

A BIG PICARD THEOREM FOR QUASIREGULAR MAPPINGS INTO MANIFOLDS WITH MANY ENDS

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ABSTRACT. We study quasiregular mappings from a punctured Euclidean ball into n -manifolds with many ends and prove, by using Harnack's inequality, a version of the big Picard theorem.

1. INTRODUCTION

The classical big Picard theorem states that a holomorphic mapping of the punctured unit disc $\{z \in \mathbb{C} : 0 < |z| < 1\}$ into the complex plane omitting two values has a meromorphic extension to the whole disc; see e.g. [A, Theorem 1-14]. In [R1] Rickman proved a counterpart of Picard's theorem for quasiregular mappings (Theorem 1.1) and its local version (Theorem 1.2) corresponding to the big Picard theorem.

Theorem 1.1 ([R1]). *For each integer $n \geq 2$ and each $K \geq 1$ there exists a positive integer $q = q(n, K)$ such that if $f: \mathbb{R}^n \rightarrow \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ is K -quasiregular and a_1, \dots, a_q are distinct points in \mathbb{R}^n , then f is constant.*

Theorem 1.2 ([R1]). *Let $G = \{x \in \mathbb{R}^n : |x| > s\}$, and let $f: G \rightarrow \mathbb{R}^n \setminus \{a_1, \dots, a_q\}$ be a K -quasiregular mapping, where a_1, \dots, a_q are distinct points in \mathbb{R}^n and $q = q(n, K)$ is the integer in Theorem 1.1. Then the limit $\lim_{|x| \rightarrow \infty} f(x)$ exists.*

Recall that a continuous mapping $f: M \rightarrow N$ between smooth, oriented, and connected Riemannian n -manifolds M and N is called K -quasiregular if it belongs to the Sobolev space $W_{\text{loc}}^{1,n}(M, N)$ and satisfies an inequality

$$\|T_x f\|^n \leq KJ(x, f)$$

for a.e. $x \in M$. Here $\|T_x f\|$ is the operator norm of the tangent map $T_x f: T_x M \rightarrow T_{f(x)} N$, and $J(x, f)$ is the Jacobian determinant of f at x uniquely defined by $(f^* \text{vol}_N)_x = J(x, f)(\text{vol}_M)_x$ almost everywhere.

In this paper we consider an analogue of the big Picard theorem for quasiregular mappings of the punctured Euclidean unit ball $B^n \setminus \{0\}$ into a Riemannian n -manifold with sufficiently many ends. Given a compact set $C \subset N$ we say that a component, E , of $N \setminus C$ is *an end of N with respect to C* if the closure \bar{E} is

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noncompact. Furthermore, we say that N has at least q ends if N has at least q ends with respect to some compact set $C \subset N$. We say that a mapping $f: B^n \setminus \{0\} \rightarrow N$ has a removable singularity at the origin, if a limit $\lim_{x \rightarrow 0} f(x) \in N$ exists or, for every compact set $C \subset N$, there exists $r \in (0, 1)$ such that $f(B^n(0, r) \setminus \{0\})$ is contained in an end with respect to C . Otherwise, we say that f has an essential singularity at the origin.

We can now state our main result, which is a generalization of Theorem 1.2.

Theorem 1.3. *Let N be a Riemannian n -manifold. For every $K \geq 1$ there exists $q = q(K, n)$ such that every K -quasiregular mapping $f: B^n \setminus \{0\} \rightarrow N$ has a removable singularity at the origin if N has at least q ends.*

It is worth noting that the proof of Theorem 1.2 makes use of both of the main tools in the theory of quasiregular mappings: the method of extremal length and the nonlinear potential theory. In particular, relatively sharp estimates for the modulus of certain path families are needed both in the domain and in the range of the mapping, thus making it hard to extend the method beyond the Euclidean setting. In [HR3] Holopainen and Rickman applied a method of Lewis ([L]) that relies on Harnack's inequality to prove the following general version of Picard's theorem; see also [EL], [HR1], and [HR2] for other proofs and versions of Picard's theorem.

