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ON THE BETTI NUMBERS OF SIGN CONDITIONS

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Abstract. Let R be a real closed field and let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathrm{R}[X_1,\ldots,X_k]$ such that the set $\mathcal{P}$ has s elements, the algebraic set Z defined by $\bigwedge_{Q\in\mathcal{Q}} Q=0$ has dimension k' and the elements of$\mathcal{Q}$ and $\mathcal{P}$ have degree at most d. For each $0\le i\le k'$, we denote the sum of the i-th Betti numbers over the realizations of all sign conditions of $\mathcal{P}$ on Z by $b_i(\mathcal{P},\mathcal{Q})$. We prove that

$$b_i(\mathcal{P},\mathcal{Q}) \le \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

This generalizes to all the higher Betti numbers the bound $\binom{s}{k'}O(d)^k$ on $b_0(\mathcal{P},\mathcal{Q})$. We also prove, using similar methods, that the sum of the Betti numbers of the intersection of Z with a closed semi-algebraic set, defined by a quantifier-free Boolean formula without negations with atoms of the form $P\ge 0$ or $P\le 0$ for $P\in\mathcal{P}$, is bounded by

$$\sum_{i=0}^{k'}\sum_{j=0}^{k'-i}\binom{s}{j}6^j d(2d-1)^{k-1},$$

making the bound $s^{k'}O(d)^k$ more precise.

1. Introduction

Let R be a real closed field. For an element $a\in\mathrm{R}$ we define

$$\mathrm{sign}(a)=\begin{cases}0 & \text{if } a=0,\\ 1 & \text{if } a>0,\\ -1 & \text{if } a<0.\end{cases}$$

Let $\mathcal{Q}$ and $\mathcal{P}$ be finite subsets of $\mathrm{R}[X_1,\ldots,X_k]$. A *sign condition* on $\mathcal{P}$ is an element of $\{0,1,-1\}^{\mathcal{P}}$.

For $r>0$ we define the sets Z and Z_r by

$$Z=\mathcal{R}(\bigwedge_{Q\in\mathcal{Q}} Q=0)=\{x\in\mathrm{R}^k \mid \bigwedge_{Q\in\mathcal{Q}} Q(x)=0\},\ \ Z_r=Z\cap B(0,r).$$

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The *realization of the sign condition σ over Z*, $\mathcal{R}(\sigma, Z)$, is the basic semi-algebraic set

$$\{x \in \mathrm{R}^k \mid \bigwedge_{Q\in\mathcal{Q}} Q(x) = 0 \wedge \bigwedge_{P\in\mathcal{P}} \mathrm{sign}(P(x)) = \sigma(P)\}.$$

The *realization of the sign condition σ over Z_r*, $\mathcal{R}(\sigma, Z_r)$, is the basic semi-algebraic set $\mathcal{R}(\sigma, Z) \cap B(0, r)$.

For the rest of the paper, we fix an open ball $B(0, r)$ with center 0 and radius r big enough so that, for every sign condition σ, $\mathcal{R}(\sigma, Z)$ and $\mathcal{R}(\sigma, Z_r)$ are homeomorphic. This is always possible by the local conical structure at infinity of semi-algebraic sets ([5], page 225).

A closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$ is semi-algebraically triangulable (see [5]), and we denote by $H_i(S)$ the i-th simplicial homology group of S with rational coefficients. The groups $H_i(S)$ are invariant under semi-algebraic homeomorphisms and coincide with the corresponding singular homology groups when $\mathrm{R} = \mathbb{R}$. We denote by $b_i(S)$ the i-th Betti number of S (that is, the dimension of $H_i(S)$ as a vector space), and by $b(S)$ the sum $\sum_i b_i(S)$. For a closed but not necessarily bounded semi-algebraic set $S \subset \mathrm{R}^k$, we will denote by $H_i(S)$ the i-th simplicial homology group of $S \cap \overline{B(0, r)}$, where r is sufficiently large. This is well-defined using the local conical structure at infinity of semi-algebraic sets ([5], page 225).

