n\geq 0}$ is the family of the q-Hermite polynomials.

Our next result shows that this problem can be solved using the Al-Salam–Chihara polynomials.

Theorem 2. *If* $\mu = \mu(dx|\rho, y)$ *satisfies* (2.1), *then its orthogonal polynomials are Al-Salam–Chihara polynomials* $\{p_n(x|q, a, b)\}$ *with* $a = \rho y$, $b = \rho^2$.

Proof. Recall that $H_n(x|q) = p_n(x|q, 0, 0)$. Thus if $\rho = 0$, then (2.1) implies that $\int p_n(x|q, a, b)\mu(dx) = 0$ for all $n = 1, 2, \ldots$. Suppose now that $\rho \neq 0$. Combining (1.8) with (2.1) we get

$$\int p_n(x|q, a, b)\mu(dx) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \rho^{n-k} B_{n-k}(y|q) \int \left(H_k(x|q) - \rho^k H_k(y|q)\right)\mu(dx) = 0$$

for all $n = 1, 2, \ldots$. Since $\{p_n\}$ satisfy a three-step recurrence, this implies $\int p_k(x)p_n(x)\mu(dx) = 0$ for all $0 \leq k < n$. $\square$

Next we answer an unresolved case from [Bry01].

Corollary 1. *Fix* $q > 1, y \in \mathbb{R}$. *Let* $\mathcal{R}_q = \{1, 1/q, 1/q^2, \dots, 1/q^n, \dots, 0\}$.

(i) *If* $\rho^2 \notin \mathcal{R}_q$, *then* (2.1) *has no probabilistic solution* μ.
(ii) *If* $\rho^2 \in \mathcal{R}_q$ *is non-zero, then the probabilistic solution of* (2.1) *exists, and is a discrete measure supported on* $1 + \log_q 1/\rho^2$ *points.*

Proof. Suppose that μ is positive and solves (2.1). Therefore its monic orthogonal polynomials satisfy the three-term recurrence relation

$$p_{n+1}(x) = (x - \rho y q^n) p_n(x) - (1 - \rho^2 q^{n-1}) p_{n-1}(x). \tag{2.2}$$

For a positive non-degenerate measure $\mu_y(dx)$, and $n \geq 1$ we have

$$\int p_n^2(x)\mu_y(dx) = (1 - \rho^2 q^{n-1}) \int p_{n-1}^2(x)\mu_y(dx). \tag{2.3}$$

If $\rho^2 \notin \mathcal{R}_q$, then $(1 - \rho^2 q^{n-1}) \neq 0$ for all n. Since $\int p_0^2(x)\mu_y(dx) > 0$, this shows that $\int p_n^2(x)\mu_y(dx) > 0$ for all $n \geq 0$. But then the coefficients $1 - \rho^2 q^{n-1}$ must be non-negative for all n, which is false. This proves (i).

To conclude the proof it remains to notice that if $\rho^2 = 1/q^m$, then from (2.2) and (the proof of) Favard's theorem, see [Fre71, Theorem II.1.5], it follows that the solution of (2.1) is given by a measure supported on the roots of the polynomial p_{m+1}. Indeed, (2.2) implies that the polynomial p_{m+2} is divisible by p_{m+1}. Therefore, p_{m+1} is the common factor of all polynomials $\{p_k : k \geq m+1\}$. It is also known, see [Fre71, Theorem I.2.2], that p_{m+1} has exactly $m+1$ distinct real roots $x_1, \dots, x_{m+1}$. Thus, any measure $\mu(dx) = \sum \lambda_j \delta_{x_j}$ supported on the roots of the polynomial p_{m+1} satisfies $\int p_{m+1+k}\mu(dx) = 0$. Solving the remaining $m+1$ equations $\int p_0 \mu(dx) = 1$, and $\int p_k(x)\mu(dx) = 0, k = 1, 2, \dots, m$ for λ_j, we get a measure that solves (2.1). This measure is non-negative since the coefficients at the third term in the recurrence (2.2) are non-negative for $n = 1, \dots, m$; see [Fre71, page 58]. ☐

From Theorem 2 it follows that if the solution of (2.1) exists, then it is given by the distribution of the Al-Salam–Chihara polynomials. The distribution of the Al-Salam–Chihara polynomials is derived in [AI84, Chapter 3]. However, in [Bry01, Proposition 8.1] we found the solution of (2.1) that relies solely on the facts about the q-Hermite polynomials. We repeat the latter argument here, and then use it to re-derive the distribution of the corresponding Al-Salam–Chihara polynomials.