Theorem 1.4 ([HR3]). *Suppose that a complete Riemannian n -manifold M admits a global Harnack's inequality for positive \mathcal{A} -harmonic functions of type n . Assume, furthermore, that M has the following covering property: for each $0 < k < 1$ there exists an integer $m = m(k)$ such that every ball $B(x, r) \subset M$ can contain at most m disjoint balls of radius kr . Then, for every $K \geq 1$, there exists a constant q such that every K -quasiregular mapping $f: M \rightarrow N$ must be constant if N has at least q ends.*

The proof of our main result (Theorem 1.3) is closely related to the proof of Theorem 1.4. Thus our proof relies on Harnack's inequality for positive \mathcal{A} -harmonic functions. We recall that a continuous function $u: N \rightarrow \mathbb{R}$ is \mathcal{A} -harmonic (of type $p \in (1, \infty)$) if $u \in W_{\text{loc}}^{1,p}(N)$ and satisfies an equation

$$-\operatorname{div} \mathcal{A}_x(\nabla u) = 0$$

weakly, that is,

$$\int_N \langle \mathcal{A}_x(\nabla u), \nabla \varphi \rangle dx = 0$$

for all $\varphi \in C_0^\infty(N)$. Here $\mathcal{A}: TN \rightarrow TN$ is a map such that $\mathcal{A}_x: T_x N \rightarrow T_x N$ is continuous for a.e. $x \in N$ and the map $x \mapsto \mathcal{A}_x(X)$ is measurable for every measurable vector field X . We assume that there exist constants $0 < \alpha \leq \beta < \infty$ such that

$$\langle \mathcal{A}_x(\xi), \xi \rangle \geq \alpha |\xi|^p$$

and

$$|\mathcal{A}_x(\xi)| \leq \beta |\xi|^{p-1}$$

for a.e. $x \in N$ and for all $\xi \in T_x N$. We assume further that for a.e. $x \in N$,

$$\langle \mathcal{A}_x(\xi) - \mathcal{A}_x(\eta), \xi - \eta \rangle > 0$$

whenever $\xi \neq \eta$, and

$$\mathcal{A}_x(\lambda \xi) = |\lambda|^{p-2} \lambda \mathcal{A}_x(\xi)$$

whenever $\lambda \in \mathbb{R} \setminus \{0\}$. In the special case $\mathcal{A}_x(\xi) = |\xi|^{p-2}\xi$, \mathcal{A} -harmonic functions are called p -harmonic.

In the sequel we use repeatedly the fact that quasiregular mappings are \mathcal{A} -harmonic morphisms (see e.g. [HKM]), especially the fact that whenever $f: M \rightarrow N$ is K -quasiregular and $u: N \rightarrow \mathbb{R}$ is n -harmonic, then $u \circ f$ is an \mathcal{A} -harmonic function, where

$$\mathcal{A}_x(\xi) = \langle G_x \xi, \xi \rangle^{n/2-1} G_x \xi,$$

with

$$G_x = \begin{cases} J(x, f)^{2/n} T_x f^{-1} (T_x f^{-1})^*, & \text{if } J(x, f) > 0; \\ \text{id}, & \text{otherwise.} \end{cases}$$

Here $(T_x f^{-1})^*$ is the transpose of the linear map $T_x f^{-1}$. We can choose $\alpha = 1/K$ and $\beta = K^{n-1}$ as the ellipticity constants of \mathcal{A} .

2. A VERSION OF LEWIS' LEMMA

For our main theorem we need uniform bounds for the growth of an \mathcal{A} -harmonic function near its zero set. This calls for a version of *Lewis' lemma* [L]. Let us first recall the definition of a Harnack function. Let M be a Riemannian manifold. A function $u: M \rightarrow \mathbb{R}$ is called a *Harnack function with constant θ* if

$$M(h, x, r) := \sup_{B(x, r)} h \leq \theta \inf_{B(x, r)} h$$

holds in each ball $B(x, r)$ whenever the function h is nonnegative in $B(x, 2r)$, has the form $h = \pm u + a$ for some $a \in \mathbb{R}$, and $\bar{B}(x, 2r) \subset M$ is compact. The original version of Lewis' lemma is stated for Harnack functions. It is well known (see [HKM, 6.2]) that \mathcal{A} -harmonic functions in the Euclidean setting are Harnack functions with some θ depending only on n and on the constants p , α , and β of \mathcal{A} . In that case θ is called the Harnack constant of \mathcal{A} . Since we consider \mathcal{A} -harmonic functions only on Euclidean domains, we state our lemmata in that context. We write $S^{n-1}(x, r) = \{y \in \mathbb{R}^n : |y - x| = r\}$ for the sphere of center x and radius r .

