ON PEIXOTO’S CONJECTURE FOR FLOWS
ON NON-ORIENTABLE 2-MANIFOLDS

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Abstract. Contrary to the case of vector fields on orientable compact 2-
manifolds, there is a smooth vector field $X$ on a non-orientable compact 2-
manifold with a dense orbit (and therefore without closed orbits) whose phase
portrait – up to topological equivalence – remains intact under a one-parameter
family of twist perturbations localized in a flow box of $X$.

1. Introduction

The importance of the $C^r$-Closing Lemma Problem lies in the fact that a positive
answer to it would lead to very deep positive conclusions in Dynamical Systems, in
topics related to the Generic, Stability and Bifurcation Theories. As a consequence
of its role, there are several useful $C^r$-Closing Lemmas. As a sample of some of the
important results, we wish to mention Peixoto’s $C^r$-Connecting Lemma [24], Pugh’s
$C^1$-Closing Lemma [26], Mañé’s $C^1$-Ergodic Closing Lemma [19], Gutierrez’s $C^r$
counterexample [9], Herman’s $C^r$-Closing Lemma [17, 18], Hayashi’s $C^1$-Connecting
Lemma [14, 15, 16]. Besides the articles that will be quoted in this paper, in the
same way as above, we wish to mention [12, 13, 20], [22]–[34]. As a sample of some
very recent results that use $C^1$-Closing Lemma results, we wish to mention [3, 5].

Let $X^r(M)$, $0 \leq r \leq \infty$, denote the space of $C^r$–vector fields (with the
$C^r$–topology) on a compact, connected, boundaryless, $C^\infty$, 2-manifold $M$. Our (com-
 pact) flow boxes $V \subset M$ of $X$ will be either the standard ones or those such that
$X|_V$ is transversal to the constant vector field $(1,0)$ on the cylinder $[0,1] \times \mathbb{R}/\mathbb{Z}$. Notice that in the second case, for all $t \in (0,1)$, \{t\} \times \mathbb{R}/\mathbb{Z} is a
transversal circle to the vector field $(1,0)$.

To state our main result we shall need the following.

Definition 1.1 (twist perturbation of a vector field). Let $M$ be a 2-manifold,
$X \in X^r(M)$ and $V \subset M$ be a compact flow box of $X$. Given $Y \in X^r(M)$ with
support in $V$, we say that $X + Y$ is a $C^r$–twist perturbation of $X$, localized in $V$
if $X|_V$ is transversal to $Y|_V$, where $V^o$ denotes the interior of $V$.

Points of $\mathbb{R}/\mathbb{Z}$ will be denoted as if they were points of $\mathbb{R}$; in this way, we shall
use expressions of the form $x + a$ and $x - a$ when referring to points of $\mathbb{R}/\mathbb{Z}$. Also, if

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$x < y$ are real numbers such that $y - x < 1$, the subinterval $(x, y)$ of $\mathbb{R}$ determines a unique subinterval of $\mathbb{R}/\mathbb{Z}$.

**Definition 1.2** (smooth/affine/isometric iet). Let $\Gamma$ be either the the unit interval $[0, 1)$ or the unit circle $\mathbb{R}/\mathbb{Z}$. Given a finite subset $S = \{a_0, a_1, \ldots, a_n\}$ of $\Gamma$, with $0 = a_0 < a_1 < \cdots < a_n = 1$, a smooth interval exchange transformation, shortly smooth iet, will be an injective transformation $T : \{0\} \cup S \to \Gamma$ such that $T|_{(a_{i-1}, a_i)}$ is a smooth diffeomorphism, $1 \leq i \leq n$, and the range of $T$ is all $\Gamma$ but $n$ points. If, moreover, $T|_{(a_{i-1}, a_i)}$ is an affine (isometric) transformation for all $1 \leq i \leq n$, we call $T$ a smooth (isometric) iet. As usual, the term iet will refer to an isometric iet.

**Definition 1.3** (quasi-minimal vector field). A $C^1$ vector field on a compact 2–manifold $M$ is quasi-minimal if its set of singularities $S$ is at most finite and any of its orbits in $M \setminus S$ is dense in $M$.

In this paper, we prove the following.