The definition of homology groups of arbitrary semi-algebraic sets in R^k requires some care, and several possibilities exist. In this paper, we define the homology groups of realizations of sign conditions as follows. Let $\sigma \in \{0, 1, -1\}^{\mathcal{P}}$, and let $S_t \subset \mathrm{R}^k, t \in (0, \infty]$ be any semi-algebraic family of closed and bounded sets, satisfying $\bigcup_{0<t} S_t = \mathcal{R}(\sigma, Z_r)$ and $t_1 > t_2 \Rightarrow S_{t_1} \subset S_{t_2}$. It follows from Hardt's triviality theorem [6] that there exists $t_0 > 0$ such that for all $t \in (0, t_0]$, S_t is homeomorphic to S_{t_0}. We define $H_i(\mathcal{R}(\sigma, Z))$ to be the simplicial homology group $H_i(S_{t_0})$ with coefficients in $\mathbb{Q}$. It is easy to see (again using Hardt's triviality theorem) that $H_i(\mathcal{R}(\sigma, Z))$ does not depend on the choice of the semi-algebraic family S_t and also that it is invariant under semi-algebraic homeomorphisms. Finally, in the case that $\mathrm{R} = \mathbb{R}$, $H_i(\mathcal{R}(\sigma, Z))$ is isomorphic to the i-th singular homology group of $\mathcal{R}(\sigma, Z)$ using the fact that the singular homology of a subset of $\mathbb{R}^k$ is isomorphic to the direct limit of the singular homology groups of its compact subsets [9].

Let $b_i(\sigma)$ denote the i-th Betti number of $\mathcal{R}(\sigma, Z)$, i.e., the dimension of $H_i(\mathcal{R}(\sigma, Z))$ as a $\mathbb{Q}$ vector space, and let $b_i(\mathcal{Q}, \mathcal{P}) = \sum_\sigma b_i(\sigma)$. Note that $b_0(\mathcal{Q}, \mathcal{P})$ is the total number of semi-algebraically connected components of the realizations of all realizable sign conditions of $\mathcal{P}$ over Z.

We write $b_i(d, k, k', s)$ for the maximum of $b_i(\mathcal{Q}, \mathcal{P})$ over all $\mathcal{Q}, \mathcal{P}$ where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of $\mathrm{R}[X_1, \ldots, X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = s$ (i.e. $\mathcal{P}$ has s elements) and the algebraic set Z has real dimension k'.

In [3], it was shown that, $b_0(d, k, k', s) = \binom{s}{k'} O(d)^k$. The main point in this paper is to prove an extension of this result by obtaining bounds for $b_i(d, k, k', s)$, for each i, $0 \le i \le k'$. Namely, we prove:

Theorem 1.1.

$$b_i(d, k, k', s) \le \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d - 1)^{k-1}.$$

The bound in [3] is proved by using a general position argument. The given polynomials are perturbed using infinitesimals so as to put them in general position – i.e. so that no more than k' of the polynomials in $\mathcal{P}$ have a common real zero in Z. The main ideas behind the proofs of the results in this paper are very different. We use an inductive argument based on the Mayer-Vietoris sequence. The starting point of the induction is a dimension argument: namely, we use the fact that the i-th Betti number of a semi-algebraic set is zero when i is greater than its dimension. Notice that for $i = 0$, Theorem 1.1 gives a more precise bound than the one in [3]. In [1] separate bounds on the individual Betti numbers of basic closed semi-algebraic sets were proved using a spectral sequence argument. The spectral sequences described there suggest the inequalities proved in Proposition 2 below, but they hide the direct induction that we are performing here.

We start with preliminaries, prove Theorem 1.1 in Section 3, and in Section 4 study the sum of Betti numbers of closed semi-algebraic sets.

2. Preliminaries

We use two main ingredients: the Oleinik-Petrovski/Thom/Milnor bound on the sum of the Betti numbers of algebraic sets and the Mayer-Vietoris long exact sequence. Additionally, we will use certain tools from real algebraic geometry.

Let $b(k, d)$ be the maximum of the sum of the Betti numbers of any algebraic set defined by polynomials of degree d in R^k. The Oleinik-Petrovski/Thom/Milnor [7, 10, 8] bound is the following:

$$b(k, d) \leq d(2d - 1)^{k-1}. \tag{2.1}$$

We use extensively the inequalities in the following Proposition 1, which are easy consequences of the exactness of the Mayer-Vietoris sequence of homology groups [9] : if S_1, S_2 are two closed and bounded semi-algebraic sets, then there exists the following long exact sequence of homology groups:

$$\cdots \to H_i(S_1 \cap S_2) \to H_i(S_1) \oplus H_i(S_2) \to H_i(S_1 \cup S_2) \to H_{i-1}(S_1 \cap S_2) \to \cdots .$$