Lemma 2.1. *Let $u: \mathbb{R}^n \setminus \bar{B}^n \rightarrow \mathbb{R}$ be an \mathcal{A} -harmonic function such that*

$$\limsup_{|x| \rightarrow \infty} u(x) = +\infty \quad \text{and} \quad \liminf_{|x| \rightarrow \infty} u(x) = -\infty.$$

Then for every $C_0 > 0$ there exist $R > 1$ and $x_0 \in S^{n-1}(0, 100R)$ such that $u(x_0) = 0$ and $M(u, x_0, R) \geq C_0$.

Proof. For further purposes let us fix a positive integer k_0 such that if $r > 0$, x and y are points in $S^{n-1}(0, r)$, and J is the shortest arc on $S^{n-1}(0, r)$ connecting the points x and y , then there are points $x_i \in J$, $i = 1, \dots, k_0$, such that the balls $B^n(x_i, r/1000)$ cover J .

Let $C_0 > 0$. By the assumptions on u , there exists $r > 1$ such that $u(z) = 0$ for some $z \in S^{n-1}(0, 100r)$ and

$$\max_{S^{n-1}(0, 100r)} u \geq \theta^{k_0} C_0,$$

where $\theta > 1$ is the Harnack constant of \mathcal{A} . Fix $y \in S^{n-1}(0, 100r)$ such that $u(y) = \max_{S^{n-1}(0, 100r)} u$. We also fix $x \in S^{n-1}(0, 100r)$ such that $u(x) = 0$ and $d(x, y) = \text{dist}(\{y\}, S^{n-1}(0, 100r) \cap u^{-1}(0))$. Here $d(x, y)$ is the distance of points x and y along the sphere $S^{n-1}(0, 100r)$. The distance from y to the set $S^{n-1}(0, 100r) \cap u^{-1}(0)$

is also measured along the sphere. Let J be the shortest arc along $S^{n-1}(0, 100r)$ connecting the points x and y . Then u is positive on $J \setminus \{x\}$ by the choice of x .

Let us fix points $x_i \in J$, $i = 1, \dots, k_0$, such that the balls $B^n(x_i, r/10)$ cover J . We arrange points x_i in such a way that $y \in B^n(x_1, r/10)$, $x \in B^n(x_{k_0}, r/10)$, and $B^n(x_i, r/10) \cap B^n(x_{i+1}, r/10) \neq \emptyset$ for all i . Let k be the smallest index such that $B^n(x_k, r/5) \cap u^{-1}(0) \neq \emptyset$, and fix a point $z \in B^n(x_k, r/5) \cap u^{-1}(0)$. Since

$$|z| \geq |x_k| - |z - x_k| \geq 100r - r/5 > 99r,$$

we have that $B^n(x_k, r/5) \subset B^n(z, 2r/5) \subset B^n(z, |z|/100)$. Thus

$$M(u, z, |z|/100) \geq M(u, x_k, r/5).$$

Next we show that

$$M(u, x_k, r/5) \geq C_0.$$

If $k = 1$, we have that

$$M(u, x_1, r/5) \geq u(y) \geq \theta^{k_0} C_0 \geq C_0.$$

If $k > 1$, we may use Harnack's inequality repeatedly in pairs of concentric balls $B^n(x_i, r/10)$ and $B^n(x_i, r/5)$ since u is positive in the set $\bigcup_{i=1}^{k-1} B^n(x_i, r/5)$ and $B^n(x_i, r/10) \cap B^n(x_{i+1}, r/10) \neq \emptyset$. Hence

$$M(u, x_k, r/5) \geq \inf_{B^n(x_{k-1}, r/10)} u \geq \theta^{1-k} M(u, x_1, r/10) \geq \theta^{1-k} \theta^{k_0} C_0 \geq C_0.$$

The claim now follows by choosing $x_0 = z$ and $R = |z|/100$. □

Lemma 2.2 (Lewis' lemma). *Let $u: \mathbb{R}^n \setminus \bar{B}^n \rightarrow \mathbb{R}$ be an \mathcal{A} -harmonic function as in Lemma 2.1. Then for every $C_0 > 0$ there exists a ball $B^n(a, r)$ such that*

- (a) $B^n(a, 6r) \subset \mathbb{R}^n \setminus \bar{B}^n$,
- (b) $u(a) = 0$, and
- (c) $C_0 \leq M(u, a, 6r) \leq \theta^6 M(u, a, r)$,

where θ is the Harnack constant of \mathcal{A} .