**Theorem 1.4** (smooth iet version). There exist a minimal isometric iet $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a family of diffeomorphisms $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$, with $G_0$ being the identity map, depending smoothly on $\mu \in [0, \epsilon)$, such that, for all $\mu \in (0, \epsilon)$, $G_\mu$ has no fixed points; $G_\mu \circ T$ is a smooth iet $C^\infty$–conjugate to $T$.

In the theorem above, since $T = T_0$ is minimal and every $T_\mu$ is topologically conjugate to $T$, we obtain that every $T_\mu$ is also minimal; in particular, all orbits of $T_\mu$ are non-trivial recurrent (and so $T_\mu$ has no closed orbits).

Given a smooth iet $T$, defined on $\mathbb{R}/\mathbb{Z}$, there exist a compact 2–manifold $M$, containing $\mathbb{R}/\mathbb{Z}$, and a vector field $X \in \mathcal{X}(M)$ transversal to $\mathbb{R}/\mathbb{Z}$ such that the first return Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by $X$ is precisely $T$ (see [5]). Using this, the result above can be extended to vector fields as follows:

**Theorem 1.5** (vector field version). Let $M$ be a non-orientable compact 2–manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale vector fields $\{X_\mu \in \mathcal{X}(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of $X_0$, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in (0, \epsilon)$,

1. $X_\mu|_V$ is a flow box;
2. If $\mu \neq \nu$, $X_\mu$ is a $C^\infty$–twist perturbation of $X_\nu$ localized in $V$;
3. $X_\mu$ and $X_\nu$ are topologically equivalent.

In the theorem above, since $X = X_0$ is quasi-minimal and every $X_\mu$ is topologically equivalent to $X$, we obtain that every $X_\mu$ is also quasi-minimal; in particular, all regular orbits of $X_\mu$ are non-trivial recurrent (and so $X_\mu$ has no closed orbits).

Now we relate our results to Peixoto’s Conjecture. Let $\Sigma^r$ be the subset of $\mathcal{X}^r(M)$ formed by the Morse-Smale $C^r$–vector fields. M. Peixoto states in [24] the following conjecture.

(1) Let $M$ be a non-orientable 2–manifold. $X \in \mathcal{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, $\Sigma^r$ is open and dense in $\mathcal{X}^r(M)$.

Peixoto [24] proved this conjecture for $M$ orientable. As a consequence of Peixoto’s work and Pugh’s $C^1$–Closing Lemma [26, 30], it follows that $\Sigma^1$ is dense.
in $\mathcal{X}^1(M)$. There are some partial results concerning (PC) in class $C^r$, $r \geq 1$: The conjecture is true both for the projective plane $\mathbb{P}^2$ and for the Klein bottle $\mathbb{K}^2$ (see the proof in [24] that flows on $\mathbb{K}^2$ do not have non-trivial recurrence). Gutierrez [6] showed that (PC) is true for the torus with one cross-cap.

By [24], (PC) is true if, and only if, it is possible to give an affirmative answer for the following $C^r$–Connecting Lemma question:

(\text{CL}) Let $X \in \mathcal{X}^r(M)$ have finitely many singularities, all hyperbolic (at least one singularity). Suppose that $X$ has a non-trivial recurrent trajectory. Does there exist an arbitrarily small $C^r$–perturbation of $X$ such that the resulting vector field has one more saddle connection than $X$?

We recall that, when $M$ is orientable, (CL) has a positive answer by arbitrarily small $C^r$–twist perturbations [24]. Also, when $M$ is non-orientable, there are many cases in which (CL) has a positive answer, again by arbitrarily small $C^r$–twist perturbations [11].

Hence, it is natural to wonder whether, in the non-orientable case, we may make use of an arbitrary $C^r$–twist perturbation in order to give an affirmative answer for (CL). Theorem [17] gives a negative answer to this question.

Finally, Theorems [14] and [15] above are relevant to the $C^r$-Closing Lemma problem, because the positive answer to the $C^r$-Closing Lemma, given in [7], for a large class of flows on orientable two-manifolds, was obtained by means of twist perturbations (see also [1] [2] [4]).

2. A FLOW ON THE TORUS WITH TWO CROSS-CAPS

Let $\varphi : \mathbb{R} \times M \to M$ be a flow on $M$ and $\gamma = \gamma(t)$ a trajectory of $\varphi$. We denote by $\omega(\gamma)(\alpha(\gamma))$ the $\omega$-limit set (\alpha-limit set) of $\gamma$. We say that $\gamma$ is $\omega$–recurrent (\alpha–recurrent) if $\gamma \subset \omega(\gamma)(\gamma \subset \alpha(\gamma))$. A recurrent trajectory is a trajectory that is $\omega$–recurrent or $\alpha$–recurrent. A fixed point and a closed orbit are called trivial recurrent trajectories.

