PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 4, Pages 1063–1074
S 0002-9939(04)07687-7

ON PEIXOTO'S CONJECTURE FOR FLOWS ON NON-ORIENTABLE 2-MANIFOLDS

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(Communicated by Michael Handel)

Abstract. Contrary to the case of vector fields on orientable compact 2-manifolds, there is a smooth vector field X on a non-orientable compact 2-manifold with a dense orbit (and therefore without closed orbits) whose phase portrait –up to topological equivalence– remains intact under a one-parameter family of twist perturbations localized in a flow box of X.

1. Introduction

The importance of the C^r-Closing Lemma Problem lies in the fact that a positive answer to it would lead to very deep positive conclusions in Dynamical Systems, in topics related to the Generic, Stability and Bifurcation Theories. As a consequence of its role, there are several useful C^r-Closing Lemmas. As a sample of some of the important results, we wish to mention Peixoto's C^r-Connecting Lemma [24], Pugh's C^1-Closing Lemma [26], Mañé's C^1-Ergodic Closing Lemma [19], Gutierrez's C^r-counterexample [9], Herman's C^r-Closing Lemma [17, 18], Hayashi's C^1-Connecting Lemma [14, 15, 16]. Besides the articles that will be quoted in this paper, in the same way as above, we wish to mention [12, 13, 20], [22]–[34]. As a sample of some very recent results that use C^1-Closing Lemma results, we wish to mention [3, 5].

Let $\mathfrak{X}^r(M)$, $0 \leq r \leq \infty$, denote the space of C^r–vector fields (with the C^r–topology) on a compact, connected, boundaryless, C^∞, 2-manifold M. Our (compact) flow boxes $V \subset M$ of X will be either the standard ones or those such that $X|_V$ is topogically equivalent to the constant vector field $(1,0)$ on the cylinder $[0,1] \times \mathbb{R}/\mathbb{Z}$. Notice that in the second case, for all $t \in (0,1)$, $\{t\} \times \mathbb{R}/\mathbb{Z}$ is a transversal circle to the vector field $(1,0)$.

To state our main result we shall need the following.

Definition 1.1 (twist perturbation of a vector field)**.** Let M be a 2-manifold, $X \in \mathfrak{X}^r(M)$ and $V \subset M$ be a compact flow box of X. Given $Y \in \mathfrak{X}^r(M)$ with support in V, we say that $X+Y$ is a C^r–twist perturbation of X, localized in V if $X|_{V^o}$ is transversal to $Y|_{V^o}$, where V^o denotes the interior of V.

Points of $\mathbb{R}/\mathbb{Z}$ will be denoted as if they were points of $\mathbb{R}$; in this way, we shall use expressions of the form $x+a$ and $x-a$ when referring to points of $\mathbb{R}/\mathbb{Z}$. Also, if

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Received by the editors November 2, 2003.
2000 *Mathematics Subject Classification.* Primary 34D30, 37E05, 37E35; Secondary 37C20.
The first author was supported in part by Pronex/CNPq/MCT grant number 66.2249/1997-6.
The second author was supported by Fapesp grant number 01/04598-0.

$x < y$ are real numbers such that $y - x < 1$, the subinterval (x, y) of $\mathbb{R}$ determines a unique subinterval of $\mathbb{R}/\mathbb{Z}$.

Definition 1.2 (smooth/affine/isometric *iet*)**.** Let Γ be either the the unit interval $[0, 1)$ or the unit circle $\mathbb{R}/\mathbb{Z}$. Given a finite subset $S = \{a_0, a_1, \cdots, a_n\}$ of Γ, with $0 = a_0 < a_1 < \cdots < a_n = 1$, a smooth interval exchange transformation, shortly smooth *iet*, will be an injective transformation $T : \Gamma \setminus S \to \Gamma$ such that $T|_{(a_{i-1},a_i)}$ (i.e. T restricted to the interval (a_{i-1}, a_i)) is a smooth diffeomorphism, $1 \leq i \leq n$, and the range of T is all Γ but n points. If, moreover, $T|_{(a_{i-1},a_i)}$ is an affine (isometric) transformation for all $1 \leq i \leq n$, we call T an affine (isometric) *iet*. As usual, the term *iet* will refer to an isometric *iet*.