Proof. Since the proof of this lemma is almost verbatim to the proof of [IM, Lemma 19.4.1], we discuss only the modifications needed at the beginning of the proof. Fix $C_0 > 0$, and let $R > 1$ and $x_0 \in S^{n-1}(0, 100R)$ be given by Lemma 2.1. Let $Z = \{x \in \mathbb{R}^n \setminus \bar{B}^n : u(x) = 0\}$ and $Z_R = Z \cap \bar{B}^n(x_0, 41R)$. For each $x \in Z_R$ we set

$$r_x = R - |x - x_0|/41.$$

As in the proof of [IM, Lemma 19.4.1] we find a point $a \in Z_R$ such that

$$M(u, a, r_a) = \sup\{M(u, x, r_x) : x \in Z_R\} \geq M(u, x_0, R) \geq C_0.$$

Now the point a and radius $r = r_a$ satisfy the conditions of the claim. Indeed, conditions (a) and (b) as well as the inequality on the left-hand side of (c) are clearly satisfied. The Harnack type inequality on the right-hand side of (c) can be proved as the corresponding inequality in [IM, Lemma 19.4.1]. □

3. BIG PICARD'S THEOREM

We say that a Riemannian manifold M is p -parabolic, or, equivalently, has zero p -capacity at infinity if, for each compact set $C \subset M$,

$$\text{cap}_p(M, C) := \inf_u \int_M |\nabla u|^p dm = 0,$$

where the infimum is taken over all functions $u \in C_0^\infty(M)$, with $u \geq 1$ in C . Otherwise, we say that M is p -hyperbolic.

The following lemma describes the behavior of a quasiregular mapping $f: B^n \setminus \{0\} \rightarrow N$ in a neighborhood of 0 assuming that f has an essential singularity at 0. See [R2, III.2.9, III.2.11] for the case $N = \mathbb{R}^n$. Since the proof in our setting involves only minor changes to the proofs in the Euclidean case, we just sketch the idea.

Lemma 3.1. *Let N be a Riemannian n -manifold and $f: B^n \setminus \{0\} \rightarrow N$ a quasiregular mapping having an essential singularity at the origin. Then $f(B^n(0, r) \setminus \{0\})$ is n -parabolic for every $r \in (0, 1)$.*

Proof. Suppose on the contrary that $f(B^n(0, r_0) \setminus \{0\})$ is n -hyperbolic for some $r_0 \in (0, 1)$. Since f has an essential singularity at the origin, we find a compact set $C \subset N$ such that $f(B^n(0, r) \setminus \{0\}) \cap C \neq \emptyset$ for every $r \in (0, 1)$. Hence we may choose a sequence (x_k) of points in $B^n(0, r_0) \setminus \{0\}$ tending to the origin such that $f(x_k) \rightarrow z$ for some $z \in C$. Let (y_k) be another sequence in $B^n(0, r_0) \setminus \{0\}$ tending to the origin. We show that $f(y_k) \rightarrow z$, too. If this is not the case, we may assume that $d(f(x_k), f(y_k)) \geq \delta > 0$ for all k . Let F_k be a continuum connecting x_k and y_k in $\bar{B}^n(0, \max\{|x_k|, |y_k|\}) \setminus \{0\}$. Then

$$(3.1) \quad \text{cap}_n(f(B^n(0, r_0) \setminus \{0\}), f(F_k)) \leq K_I(f) \text{cap}_n(B^n(0, r_0) \setminus \{0\}, F_k)$$

by the K_I -inequality (see [R2, II.10.10]). Here $K_I(f)$ is the inner dilatation of f . On the other hand,

$$(3.2) \quad \text{cap}_n(B^n(0, r_0) \setminus \{0\}, F_k) \rightarrow 0$$

as $k \rightarrow \infty$. However, there exists $s > 0$ such that $f(F_k)$ connects $B(z, s)$ and $N \setminus B(z, 4s)$ for every sufficiently large k and such that $B(z, 4s)$ is contained in a bilipschitz chart. Hence there exists a constant $c > 0$, independent of k , such that, for all sufficiently large k ,

$$\text{cap}_n(f(B^n(0, r_0) \setminus \{0\}), f(F_k)) \geq c > 0$$

by the assumption that $f(B^n(0, r_0) \setminus \{0\})$ be n -hyperbolic. This contradicts (3.1) and (3.2). Hence $f(y_k) \rightarrow z$, but this contradicts the essential singularity assumption on f . Therefore $f(B^n(0, r) \setminus \{0\})$ is n -parabolic for every $r \in (0, 1)$. \square

The following lemma is originally given in [HR1]. See also the remark after [HR3, Lemma 3.2].