Proposition 1. *Let S_1, S_2 be two closed and bounded semi-algebraic sets. Then,*

$$b_i(S_1) + b_i(S_2) \leq b_i(S_1 \cup S_2) + b_i(S_1 \cap S_2), \tag{2.2}$$

$$b_i(S_1 \cup S_2) \leq b_i(S_1) + b_i(S_2) + b_{i-1}(S_1 \cap S_2), \tag{2.3}$$

$$b_i(S_1 \cap S_2) \leq b_i(S_1) + b_i(S_2) + b_{i+1}(S_1 \cup S_2). \tag{2.4}$$

We perturb polynomials by various infinitesimals so that our geometric objects live over the field of algebraic Puiseux series in these infinitesimals. We denote by $\mathrm{R}\langle\zeta\rangle$ the real closed field of algebraic Puiseux series in ζ with coefficients in R [4]. The sign of a Puiseux series in $\mathrm{R}\langle\zeta\rangle$ agrees with the sign of the coefficient of the lowest degree term in ζ. This order makes ζ infinitesimal: ζ is positive and smaller than any positive element of R. When $a \in \mathrm{R}\langle\zeta\rangle$ is bounded by an element of R, $\lim_\zeta(a)$ is the constant term of a, obtained by substituting 0 for ζ in a.

Let R denote a real closed field and R$'$ a real closed field containing R. Given a semi-algebraic set S in R^k, the *extension* of S to R', denoted $\mathrm{Ext}(S, \mathrm{R}')$, is the semi-algebraic subset of ${\mathrm{R}'}^k$ defined by the same quantifier free formula that defines S. The set $\mathrm{Ext}(S, \mathrm{R}')$ is well defined (i.e. it only depends on the set S and not on the quantifier free formula chosen to describe it). This is an easy consequence of the transfer principle [5]. Moreover, the Betti numbers are not changed after extension: $b_i(S) = b_i(\mathrm{Ext}(S, \mathrm{R}'))$ (see [4], Chapter 6).

3. Bounds on Betti numbers of basic semi-algebraic sets: Proof of Theorem 1.1

Let $S_1, \ldots, S_s \subset \mathrm{R}^k$ be closed semi-algebraic sets, contained in a closed bounded semi-algebraic set T of dimension k'. For $1 \le t \le s$, we let

$$S_{\le t} = \bigcap_{1 \le j \le t} S_j, \quad S^{\le t} = \bigcup_{1 \le j \le t} S_j.$$

Also, for $J \subset \{1, \ldots, s\}$, $J \ne \emptyset$, let

$$S_J = \bigcap_{j \in J} S_j, \ S^J = \bigcup_{j \in J} S_j.$$

Finally, let $S^{\emptyset} = T$.

The following proposition, Proposition 2, plays a key role in the proofs of our theorems. The first part of the proposition bounds the Betti numbers of a union of s semi-algebraic sets in R^k in terms of the Betti numbers of the intersections of the sets taken at most k at a time. In some simple situations the Betti numbers of a union of s sets are easy to bound. For instance, when the sets are such that all non-empty intersections amongst them are contractible, a classical result of topology, the nerve lemma, gives us a bound on the individual Betti numbers of the union. The nerve lemma states that the homology groups of such a union are isomorphic to the homology groups of a combinatorially defined simplicial complex, the nerve complex. The nerve complex has s vertices, and thus the i-th Betti number is bounded by $\binom{s}{i+1}$. The first part of the proposition can be thought of as a generalization of this bound to the case when the intersections are not topologically trivial. The second part of the proposition is a dual version of the first, with unions being replaced by intersections and vice-versa, with an additional complication arising from the fact that the empty intersection, corresponding to the base case of the induction, is an arbitrary real algebraic variety of dimension k', which is generally not contractible.

Proposition 2. *For* $0 \le i \le k'$,

$$b_i(S^{\le s}) \le \sum_{j=1}^{i+1} \sum_{J \subset \{1, \ldots s\}, \#(J) = j} b_{i-j+1}(S_J), \tag{3.1}$$

$$b_i(S_{\le s}) \le b_{k'}(S^{\emptyset}) + \sum_{j=1}^{k'-i} \sum_{J \subset \{1, \ldots, s\}, \#(J) = j} \left(b_{i+j-1}(S^J) + b_{k'}(S^{\emptyset}) \right). \tag{3.2}$$

Proof of inequality (3.1). We prove the claim by induction on s. The statement is clearly true for $s = 1$.