Corollary 2. *If* $\rho, q, y \in \mathbb{R}$ *are such that* $|\rho| < 1$, $|q| < 1$, *and* $y^2(1-q) < 4$, *then the probabilistic solution of* (2.1) *is given by the absolutely continuous measure* μ *with the density on* $x^2 < 4/(1-q)$ *given by*

$$\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \frac{(1-\rho^2 q^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - (1-q)x^2 q^k\right)}{(1-\rho^2 q^{2k})^2 - (1-q)\rho q^k (1+\rho^2 q^{2k}) xy + (1-q)\rho^2 (x^2+y^2) q^{2k}}.$$

Proof. The distribution of the q-Hermite polynomials $H_n(x|q)$ is supported on $x^2 < 4/(1-q)$ with the density

$$f_H(x) = \frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \left((1+q^k)^2 - (1-q)x^2 q^k\right) \prod_{k=0}^{\infty} (1-q^{k+1});$$

see [ISV87, (2.15)]. Moreover, since $|H_n(x)| \le C_q(n+1)(1-q)^{-n/2}$ when $x^2, y^2 \le 4/(1-q)$, the series

$$g_H(x,y,\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x) H_n(y) \tag{2.4}$$

converges uniformly and defines the Poisson-Mehler kernel, which is given by

$$g_H(x,y,\rho) = \prod_{k=0}^{\infty} \frac{(1-\rho^2 q^k)}{(1-\rho^2 q^{2k})^2 - (1-q)\rho q^k (1+\rho^2 q^{2k}) xy + (1-q)\rho^2 (x^2+y^2) q^{2k}}; \tag{2.5}$$

this is the renormalized version of the well-known result; see e.g. [IS88, (2.2)], which considers the q-Hermite polynomials given by $\{(1-q)^{n/2} H_n(2x/\sqrt{1-q}|q)\}$ instead of our $\{H_n(x|q)\}$.

Since (1.3) implies that $\int H_n^2(x|q) f_H(x) dx = [n]_q!$, it follows from (2.4) that

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x|q) g_H(x,y,\rho) f_H(x)\, dx = \rho^n H_n(y).$$

□

Corollary 3. *If $q, a, b \in \mathbb{R}$ are such that $|q| < 1$, $0 < b < 1$, and $a^2(1-q) < 4b$, then the distribution of the Al-Salam–Chihara polynomials $\{p_n(x|q,a,b)\}$ is absolutely continuous with the density on $x^2 < 4/(1-q)$ given by*

$$\frac{\sqrt{1-q}}{2\pi\sqrt{4-(1-q)x^2}} \prod_{k=0}^{\infty} \frac{(1-bq^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - (1-q)x^2 q^k\right)}{(1-bq^{2k})^2 - (1-q) a q^k (1+bq^{2k}) x + (1-q)(bx^2+a^2) q^{2k}}.$$

Proof. By Theorem 2, the distribution of polynomials p_n solves (2.1) with $\rho = \sqrt{b}, y = a/\rho$. Thus the formula follows from Corollary 2. □

Remark 2. Iterating (2.1) we see that the measure corresponding to the parameter $\rho_1\rho_2$ instead of ρ is given by

$$\mu(\cdot|\rho_1\rho_2, x) = \int \mu(\cdot|\rho_1, y) \mu(dy|\rho_2, x). \tag{2.6}$$

For $|q| < 1, |\rho| < 1$ the density of μ is given in Corollary 2; hence, after simplifying common factors and substituting $x = 2\xi/\sqrt{1-q}$, $y = 2\eta/\sqrt{1-q}$, $z = 2\zeta/\sqrt{1-q}$, the relationship (2.6) takes the following form:

$$\begin{aligned} &\int_{-1}^{1} \prod_{k=0}^{\infty} \frac{(1-\rho_1^2 q^k)\left(1-q^{k+1}\right)\left((1+q^k)^2 - 4\eta^2 q^k\right)}{(1-\rho_1^2 q^{2k})^2 - 4\rho_1 q^k (1+\rho_1^2 q^{2k}) \eta\zeta + 4\rho_1^2(\eta^2+\zeta^2) q^{2k}} \\ &\quad \times \prod_{k=0}^{\infty} \frac{(1-\rho_2^2 q^k)}{(1-\rho_2^2 q^{2k})^2 - 4\rho_2 q^k (1+\rho_2^2 q^{2k}) \xi\eta + 4\rho_2^2(\eta^2+\xi^2) q^{2k}} \frac{d\eta}{2\pi\sqrt{1-\eta^2}} \\ = &\prod_{k=0}^{\infty} \frac{(1-\rho_1^2\rho_2^2 q^k)}{(1-\rho_1^2\rho_2^2 q^{2k})^2 - 4\rho_1\rho_2 q^k (1+\rho_1^2\rho_2^2 q^{2k}) \xi\zeta + 4\rho_1^2\rho_2^2(\zeta^2+\xi^2) q^{2k}}. \end{aligned}$$

3. Determinants of Hankel matrices

In this section we are interested in calculating the determinants of the Hankel matrices

$$M_n = [m_{i+j}]_{i,j=0,\dots n-1},$$

where $m_i = \int x^i \mu(dx)$ are the moments of a certain (perhaps signed) measure μ. It is well known that for positive measures we must have $\det M_n \geq 0$, and that these determinants can be read out from the three-term recurrence for the corresponding monic orthogonal polynomials.