Lemma 3.2 ([HR1],[HR3]). *Let N be an n -parabolic Riemannian manifold. Suppose that $C \subset N$ is compact such that N has q ends V_1, \dots, V_q with respect to C . Then there exist n -harmonic functions v_j , $j = 2, \dots, q$, and a positive constant κ*

such that

- (3.3) $|v_j| \leq \kappa$ in C ,
- (3.4) $|v_j - v_i| \leq 2\kappa$ in V_1 ,
- (3.5) $\sup_{V_1} v_j = \infty$,
- (3.6) $\inf_{V_j} v_j = -\infty$,
- (3.7) v_j is bounded in V_k for $k \neq 1, j$,
- (3.8) if $v_j(x) > \kappa$, then $x \in V_1$,
- (3.9) if $v_j(x) < -\kappa$, then $x \in V_j$.

Proof of Theorem 1.3. The proof is similar to that of Theorem 1.4 given in [HR3]. However, for the reader's convenience we give all the details. Suppose that there exists a K -quasiregular mapping $f: B^n \setminus \{0\} \rightarrow N$ having an essential singularity at the origin. Let $C \subset N$ be a compact set such that N has q ends V_1, \dots, V_q with respect to C . For every $j = 2, \dots, q$ let us fix an n -harmonic function v_j with properties (3.3) - (3.9) given in Lemma 3.2. For every $j = 2, \dots, q$ we set $u_j = v_j \circ f \circ \sigma$, where σ is a Möbius transformation such that $\sigma(\mathbb{R}^n \setminus B^n) = B^n \setminus \{0\}$. Thus functions u_j are \mathcal{A} -harmonic in $\mathbb{R}^n \setminus \bar{B}^n$. Next we show that they satisfy the assumptions of Lemma 2.1, that is,

$$(3.10) \quad \limsup_{|x| \rightarrow \infty} u_j(x) = +\infty \quad \text{and} \quad \liminf_{|x| \rightarrow \infty} u_j(x) = -\infty.$$

This can be seen by observing that the sets $\{x \in N: v_j(x) > c\}$ and $\{x \in N: v_j(x) < -c\}$ are nonempty and open for every $c > 0$ and $j = 2, \dots, q$. By Lemma 3.1, $f(B^n(0, r) \setminus \{0\})$ intersects these sets for every $r \in (0, 1)$, and therefore (3.10) follows. By Lemma 2.2 there are sequences (x_i) and (r_i) of $\mathbb{R}^n \setminus \bar{B}^n$ and $(0, \infty)$, respectively, such that $u_2(x_i) = 0$, $B^n(x_i, 3r_i) \subset \mathbb{R}^n \setminus \bar{B}^n$,

$$M(u_2, x_i, 3r_i) \leq \theta^6 M(u_2, x_i, r_i/2),$$

and $M(u_2, x_i, r_i/2) \rightarrow \infty$ as $i \rightarrow \infty$. Let us fix an index i such that $M(u_2, x_i, r_i/2) \geq 4\theta\kappa$, where $\theta > 1$ is the Harnack constant of \mathcal{A} and κ is the constant in Lemma 3.2. We write $x = x_i$ and $r = r_i$. By (3.8), $(f \circ \sigma)(B^n(x, r/2)) \cap V_1 \neq \emptyset$. Thus, by (3.4), we have

$$(3.11) \quad M(u_2, x, s) - 2\kappa \leq M(u_j, x, s) \leq M(u_2, x, s) + 2\kappa$$

whenever $s \geq r/2$. Next we conclude by using Harnack's inequality that

$$(3.12) \quad M(u_j, x, r) \leq (\theta - 1)M(-u_j, x, 2r)$$

for all j . Let us first show that $u_j(z) = 0$ for some $z \in B^n(x, r)$. Suppose on the contrary that $u_j > 0$ in $B^n(x, r)$. Then $u_j(y) \leq \theta u_j(x)$ for all $y \in B^n(x, r/2)$ by Harnack's inequality. Since $M(u_2, x, r/2) \geq 4\theta\kappa$, there exists $y \in B^n(x, r/2)$ such that $u_j(y) > 2\theta\kappa$ by (3.11). Thus $u_j(x) > 2\kappa$, and so $x \in V_1$. By (3.4), $u_2(x) \geq u_j(x) - 2\kappa > 0$, contradicting the assumption $u_2(x) = 0$. Therefore there exists $z \in B^n(x, r)$ such that $u_j(z) = 0$. Thus $\inf_{B^n(x, r)} u_j \leq 0$. Inequality (3.12)