Using Proposition 1(2.3), we have that

$$b_i(S^{\le s}) \le b_i(S^{\le s-1}) + b_i(S_s) + b_{i-1}(S^{\le s-1} \cap S_s).$$

Applying the induction hypothesis to the set $S^{\leq s-1}$, we deduce that

$$b_i(S^{\leq s-1}) \leq \sum_{j=1}^{i+1} \sum_{J\subset\{1,\ldots,s-1\},\#(J)=j} b_{i-j+1}(S_J).$$

Next, we apply the induction hypothesis to the set

$$S^{\leq s-1} \cap S_s = \cup_{1\leq j\leq s-1}(S_j \cap S_s)$$

and get that

$$b_{i-1}(S^{\leq s-1} \cap S_s) \leq \sum_{j=1}^{i} \sum_{J\subset\{1,\ldots,s-1\},\#(J)=j} b_{i-j}(S_{J\cup\{s\}}).$$

Adding the inequalities obtained above we get

$$b_i(S^{\leq s-1}) + b_i(S_s) + b_{i-1}(S^{\leq s-1} \cap S_s) \leq \sum_{j=1}^{i+1} \sum_{J\subset\{1,\ldots,s\},\#(J)=j} b_{i-j+1}(S_J).$$

□

Proof of inequality (3.2). We first prove the claim when $s = 1$. If $0 \leq i \leq k' - 1$, the claim is

$$b_i(S_1) \leq b_{k'}(S^{\emptyset}) + \left(b_i(S_1) + b_{k'}(S^{\emptyset})\right).$$

If $i = k'$, the claim is $b_{k'}(S_1) \leq b_{k'}(S^{\emptyset})$. If the dimension of S_1 is k', consider the closure V of the complement of S_1 in T. The intersection W of V with S_1, which is the boundary of S_1, has dimension strictly smaller than k' [5] (page 53); thus $b_{k'}(W) = 0$. Using Proposition 1 (2.2), $b_{k'}(S_1) + b_{k'}(V) \leq b_{k'}(S^{\emptyset}) + b_{k'}(W)$, and the claim follows. On the other hand, if the dimension of S_1 is strictly smaller than k', $b_{k'}(S_1) = 0$.

The claim is now proved by induction on s. Assume that the induction hypothesis (3.2) holds for $s - 1$ and for all $0 \leq i \leq k'$. From Proposition 1(2.4) we have

$$b_i(S_{\leq s}) \leq b_i(S_{\leq s-1}) + b_i(S_s) + b_{i+1}(S_{\leq s-1} \cup S_s).$$

Applying the induction hypothesis to the set $S_{\leq s-1}$, we deduce that

$$b_i(S_{\leq s-1}) \leq b_{k'}(S^{\emptyset}) + \sum_{j=1}^{k'-i} \sum_{J\subset\{1,\ldots,s-1\},\#(J)=j} \left(b_{i+j-1}(S^J) + b_{k'}(S^{\emptyset})\right).$$

Next, applying the induction hypothesis to the set, $S_{\leq s-1} \cup S_s = \bigcap_{1\leq j\leq s-1}(S_j \cup S_s)$, we get that

$$b_{i+1}(S_{\leq s-1} \cup S_s) \leq b_{k'}(S^{\emptyset}) + \sum_{j=1}^{k'-i-1} \sum_{J\subset\{1,\ldots,s-1\},\#(J)=j} \left(b_{i+j}(S^{J\cup\{s\}}) + b_{k'}(S^{\emptyset})\right).$$

Adding the inequalities obtained above we get

$$b_i(S_{\leq s}) \leq b_{k'}(S^{\emptyset}) + \sum_{j=1}^{k'-i} \sum_{J\subset\{1,\ldots,s\},\#(J)=j} \left(b_{i+j-1}(S^J) + b_{k'}(S^{\emptyset})\right).$$

□

Let $\mathcal{P} = \{P_1, \ldots, P_s\}$, and let δ be a new variable. We consider the field, $\mathrm{R}\langle\delta\rangle$, of algebraic Puiseux series in δ, in which δ will be an infinitesimal. Let W_0 (resp. W_1) be the union of the sets $\mathcal{R}\left(P_i^2(P_i^2 - \delta^2) = 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right)$ (resp. $\mathcal{R}\left(P_i^2(P_i^2 - \delta^2 \geq 0), \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right)$ with $1 \leq i \leq j$.