Given a $C^1$ flow $\varphi : \mathbb{R} \times M \to M$ with a recurrent trajectory $\gamma$ and a point $p \in \gamma$, Peixoto proved that there exists a smooth circle $\Gamma$, transversal to $\varphi$, passing through the point $p$ (see [6] [24]). We shall analyse flows in terms of their action on transverse circles. Let us recall now the example of a flow given by Gutierrez in [10].

Gutierrez constructed in [10] an \textit{iet} $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (cf. Figures [1] and [2]) and a suspension of $T$ that is a quasi-minimal $C^\infty$–flow $\varphi : \mathbb{R} \times M \to M$, on the torus with two cross-caps $M$, in such a way that:

1. $\varphi$ has two hyperbolic saddle points as its only singularities;
2. $\mathbb{R}/\mathbb{Z}$ is a subset of $M$, $\varphi$ is transversal to $\mathbb{R}/\mathbb{Z}$, and the \textit{iet} $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is precisely the Poincaré return map induced by $\varphi$ on $\mathbb{R}/\mathbb{Z}$; moreover, for some real positive numbers $a, b, c, d, e$,
   1. $\text{dom}(T) = \mathbb{R}/\mathbb{Z} \setminus \{a, a + b, a + b + c, 1\}$, where $\text{dom}(T)$ is the domain of definition of $T$;
   2. the numbers $a, a + e - c, a - c - d + 2e, \cdots$, are ordered (modulo 1) according to Figures [1] and [2] In particular,
      - $0 < a < b < e < a + b < 1 - e < a + b + c < a + b + c + d = 1$, $d < e$,
      - $a + b + c + e = 1 + e - d$;
Figure 1. The first return map $T$ induced by $\varphi$ on $\mathbb{R}/\mathbb{Z}$

Figure 2. The isometric $\text{iet} T$
Proposition 3.2 (topological conjugacy)

- $T$ operates upon the intervals below, by reversing orientation, in the following way:
  
  $$(a, a + b) \mapsto (a + e, a + b + e),$$
  $$(a + b + c, 1) \mapsto (e - d, e);$$

- $T$ operates upon the intervals below, by preserving orientation, in the following way:
  
  $$(0, a) \mapsto (e, a + e);$$
  $$(a + b, a + b + c) \mapsto (a + b + e, e - d).$$

In the next section, given a small number $\delta > 0$, we will introduce an affine iet $T_\delta$ that is topologically conjugate to $T$ and $2\delta$-close to $T$, in the uniform $C^0$-topology.

3. Topological conjugacy

Definition 3.1 ($T_\delta$). Let $\delta > 0$ be small and let $T_\delta$ (cf. Figure 3) be the affine iet satisfying

- $T_\delta$ operates upon the intervals below, linearly, diffeomorphically, and reversing orientation, in the following way:
  
  $$(a, a + e - c + \delta) \mapsto [a + b + c, a + b + e + \delta),$$
  $$[a + e - c + \delta, a - c - d + 2e + 2\delta] \mapsto [1 - e - \delta, a + b + c],$$
  $$[a - c - d + 2e + 2\delta, 2a - c + e] \mapsto [b + c + \delta, 1 - e - \delta],$$
  $$[2a - c + e, a + b] \mapsto (a + e + \delta, b + c + \delta),$$
  $$(a + b + c, a + b + e + \delta) \mapsto [c, e + \delta),$$
  $$[a + b + e + \delta, a + 2e + 2\delta] \mapsto [c + b - e, c],$$
  $$[a + 2e + 2\delta, b + c + e + 2\delta] \mapsto [a, c + b - e],$$
  $$[b + c + e + 2\delta, 1) \mapsto (e - d + \delta, a);$$

- $T_\delta$ operates upon the intervals below, linearly, diffeomorphically and preserving orientation, in the following way:
  
  $$(a + b, a + e + \delta) \mapsto (a + b + e + \delta, a + 2e + 2\delta],$$
  $$[a + e + \delta, b + c + \delta] \mapsto [a + 2e + 2\delta, b + c + e + 2\delta],$$
  $$[b + c + \delta, 1 - e - \delta) \mapsto [b + c + e + 2\delta, 1),$$
  $$(1 - e - \delta, a + b + c) \mapsto (0, e - d + \delta),$$
  $$(0, e - d + \delta] \mapsto (e + \delta, 2e - d + 2\delta],$$
  $$[e - d + \delta, a) \mapsto [2e - d + 2\delta, a + e + \delta).$$