follows now from the calculation

$$\begin{aligned}
 M(u_j, x, r) &= \sup_{B^n(x, r)} u_j = \sup_{B^n(x, r)} \left(u_j - \inf_{B^n(x, 2r)} u_j \right) + \inf_{B^n(x, 2r)} u_j \\
 &\leq \theta \inf_{B^n(x, r)} \left(u_j - \inf_{B^n(x, 2r)} u_j \right) + \inf_{B^n(x, 2r)} u_j \\
 &= \theta \inf_{B^n(x, r)} u_j + (1 - \theta) \inf_{B^n(x, 2r)} u_j \\
 &\leq -(\theta - 1) \inf_{B^n(x, 2r)} u_j = (\theta - 1) \sup_{B^n(x, 2r)} (-u_j) \\
 &= (\theta - 1)M(-u_j, x, 2r),
 \end{aligned}$$

since $u_j - \inf_{B^n(x, 2r)} u_j \geq 0$ in $B^n(x, 2r)$.

Inequalities (3.11) and (3.12), and the assumption $M(u_2, x, r/2) \geq 4\theta\kappa$ together yield the inequality

$$(3.13) \quad M(u_2, x, r) \leq \theta M(-u_j, x, 2r).$$

Indeed,

$$\begin{aligned}
 M(u_2, x, r) &\leq M(u_j, x, r) + \theta^{-1}M(u_2, x, r) \\
 &\leq (\theta - 1)M(-u_j, x, 2r) + \theta^{-1}M(u_2, x, r),
 \end{aligned}$$

which is equivalent to (3.13). We fix $z_j \in \bar{B}^n(x, 2r)$ such that

$$(3.14) \quad M(-u_j, x, 2r) = -u_j(z_j).$$

The well-known oscillation estimate (see e.g. [HKM, 6.6])

$$\operatorname{osc}_{B^n(y, \rho)} u_j \leq c(\rho/r)^\gamma \operatorname{osc}_{B^n(y, r)} u_j$$

together with [HR1, Lemma 4.2] and (3.11) imply that

$$(3.15) \quad \operatorname{osc}_{B^n(z_j, \rho)} u_j \leq c_1(\rho/r)^\gamma M(u_2, x, 3r)$$

for $\rho \in (0, r)$. See [HR1, (5.5)] for details. Here c , c_1 , and γ depend only on n and K , since $\theta = \theta(n, K)$. Thus

$$\begin{aligned}
 \max_{\bar{B}^n(z_j, \rho)} u_j &= \operatorname{osc}_{B^n(z_j, \rho)} u_j + \min_{\bar{B}^n(z_j, \rho)} u_j \\
 &\leq c_1(\rho/r)^\gamma M(u_2, x, 3r) + u_j(z_j) \\
 &\leq c_1(\rho/r)^\gamma M(u_2, x, 3r) - \theta^{-1}M(u_2, x, r)
 \end{aligned}$$

by (3.15), (3.14), and (3.13). Since $M(u_2, x, 3r) \leq \theta^6 M(u_2, x, r)$, we obtain

$$c_2(\rho/r)^\gamma M(u_2, x, 3r) \leq (2\theta)^{-1}M(u_2, x, r)$$

by choosing $\rho = (2\theta^7 c_2)^{-1/\gamma} r$. Hence

$$\max_{\bar{B}^n(z_j, \rho)} u_j \leq -(2\theta)^{-1}M(u_2, x, r) \leq -2\kappa.$$

By (3.9), we conclude that $(f \circ \sigma)(B^n(z_j, \rho)) \subset V_j$ and hence the balls $B^n(z_j, \rho)$ are disjoint. Since $B^n(z_j, \rho) \subset B^n(x, 3r)$, there can be at most $(3r/\rho)^n$ of them. Hence $q \leq (3r/\rho)^n$, which is an upper bound that depends only on n and K . \square

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