Lemma 3.1.

$$b_i(W_0) \leq (4^j - 1)d(2d-1)^{k-1}.$$

Proof. Each of the sets $\mathcal{R}\left(P_i^2(P_i^2 - \delta^2) = 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right)$ is the disjoint union of three algebraic sets, namely

$$\mathcal{R}\left(P_i = 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right),$$
$$\mathcal{R}\left(P_i = \delta, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right),$$

and

$$\mathcal{R}\left(P_i = -\delta, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right).$$

Moreover, each Betti number of their union is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking, for $1 \leq \ell \leq j$, ℓ-ary intersections of these algebraic sets using inequality 3.1 of Proposition 2. The number of possible ℓ-ary intersections is $\binom{j}{\ell}$. Each such intersection is a disjoint union of 3^ℓ algebraic sets. The sum of the Betti numbers of each of these algebraic sets is bounded by $d(2d-1)^{k-1}$ by the Oleinik-Petrovski/Thom/Milnor bound (2.1).

Thus, $b_i(W_0) \leq \displaystyle\sum_{\ell=1}^{j} \binom{j}{\ell} 3^\ell d(2d-1)^{k-1} = (4^j - 1)d(2d-1)^{k-1}$. □

Lemma 3.2.

$$b_i(W_1) \leq (4^j - 1)d(2d-1)^{k-1} + b_i(Z_r).$$

Proof. Let $Q_i = P_i^2(P_i^2 - \delta^2)$ and

$$F = \mathcal{R}\left(\bigwedge_{1 \leq i \leq j} (Q_i \leq 0 \vee \bigvee_{1 \leq i \leq j} Q_i = 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right).$$

Now apply inequality (2.2), noting that $W_1 \cup F = \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle), W_1 \cap F = W_0$, since $b_i(Z_r) = b_i(\mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle))$. We get that $b_i(W_1) \leq b_i(W_1 \cap F) + b_i(W_1 \cup F) = b_i(W_0) + b_i(Z_r)$. We conclude using Lemma 3.1. □

Let $S_i = \mathcal{R}\left(P_i^2(P_i^2 - \delta^2) \geq 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)\right), 1 \leq i \leq \ell$, and S be the intersection of the S_i. Then

Lemma 3.3.

$$b_i(\mathcal{P}, \mathcal{Q}) = b_i(S).$$

Proof. Consider a sign condition σ on $\mathcal{P}$ such that, without loss of generality,
$\sigma(P_i) = 0$ if $i = 1, \ldots, j$,
$\sigma(P_i) = 1$ if $i = j+1, \ldots, \ell$,
$\sigma(P_i) = -1$ if $i = \ell+1, \ldots, s$,
and denote by $\bar{\mathcal{R}}(\sigma)$ the subset of $\mathrm{Ext}(Z_r, \mathrm{R}\langle\delta\rangle)$ defined by

$$\bigwedge_{i=1,\ldots,j} P_i(x) = 0 \wedge \bigwedge_{i=j+1,\ldots,\ell} P_i(x) \geq \delta \wedge \bigwedge_{\ell+1,\ldots,s} P_i(x) \leq -\delta. \tag{3.3}$$

It follows from our definition of $b_i(\sigma)$ and Hardt's triviality theorem [5] that $b_i(\sigma) = b_i(\bar{\mathcal{R}}(\sigma))$. Note that S is the disjoint union of the $\bar{\mathcal{R}}(\sigma)$ (for the σ realizable sign condition) so that $\sum_\sigma b_i(\sigma) = b_i(S)$. On the other hand, by definition, $\sum_\sigma b_i(\sigma) = b_i(\mathcal{P},\mathcal{Q})$. □

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Using inequality 3.2 of Proposition 2, Lemma 3.2, and (2.1) which implies, for all $i < k'$, $b_i(Z_r) + b_{k'}(Z_r) \le d(2d-1)^{k-1}$, we deduce that

$$b_i(S) \le b_{k'}(Z_r) + \sum_{j=1}^{k'-i} \binom{s}{j} \left(4^j d(2d-1)^{k-1}\right).$$

Thus, we have $b_i(S) \le \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}$.

It now follows, using Lemma 3.3, that

$$b_i(\mathcal{P},\mathcal{Q}) \le \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1},$$

and finally

$$b_i(d,k,k',s) \le \sum_{j=0}^{k'-i} \binom{s}{j} 4^j d(2d-1)^{k-1}.$$

□

4. Sum of Betti numbers of closed semi-algebraic sets

A $(\mathcal{Q},\mathcal{P})$-*closed formula* is a formula defined as follows:

- For each $P \in \mathcal{P}$, $\bigwedge_{Q\in\mathcal{Q}} Q = 0 \wedge P = 0, \bigwedge_{Q\in\mathcal{Q}} Q = 0 \wedge P \ge 0, \bigwedge_{Q\in\mathcal{Q}} Q = 0 \wedge P \le 0$ are $(\mathcal{Q},\mathcal{P})$-closed formulas.
- If Φ_1 and Φ_2 are $(\mathcal{Q},\mathcal{P})$-closed formulas, $\Phi_1 \wedge \Phi_2$ and $\Phi_1 \vee \Phi_2$ are $(\mathcal{Q},\mathcal{P})$-closed formulas.