Definition 1.3 (quasi-minimal vector field)**.** A C^1 vector field on a compact 2–manifold M is quasi-minimal if its set of singularities S is at most finite and any of its orbits in $M \setminus S$ is dense in M.

In this paper, we prove the following.

Theorem 1.4 (smooth *iet* version)**.** *There exist a minimal isometric iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *and a family of diffeomorphisms* $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$*, with* G_0 *being the identity map, depending smoothly on* $\mu \in [0, \epsilon)$*, such that, for all* $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$*,*

1. $\frac{d}{d\mu}\big|_{\mu=\mu_0} G_\mu(p_0) > 0$ *(in particular for all* $\mu \in (0, \epsilon)$*,* G_μ *has no fixed points);*
2. $G_{\mu_0} \circ T$ *is a smooth iet* C^∞*–conjugate to* T*.*

In the theorem above, since $T = T_0$ is minimal and every T_μ is topologically conjugate to T, we obtain that every T_μ is also minimal; in particular, all orbits of T_μ are non-trivial recurrent (and so T_μ has no closed orbits).

Given a smooth *iet* T, defined on $\mathbb{R}/\mathbb{Z}$, there exist a compact 2–manifold M, containning $\mathbb{R}/\mathbb{Z}$, and a vector field $X \in \mathfrak{X}^\infty(M)$ transversal to $\mathbb{R}/\mathbb{Z}$ such that the first return Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by X is precisely T (see [8]). Using this, the result above can be extended to vector fields as follows:

Theorem 1.5 (vector field version)**.** *Let* M *be a non-orientable compact* 2*-manifold of genus* 4*. Then there exists a family of quasi-minimal, Kupka-Smale vector fields* $\{X_\mu \in \mathfrak{X}^\infty(M)\}$*, depending smoothly on* $\mu \in [0, \epsilon)$*, such that, for some flow box* $V \subset M$ *of* X_0*, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all* $\mu, \nu \in [0, \epsilon)$*,*

1. $X_\mu|_V$ *is a flow box;*
2. *If* $\mu \neq \nu$*,* X_μ *is a* C^∞*–twist perturbation of* X_ν *localized in* V*;*
3. X_μ *and* X_ν *are topologically equivalent.*

In the theorem above, since $X = X_0$ is quasi-minimal and every X_μ is topologically equivalent to X, we obtain that every X_μ is also quasi-minimal; in particular, all regular orbits of X_μ are non-trivial recurrent (and so X_μ has no closed orbits).

Now we relate our results to Peixoto's Conjecture. Let Σ^r be the subset of $\mathfrak{X}^r(M)$ formed by the Morse-Smale C^r–vector fields. M. Peixoto states in [24] the following conjecture.

(PC) Let M be a non-orientable 2-manifold. $X \in \mathfrak{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, Σ^r is open and dense in $\mathfrak{X}^r(M)$.

Peixoto [24] proved this conjecture for M orientable. As a consequence of Peixoto's work and Pugh's C^1–Closing Lemma [26, 30], it follows that Σ^1 is dense in $\mathfrak{X}^1(M)$. There are some partial results concerning **(PC)** in class C^r, $r \geq 1$: The conjecture is true both for the projective plane $\mathbf{P}^2$ and for the Klein bottle $\mathbf{K}^2$ (see the proof in [21] that flows on $\mathbf{K}^2$ do not have non-trivial recurrence). Gutierrez [6] showed that **(PC)** is true for the torus with one cross-cap.