Proposition 3.2 (topological conjugacy). Given $\delta > 0$, there exist a fixed-point-free homeomorphism

$$H = H_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

and a piecewise affine homeomorphism (cf. Figure 4)

$$h = h_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

such that

1. $T_\delta = H \circ T$;
Figure 3. The affine iet $T_\delta$

Figure 4. The homeomorphism $h$ in local coordinates

(2) $\text{fix}(h) \supset \text{dis}(T_\delta) = \text{dis}(T) = \mathbb{R}/\mathbb{Z} \setminus \text{dom}(T_\delta) = \{a, a + b, a + b + c, 1\}$, where $\text{fix}(h)$ (resp. $\text{dis}(T)$) denotes the set of fixed points of $h$ (resp. the discontinuity point set of $T$);
We may extend $h$ linearly to the other points so that $h$ becomes a piecewise affine homeomorphism of the unit circle $\mathbb{R}/\mathbb{Z}$. It follows at once that $h$ satisfies (2)--(5).

\section*{4. $C^\infty$-Conjugacy}

In this section we show that the homeomorphism $h$ conjugating $T$ and $T_\delta$ and the fixed-point-free homeomorphism $H$ can be substituted by a $C^\infty$-diffeomorphism $g$ and a fixed-point-free $C^\infty$-diffeomorphism $G$, respectively, in such a way that the relation $g^{-1} \circ T \circ g = G \circ T$ remains true.

Recall \( \text{dis}(T) \subseteq \text{fix}(h) \) and that

\[
\begin{align*}
\text{dom}(T) &= \mathbb{R}/\mathbb{Z}\backslash\{a, a+b, a+b+c, 1\}, \\
\text{dis}(T) &= \{a, a+b, a+b+c, 1\}, \\
\text{dom}(T^{-1}) &= \mathbb{R}/\mathbb{Z}\backslash\{e, a+e, a+b+e, a+b+e+c, e-d\}, \\
\text{dis}(T^{-1}) &= \{e, a+e, a+b+e, a+b+e+c, e-d\}.
\end{align*}
\]

Let $\mathcal{G}$ be the set of $C^\infty$-diffeomorphisms $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that

\begin{enumerate}[(P1)]
\item $g|_U = I|_U$, where $U$ is a small neighborhood of $\text{fix}(h)$,
\item $g'(x) \geq \frac{1}{2}$, $\forall x \in \mathbb{R}/\mathbb{Z}$;
\end{enumerate}

\begin{lemma}
If $g \in \mathcal{G}$, then
\begin{enumerate}[(P3)]
\item $T(g(x)) - T(x) = T'(x)(g(x) - x)$, $\forall x \in \text{dom}(T)$
\end{enumerate}

(where $T'(x) \in \{-1, 1\}$).
Proof. An immediate consequence of (P1) is that for any \( x \in \text{dom}(T) \), \( x \) and \( g(x) \) lie in the same interval of the partition associated to the iet \( T \); that is, for each \( x \in \text{dom}(T) \), there exists \( 1 \leq i \leq n \) such that \( x, g(x) \in (a_{i-1}, a_i) \). This implies the lemma.

We prove now the main result of this section.

**Proposition 4.2.** Let \( g \in \mathcal{G} \).

1. The map
   \[ G = g^{-1} \circ T \circ g \circ T^{-1} \]
   is well defined in \( \text{dom}(T^{-1}) \) and extends to a \( C^\infty \)-diffeomorphism \( G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) such that \( \forall x \in \text{dom}(T) \), \( (g^{-1} \circ T \circ g)(x) = (G \circ T)(x) \). In particular, the isometric iet \( T \) and the smooth iet \( G \circ T \) are \( C^\infty \)-conjugate.

2. Fix \( \delta > 0 \) small, and let \( h = h_\delta \) be as in Proposition 3.3. If \( g \in \mathcal{G} \) is \( C^0 \)-close enough to \( h \) and \( G \) is as above, then \( G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) is a fixed-point-free \( C^\infty \)-diffeomorphism.