Clearly, $\mathcal{R}(\Phi)$, the intersection of the realization of a $(\mathcal{Q},\mathcal{P})$-closed formula Φ with $B(0,r)$ is a closed semi-algebraic set. We denote by $b(\Phi)$ the sum of its Betti numbers.

We write $\bar{b}(d,k,k',s)$ for the maximum of $b(\Phi)$, where Φ is a $(\mathcal{Q},\mathcal{P})$-closed formula, $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of $\mathrm{R}[X_1,\ldots,X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = s$ and the algebraic set $\mathcal{R}(\bigwedge_{Q\in\mathcal{Q}} Q = 0)$ has dimension k'.

In [2], it was shown that $\bar{b}(d,k,k',s)$ is bounded by $s^{k'}O(d)^k$. In this section, we prove a more precise bound:

Theorem 4.1.

$$\bar{b}(d,k,k',s) \le \sum_{i=0}^{k'} \sum_{j=0}^{k'-i} \binom{s}{j} 6^j d(2d-1)^{k-1}.$$

For the proof of Theorem 4.1, we are going to introduce several infinitesimals. Given an ordered list of polynomials $\mathcal{P} = \{P_1, \ldots, P_s\}$ with coefficients in R, we introduce s new variables $\delta_1, \cdots, \delta_s$, and inductively define: $\mathrm{R}\langle\delta_1, \ldots, \delta_{i+1}\rangle = \mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle\langle\delta_{i+1}\rangle$. Note that δ_{i+1} is infinitesimal with respect to δ_i, which is denoted by

$$\delta_1 \gg \ldots \gg \delta_s.$$

We define $\mathcal{P}_{>i} = \{P_{i+1}, \ldots, P_s\}$ and

$$\Sigma_i = \{P_i = 0, P_i = \delta_i, P_i = -\delta_i, P_i \geq 2\delta_i, P_i \leq -2\delta_i\},$$
$$\Sigma_{\leq i} = \{\Psi \mid \Psi = \bigwedge_{j=1,\ldots,i} \Psi_i, \Psi_i \in \Sigma_i\}.$$

If Φ is a $(\mathcal{Q}, \mathcal{P})$-closed formula, we denote by $\mathcal{R}_i(\Phi)$ the extension of $\mathcal{R}(\Phi)$ to $\mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle^k$. For $\Psi \in \Sigma_{\leq i}$, we denote by $\mathcal{R}_i(\Phi \wedge \Psi)$ the intersection of the realization of Ψ with $\mathcal{R}_i(\Phi)$ and by $b(\Phi \wedge \Psi)$ the sum of the Betti numbers of $\mathcal{R}_i(\Phi \wedge \Psi)$.

Proposition 3. *For every $(\mathcal{Q}, \mathcal{P})$-closed formula Φ,*

$$b(\Phi) \leq \sum_{\Psi \in \Sigma_{\leq s}, \mathcal{R}_s(\Psi) \subset \mathcal{R}_s(\Phi)} b(\Psi).$$

The main ingredient of the proof of Proposition 3 is the following lemma.

Lemma 4.2. *For every $(\mathcal{Q}, \mathcal{P})$-closed formula Φ, and every $\Psi \in \Sigma_{\leq i}$, $b(\Phi \wedge \Psi) \leq \sum_{\psi \in \Sigma_{i+1}} b(\Phi \wedge \Psi \wedge \psi)$.*

Proof. Consider the formulas

$$\Phi_1 = \Phi \wedge \Psi \wedge (P_{i+1}^2 - \delta_{i+1}^2) \geq 0,$$
$$\Phi_2 = \Phi \wedge \Psi \wedge (0 \leq P_{i+1}^2 \leq \delta_{i+1}^2).$$

Clearly, $\mathcal{R}_{i+1}(\Phi \wedge \Psi) = \mathcal{R}_{i+1}(\Phi_1 \vee \Phi_2)$. Using Proposition 1, we have that $b(\Phi \wedge \Psi) \leq b(\Phi_1) + b(\Phi_2) + b(\Phi_1 \wedge \Phi_2)$.