By [24], **(PC)** is true if, and only if, it is possible to give an affirmative answer for the following C^r–Connecting Lemma question:

(CL) Let $X \in \mathfrak{X}^r(M)$ have finitely many singularities, all hyperbolic (at least one singularity). Suppose that X has a non-trivial recurrent trajectory. Does there exist an arbitrarily small C^r–perturbation of X such that the resulting vector field has one more saddle connection than X?

We recall that, when M is orientable, **(CL)** has a positive answer by arbitrarily small C^r–twist perturbations [24]. Also, when M is non-orientable, there are many cases in which **(CL)** has a positive answer, again by arbitrarily small C^r–twist perturbations [11].

Hence, it is natural to wonder whether, in the non-orientable case, we may make use of an arbitrary C^r–twist perturbation in order to give an affirmative answer for **(CL)**. Theorem 1.5 gives a negative answer to this question.

Finally, Theorems 1.4 and 1.5 above are relevant to the C^r-Closing Lemma problem, because the positive answer to the C^r–Closing Lemma, given in [7], for a large class of flows on orientable two-manifolds, was obtained by means of twist perturbations (see also [1, 2, 4]).

2. A flow on the torus with two cross-caps

Let $\varphi : \mathbb{R} \times M \to M$ be a flow on M and $\gamma = \gamma(t)$ a trajectory of φ. We denote by $\omega(\gamma)(\alpha(\gamma))$ the ω-limit set (α-limit set) of γ. We say that γ is ω–recurrent (α–recurrent) if $\gamma \subset \omega(\gamma)(\gamma \subset \alpha(\gamma))$. A recurrent trajectory is a trajectory that is ω–recurrent or α–recurrent. A fixed point and a closed orbit are called trivial recurrent trajectories.

Given a C^1 flow $\varphi : \mathbb{R} \times M \to M$ with a recurrent trajectory γ and a point $p \in \gamma$, Peixoto proved that there exists a smooth circle Γ, transversal to φ, passing through the point p (see [6, 24]). We shall analyse flows in terms of their action on transverse circles. Let us recall now the example of a flow given by Gutierrez in [10].

Gutierrez constructed in [10] an *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (cf. Figures 1 and 2) and a suspension of T that is a quasi-minimal C^∞–flow $\varphi : \mathbb{R} \times M \to M$, on the torus with two cross-caps M, in such a way that:

(1) φ has two hyperbolic saddle points as its only singularities;
(2) $\mathbb{R}/\mathbb{Z}$ is a subset of M, φ is transversal to $\mathbb{R}/\mathbb{Z}$, and the *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is precisely the Poincaré return map induced by φ on $\mathbb{R}/\mathbb{Z}$; moreover, for some real positive numbers a, b, c, d, e,
  - $\operatorname{dom}(T) = \mathbb{R}/\mathbb{Z} \setminus \{a, a+b, a+b+c, 1\}$, where $\operatorname{dom}(T)$ is the domain of definition of T;
  - the numbers $a, a+e-c, a-c-d+2e, \cdots$, are ordered (modulo 1) according to Figures 1 and 2. In particular,
    $0 < a < b < e < a+b < 1-e < a+b+c < a+b+c+d = 1\,,\ d < e,$
    $a+b+c+e = 1+e-d;$