Proof. Extend \( G \) to the whole \( \mathbb{R}/\mathbb{Z} \) by defining \( G(q) = g^{-1}(q), \forall q \in \text{dis}(T^{-1}) \).

Therefore, \( G \) is smooth at any point \( q \in \text{dis}(T^{-1}) \) and as, by (P2), \( G'(q) \neq 0 \), we obtain that \( G^{-1} \) is a \( C^\infty \)-diffeomorphism. By definition of \( G \), \( (g^{-1} \circ T \circ g)(x) = (G \circ T)(x), \forall x \in \text{dom}(T) \). This proves (1). If \( g \) is \( C^0 \)-close to \( h \), then \( G \) will also be \( C^0 \)-close to \( H \); therefore, since \( H \) is a fixed-point-free homeomorphism, we will obtain that \( G \) is also fixed-point-free. □

5. Main Results

In this section we prove Theorems 1.4 and 1.5. We start by proving Theorem 1.4 (smooth iet version). There exist a minimal isometric iet \( T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) and a family of diffeomorphisms \( \{ G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon \} \), with \( G_0 \) being the identity map, depending smoothly on \( \mu \in [0, \epsilon) \), such that, for all \( (\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z} \),

1. \( \frac{d}{d\mu} \bigg|_{\mu = \mu_0} G_\mu(p_0) > 0 \);
2. \( G_{\mu_0} \circ T \) is a smooth iet \( C^\infty \)-conjugate to \( T \).

Proof. By Proposition 4.3 there exist a diffeomorphism \( g \) with Properties (P1) to (P3) and a fixed-point-free diffeomorphism \( G \) such that \( g^{-1} \circ T \circ g = G \circ T \) at every point of \( \text{dom}(T) \). Given \( \mu \in [0, 1] \), let \( g_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be defined by

\[ g_\mu(x) = \mu g(x) + (1 - \mu)x. \]

Notice that \( \{ g_\mu : \mu \in [0, 1] \} \) provides a smooth isotopy between the identity map \( g_0 = I \) and \( g_1 = g \); also by (P2),

\[ \frac{dg_\mu}{dx}(x) = \mu g'(x) + 1 - \mu \geq \frac{\mu}{2} + 1 - \mu = 1 - \frac{\mu}{2} \geq \frac{1}{2}, \forall x \in \mathbb{R}/\mathbb{Z}. \]
Thus, by construction, \( g_\mu \) is a diffeomorphism of the unit circle satisfying (P1)–(P3), for each \( \mu \in [0,1] \), that is, \( g_\mu \in G \). Given \( \mu \in [0,1] \), by Proposition 4.2 there exists a diffeomorphism \( G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) satisfying
\[
(G_\mu \circ T)(x) = ((g_\mu)^{-1} \circ T \circ g_\mu)(x), \ \forall x \in \text{dom}(T).
\]

The proof of this theorem follows at once from the following lemma.

**Lemma 5.1.** Given \( x \in \mathbb{R}/\mathbb{Z} \), the map \( \mu \in [0,1] \mapsto G_\mu(x) \in \mathbb{R}/\mathbb{Z} \) is differentiable. Moreover, there exist \( \epsilon > 0 \) and \( \sigma > 0 \) such that \( \forall (\mu_0, x) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z} \),
\[
\frac{d}{d\mu} \big|_{\mu=\mu_0} G_\mu(x) > \sigma.
\]