Now, since $\mathcal{R}_{i+1}(\Phi_1 \wedge \Phi_2)$ is the disjoint union of

$$\mathcal{R}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = \delta_{i+1})) \text{ and } \mathcal{R}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = -\delta_{i+1})),$$
$$b(\Phi_1 \wedge \Phi_2) = b(\Phi \wedge \Psi \wedge (P_{i+1} = \delta_{i+1}) + b(\Phi \wedge \Psi \wedge (P_{i+1} = -\delta_{i+1})).$$

Moreover,

$$b(\Phi_1) = b(\Phi \wedge \Psi \wedge (P_{i+1} \geq 2\delta_{i+1})) + b(\Phi \wedge \Psi \wedge (P_{i+1} \leq -2\delta_{i+1})),$$
$$b(\Phi_2) = b(\Phi \wedge \Psi \wedge (P_{i+1} = 0)).$$

Indeed, by Hardt's triviality theorem [5], denoting $F_t = \{x \in \mathcal{R}_i(\Phi \wedge \Psi) \mid P_{i+1}(x) = t\}$, there exists $t_0 \in \mathrm{R}\langle\delta_1, \ldots, \delta_i\rangle$ such that $F_{[-t_0,0)\cup(0,t_0]} = \{x \in \mathcal{R}_i(\Phi \wedge \Psi) \mid t_0^2 \geq P_{i+1}(x) > 0\}$ and $([-t_0, 0) \times F_{-t_0}) \cup ((0, t_0] \times F_{t_0})$, are homeomorphic, and moreover the homeomorphism can be chosen such that it is the identity on the fibers F_{-t_0} and F_{t_0}.

This clearly implies that $F_{[\delta,t_0]} = \{x \in \mathcal{R}_{i+1}(\Phi \wedge \Psi) \mid t_0 \geq P_{i+1}(x) \geq \delta\}$ and $F_{[2\delta,t_0]} = \{x \in \mathcal{R}_{i+1}(\Phi \wedge \Psi) \mid t_0 \geq P_{i+1}(x) \geq 2\delta\}$ are homeomorphic.

Hence, $b(\Phi_1) = b(\Phi \wedge \Psi \wedge (P_{i+1} \geq 2\delta_{i+1})) + b(\Phi \wedge \Psi \wedge (P_{i+1} \leq -2\delta_{i+1}))$.

Note that $F_0 = \mathcal{R}_{i+1}(\Phi \wedge \Psi \wedge (P_{i+1} = 0))$ and $F_{[-\delta,\delta]} = \mathcal{R}_{i+1}(\Phi_2)$. Thus, it remains to prove that $b(F_{[-\delta,\delta]}) = b(F_0)$. By Hardt's triviality theorem [5], for every $0 < u < 1$ there is a fiber-preserving semi-algebraic homeomorphism ϕ_u from

$F_{[-\delta,-u\delta]}$ to $[-\delta,-u\delta]\times F_{-u\delta}$ (resp. a semi-algebraic homeomorphism ψ_u from $F_{[u\delta,\delta]}$ to $[u\delta,\delta]\times F_{u\delta}$). We define a continuous semi-algebraic homotopy g from the identity of $F_{[-\delta,\delta]}$ to $\lim_{\delta_{i+1}}$ from $F_{[-\delta,\delta]}$ to F_0 as follows:

- $g(0,-)$ is $\lim_{\delta_{i+1}}$,
- for $0<u\le 1$, $g(u,-)$ is the identity on $F_{[-u\delta,u\delta]}$ and sends $F_{[-\delta,-u\delta]}$ (resp. $F_{[u\delta,\delta]}$) to $F_{-u\delta}$ (resp. $F_{u\delta}$) by ϕ_u (resp. ψ_u)followed by the projection on $F_{u\delta}$ (resp. $F_{-u\delta}$).

Thus $b(F_{[-\delta,\delta]})=b(F_0)$. Finally, $b(\Phi\wedge\Psi)\le \sum_{\psi\in\Sigma_{i+1}} b(\Phi\wedge\Psi\wedge\psi)$. □

Proof of Proposition 3. Starting from the formula Φ, apply Lemma 4.2 with Ψ the empty formula. Now, repeatedly apply Lemma 4.2 to the terms appearing on the right-hand side of the inequality obtained, noting that for any $\Psi\in\Sigma_{\le s}$,

- either $\mathcal{R}_s(\Phi\wedge\Psi)=\mathcal{R}_s(\Psi)$, and $\mathcal{R}_s(\Psi)\subset\mathcal{R}_s(\Phi)$,
- or $\mathcal{R}_s(\Phi\wedge\Psi)=\emptyset$. □

Using an argument analogous to that used in the proof of Theorem 1.1 we prove the following proposition.