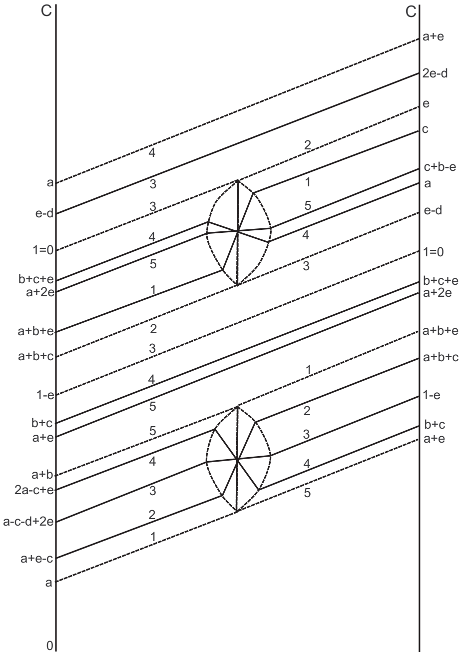

FIGURE 1. The first return map T induced by φ on $\mathbf{R}/\mathbf{Z}$

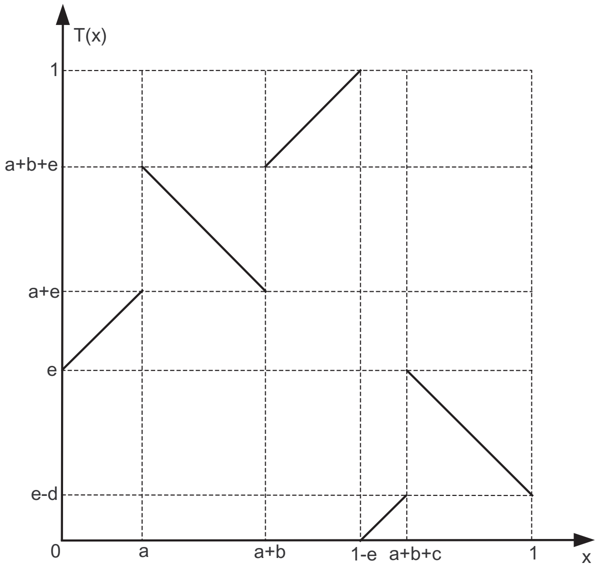

FIGURE 2. The isometric *iet* T

- T operates upon the intervals below, by reversing orientation, in the following way:

$$\begin{aligned}(a, a+b) &\mapsto (a+e, a+b+e),\\ (a+b+c, 1) &\mapsto (e-d, e);\end{aligned}$$

- T operates upon the intervals below, by preserving orientation, in the following way:

$$\begin{aligned}(0, a) &\mapsto (e, a+e);\\ (a+b, a+b+c) &\mapsto (a+b+e, e-d).\end{aligned}$$

In the next section, given a small number $\delta > 0$, we will introduce an affine *iet* T_δ that is topologically conjugate to T and 2δ-close to T, in the uniform C^0-topology.

3. Topological conjugacy

Definition 3.1 (T_δ)**.** Let $\delta > 0$ be small and let T_δ (cf. Figure 3) be the affine *iet* satisfying

- T_δ operates upon the intervals below, linearly, diffeomorphically, and reversing orientation, in the following way:

$$\begin{aligned}(a, a+e-c+\delta] &\mapsto [a+b+c, a+b+e+\delta),\\ [a+e-c+\delta, a-c-d+2e+2\delta] &\mapsto [1-e-\delta, a+b+c],\\ [a-c-d+2e+2\delta, 2a-c+e] &\mapsto [b+c+\delta, 1-e-\delta],\\ [2a-c+e, a+b) &\mapsto (a+e+\delta, b+c+\delta],\\ (a+b+c, a+b+e+\delta] &\mapsto [c, e+\delta),\\ [a+b+e+\delta, a+2e+2\delta] &\mapsto [c+b-e, c],\\ [a+2e+2\delta, b+c+e+2\delta] &\mapsto [a, c+b-e],\\ [b+c+e+2\delta, 1) &\mapsto (e-d+\delta, a];\end{aligned}$$

- T_δ operates upon the intervals below, linearly, diffeomorphically and preserving orientation, in the following way:

$$\begin{aligned}(a+b, a+e+\delta] &\mapsto (a+b+e+\delta, a+2e+2\delta],\\ [a+e+\delta, b+c+\delta] &\mapsto [a+2e+2\delta, b+c+e+2\delta],\\ [b+c+\delta, 1-e-\delta) &\mapsto [b+c+e+2\delta, 1),\\ (1-e-\delta, a+b+c) &\mapsto (0, e-d+\delta),\\ (0, e-d+\delta] &\mapsto (e+\delta, 2e-d+2\delta],\\ [e-d+\delta, a) &\mapsto [2e-d+2\delta, a+e+\delta).\end{aligned}$$