Consider first the moments $m_k(y) = \int x^k \mu(dx)$ of the (perhaps signed) measure $\mu = \mu_{y,\rho}$, which solves (2.1). Then $m_k(y)$ are polynomials of degree k in the variable y and can be written as follows. Let $a_{n,2i}$, $i \leq \lfloor n/2 \rfloor$ be the coefficients in the expansion of the monomial x^n into the q-Hermite polynomials,

$$x^n = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,2i} H_{n-2i}\left(x|q\right),\ n \geq 0.$$

Then

$$m_n(y) = \int x^n d\mu\left(x\right) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho^{n-2k} a_{n,2k} H_{n-2k}\left(y|q\right).$$

Let S_n be the Hankel matrix of moments $m_k(y)$,

$$S_n(y|q,\rho) = \begin{bmatrix} m_0(y) & m_1(y) & \dots & m_{n-1}(y) \\ m_1(y) & m_2(y) & & \\ \vdots & & \ddots & \vdots \\ m_{n-1}(y) & & \dots & m_{2n-2}(y) \end{bmatrix}.$$

It is well known that $\det S_n$ is the product of the coefficients at the third term of (2.2), which implies the following.

Corollary 4. $\det S_{n+1}/\det S_n = [n]_q! \prod_{i=1}^n \left(1-\rho^2 q^{i-1}\right)$.

Our second Hankel matrix has an even simpler form. As indicated in [IS97], [IS02] the q-Hermite polynomials can be viewed as moments of a signed measure, $H_n(x|q) = \int u^n \mu(du|x,q)$. It turns out that if $q \neq 0$, the measure $\mu(du|x,q)$ cannot be positive even for a single value of x. To see this, consider the following $n \times n$ matrices:

$$M_n(x|q) = \begin{bmatrix} H_0(x|q) & H_1(x|q) & H_2(x|q) & \dots & H_{n-1}(x|q) \\ H_1(x|q) & H_2(x|q) & & & H_n(x|q) \\ H_2(x|q) & & \ddots & & H_{n+1}(x|q) \\ \vdots & & & \ddots & \vdots \\ H_{n-1}(x|q) & H_n(x|q) & & \dots & H_{2n-2}(x|q) \end{bmatrix}.$$

The following q-generalization of [Kra99, (3.55)] shows that the determinants $\det M_n(x|q)$ are free of the variable x and take negative values.

Theorem 3.

$$\frac{\det M_{n+1}}{\det M_n} = (-1)^n q^{n(n-1)/2} [n]_q!.$$

Proof. Using (1.3), we row-reduce the first column of the matrix. Namely, from the second row of M_{n+1}, we subtract the first one multiplied by x. Similarly, for $i \geq 3$, we subtract x times row $i-1$ and add the $(i-2)$-th row multiplied by $[i-1]_q$. Taking (1.3) into account, $\det M_{n+1}(x|q)$ becomes

$$\det \begin{bmatrix} H_0 & H_1 & H_2 & \dots & H_n \\ 0 & ([0]-[1])H_0 & ([0]-[2])H_1 & & ([0]-[n])H_{n-1} \\ 0 & ([1]-[2])H_1 & ([1]-[3])H_2 & & ([1]-[n+1])H_n \\ \vdots & & & \ddots & \vdots \\ 0 & ([n-1]-[n])H_{n-1} & ([n-1]-[n+1])H_n & \dots & ([n-1]-[2n-1])H_{2n-2} \end{bmatrix}.$$

Now, we use the fact that for $m \leq n$ we have $[n]_q - [m]_q = q^m [n-m]_q$. Thus $\det M_{n+1}(x|q)$ becomes

$$\det \begin{bmatrix} H_0 & H_1 & H_2 & \dots & H_{n-1} \\ 0 & -[1]H_0 & -[2]H_1 & & -[n]H_{n-1} \\ 0 & -q[1]H_1 & -q[2]H_2 & & -q[n]H_n \\ \vdots & & & \ddots & \vdots \\ 0 & -q^{n-1}[1]H_{n-1} & -q^{n-1}[2]H_n & \dots & -q^{n-1}[n]H_{2n-2} \end{bmatrix}.$$

Expanding $\det M_{n+1}$ with respect to the first column, and factoring out the common factors $-q^{i-1}$ from the i-th row and $[j]_q$ from the j-th column of the resulting matrix, we get

$$\det M_{n+1} = (-1)^n q^{\sum_{i=1}^{n-1} i} \prod_{j=1}^{n} [j] \det M_n = (-1)^n q^{n(n-1)/2} [n]_q! \det M_n. \qquad \square$$

The formula stated in Corollary 4 was originally discovered through symbolic computations and motivated this paper. We were unable to find a direct algebraic proof along the lines of the proof of Theorem 3, and our search for the explanation of why $\det S_n(y)$ does not depend on y led us to Al-Salam–Chihara polynomials and identity (1.8).

The fact that Hankel determinants formed of certain linear combinations of the q-Hermite polynomials do not depend on the argument of these polynomials as exposed in Theorem 3 and Corollary 4 is striking and unexpected to us. A natural question arises whether other linear combinations have this property.

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