Proposition 4.

$$\sum_{\Psi\in\Sigma_{\le s}} b(\Psi)\le\sum_{j=0}^{k'-i}\binom{s}{j}6^j d(2d-1)^{k-1}.$$

We first prove the following Lemma 4.3 and Lemma 4.4.

Let $\mathcal{P}=\{P_1,\dots,P_j\}\subset R[X_1,\dots,X_k]$, and let

$$Q_i=P_i^2(P_i^2-\delta_i^2)^2(P_i^2-4\delta_i^2).$$

Let W_0 (resp. W_1) be the union of the sets $\mathcal{R}\left(Q_i=0,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)\right)$ (resp. $\mathcal{R}\left(Q_i\ge 0,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)\right)$), with $1\le i\le j$.

Notice that $W_1=\bigcup_{\Psi\in\Sigma_{\le s}}\mathcal{R}(\Psi)$.

Lemma 4.3.

$$b_i(W_0)\le(6^j-1)d(2d-1)^{k-1}.$$

Proof. The set $\mathcal{R}((P_i^2(P_i^2-\delta_i^2)^2(P_i^2-4\delta_i^2)=0),Z_r)$ is the disjoint union of

$$\mathcal{R}(P_i=0,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)),$$
$$\mathcal{R}(P_i=\delta_i,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)),$$
$$\mathcal{R}(P_i=-\delta_i,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)),$$
$$\mathcal{R}(P_i=2\delta_i,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)),$$

and

$$\mathcal{R}(P_i=-2\delta_i,\mathrm{Ext}(Z_r,\mathrm{R}\langle\delta_1,\dots,\delta_j\rangle)).$$

Moreover, the i-th Betti numbers of their union W_0 is bounded by the sum of the Betti numbers of all possible non-empty sets that can be obtained by taking intersections of these sets using inequality 3.1 of Proposition 2.

The number of possible ℓ-ary intersections is $\binom{j}{\ell}$. Each such intersection is a disjoint union of 5^ℓ algebraic sets. The i-th Betti number of each of these algebraic sets is bounded by $d(2d-1)^{k-1}$ by (2.1).

Thus, $b_i(W_0)\le\sum_{\ell=1}^{j}\binom{j}{\ell}5^\ell d(2d-1)^{k-1}=(6^j-1)d(2d-1)^{k-1}$. □

Lemma 4.4.

$$b_i(W_1) \leq (6^j - 1)d(2d-1)^{k-1} + b_i(Z_r).$$

Proof. Let $F = \mathcal{R}\left(\bigwedge_{1\leq i\leq j} Q_i \leq 0 \vee \bigvee_{1\leq i\leq j} Q_i = 0, \mathrm{Ext}(Z_r, \mathrm{R}\langle\delta_1,\ldots,\delta_i\rangle)\right)$. Now, $W_1 \cup F = Z_r$ and $W_1 \cap F = W_0$. Using inequality (2.2) and the fact that

$$b_i(Z_r) = b_i(\mathrm{Ext}(Z_r, \mathrm{R}\langle\delta_1,\ldots,\delta_i\rangle)),$$

we deduce that $b_i(W_1) \leq b_i(W_1\cap F) + b_i(W_1 \cup F) = b_i(W_0) + b_i(Z_r)$. We conclude using Lemma 4.3. □

Proof of Proposition 4. Since for all $i < k'$, $b_i(Z_r) + b_{k'}(Z_r) \leq d(2d-1)^{k-1}$ by (2.1), we have that

$$\sum_{\Psi\in\Sigma_{\leq s}} b(\Psi) = b(W_1) \leq b_{k'}(Z_r) + \sum_{j=1}^{k'-i}\binom{s}{j}\left(6^j d(2d-1)^{k-1}\right)$$

using inequality 3.2 of Proposition 2 and Lemma 4.4. Thus,

$$\sum_{\Psi\in\Sigma_{\leq s}} b(\Psi) \leq \sum_{j=0}^{k'-i}\binom{s}{j} 6^j d(2d-1)^{k-1}.$$ □

Proof of Theorem 4.1. The statement follows from Proposition 4 and Proposition 3. □

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