**Proof.** Let \( u \in (0,1] \) be a real number. Then, from (5.2),
\[
(T \circ g_u)(x) = (g_u \circ G_u \circ T)(x), \ \forall x \in \text{dom}(T).
\]
From (5.4), we reach
\[
g_u(y) = y + u(g(y) - y), \ \forall y \in \mathbb{R}/\mathbb{Z}.
\]
From (5.4) and from property (P3) of \( g_u \), we obtain
\[
\frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{T'(x) \cdot (g_u(x) - x)}{u} = T'(x) \cdot (g(x) - x).
\]
From equations (5.3) and (5.5), we get
\[
T'(x) \cdot (g(x) - x) = \frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{(g_u \circ G_u \circ T)(x) - (g_0 \circ G_0 \circ T)(x)}{u} = \frac{G_u(T(x)) - G_0(T(x))}{u} + g(G_u(T(x))) - G_u(T(x)).
\]
When \( u \to 0 \), we get
\[
\frac{d}{d\mu} \big|_{\mu=0} G_\mu(T(x)) = T'(x) \cdot (g(x) - x) - (g(T(x)) - T(x)),
\]
and from Property (P3) of \( g \), we reach
\[
\frac{d}{d\mu} \big|_{\mu=0} G_\mu(T(x)) = T(g(x)) - g(T(x)), \ \forall x \in \text{dom}(T).
\]
Since \( G = g^{-1} \circ T \circ g \circ T^{-1} \) has no fixed points, we have that \( T(g(x)) - g(T(x)) \neq 0 \), \( \forall x \in \text{dom}(T) \). Therefore,
\[
\frac{d}{d\mu} \big|_{\mu=0} G_\mu(T(x)) \neq 0, \ \forall x \in \text{dom}(T).
\]
That is,
\[
\frac{d}{d\mu} \big|_{\mu=0} G_\mu(T(x)) \neq 0, \ \forall x \in \text{dom}(T). \]
If \( q \in \text{dis}(T^{-1}) \), then \( G_\mu(q) = (g_\mu)^{-1}(q) \), \( \forall \mu \). Besides, we have by Figure 4 that \( q - g(q) > 0 \) (the points \((q,g(q))\) with \( q \in \text{dis}(T^{-1}) \) lie below the diagonal). Observe that
\[
(g_\mu \circ (g_\mu)^{-1})(x) = x, \ \forall x \in \mathbb{R}/\mathbb{Z},
\]
\[
\mu \cdot g((g_\mu)^{-1}(x)) + (1 - \mu)(g_\mu)^{-1}(x) = x, \ \forall x \in \mathbb{R}/\mathbb{Z}.
\]
Therefore, by differentiating the previous equation with respect to $\mu$ at $\mu = 0$, we get
\begin{equation}
\frac{d}{d\mu}\bigg|_{\mu=0} G_\mu(q) = \frac{d}{d\mu}\bigg|_{\mu=0} (g_\mu)^{-1}(q) = q - g(q) > 0.
\end{equation}

We remark that equations (5.7) and (5.9) are compatible; that is, for any $q \in \text{dis}(T^{-1}),$
\begin{equation}
\lim_{y \to q} \frac{d}{d\mu}\bigg|_{\mu=0} G_\mu(y) = q - g(q).
\end{equation}

Hence, the map $(\mu, x) \in [0, 1] \times \mathbb{R}/\mathbb{Z} \mapsto \frac{d}{d\mu}\bigg|_{\mu=0} G_\mu(x)$ is continuous. This implies the lemma.

\begin{corollary}
For all $\mu \in (0, \epsilon), G_\mu$ has no fixed points.
\end{corollary}

\begin{proof}
This follows immediately from Theorem 1.4.
\end{proof}

We now prove Theorem 1.5.

\textbf{Theorem 1.5} (Vector field version). Let $M$ be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale flows $\{X_\mu \in \mathcal{X}_\infty(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of $X_0$, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,
\begin{enumerate}
\item $X_\mu|_V$ is a flow box;
\item if $\mu \neq \nu$, $X_\mu$ is a $C^\infty$-twist perturbation of $X_\nu$ localized in $V$;
\item $X_\mu$ and $X_\nu$ are topologically equivalent.
\end{enumerate}

\begin{proof}
Let $T$ be as in Theorem 1.4. From Theorem 1.4, we know that $\{G_\mu \circ T\}$ is a family of smooth ict's conjugate to $T$. Each ict $\{G_\mu \circ T\}$ may be suspended to obtain a smooth vector field $X_\mu$ on a non-orientable compact manifold $M$ of genus 4 (see [R]). We may assume that $M$ contains $\mathbb{R}/\mathbb{Z}$ and does not depend on $\mu$. By definition of suspension, each $X_\mu$ is transversal to $\mathbb{R}/\mathbb{Z}$ and $G_\mu \circ T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the forward Poincaré map induced by $X_\mu$. This family $\{X_\mu\}$ can be constructed to satisfy the condition of the theorem in the case in which $V$ is a cylinder. The other case is similar. In both cases, the fact that $T$ is minimal and every $G_\mu \circ T$ is conjugate to $T$ ensures that the family $\{X_\mu\}$ has the required properties of quasi-minimality and topological equivalence.
\end{proof}

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\begin{thebibliography}
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