Proposition 3.2 (topological conjugacy)**.** *Given $\delta > 0$, there exist a fixed-point-free homeomorphism*

$$H = H_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

and a piecewise affine homeomorphism (cf. Figure 4)

$$h = h_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

such that

(1) $T_\delta = H \circ T$;

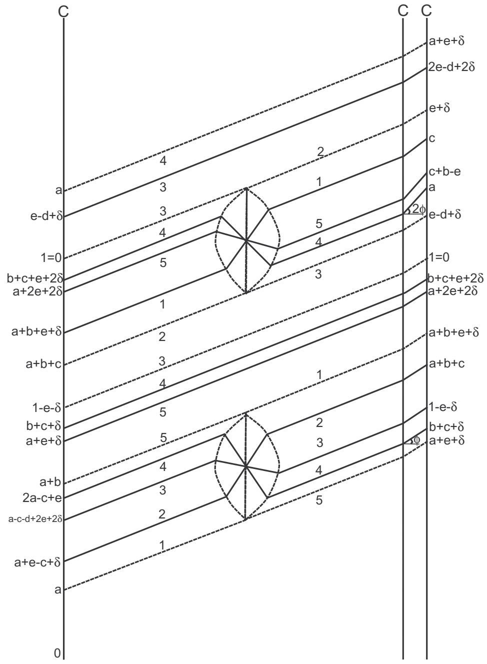

FIGURE 3. The affine *iet* T_δ

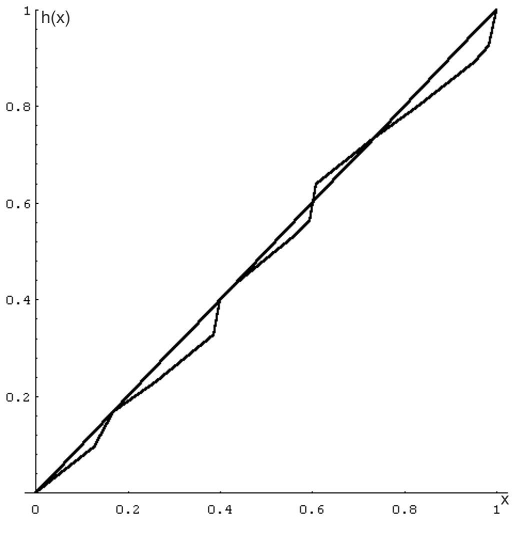

FIGURE 4. The homeomorphism h in local coordinates

(2) $\mathrm{fix}(h) \supset \mathrm{dis}(T_\delta) = \mathrm{dis}(T) = \mathbb{R}/\mathbb{Z} \setminus \mathrm{dom}(T_\delta) = \{a, a+b, a+b+c, 1\}$, *where* $\mathrm{fix}(h)$ *(resp.* $\mathrm{dis}(T)$*) denotes the set of fixed points of* h *(resp. the discontinuity point set of* T*);*

(3) $(h^{-1}\circ T\circ h)(x)=T_\delta(x)=(H\circ T)(x), \forall x\in \mathrm{dom}(T_\delta)$; *in particular, T and T_δ are topologically conjugate;*

(4) $h=h_\delta\to I$, *in the uniform C^0–topology, as δ goes to* 0, *where $I:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ denotes the identity transformation; and*

(5) *the derivative $(h_\delta)'$ of h_δ is a piecewise constant map that converges uniformly to the constant map* 1 *as $\delta\to 0$.*

Proof. The existence of h follows from the definition of T_δ. Put

$$\begin{aligned}
h(a) &= a,\\
h(a+e-c+\delta) &= a+e-c,\\
h(a-c-d+2e+2\delta) &= a-c-d+2e,\\
h(2a-c+e) &= 2a-c+e,\\
h(a+b) &= a+b,\\
h(a+e+\delta) &= a+e,\\
h(b+c+\delta) &= b+c,\\
h(1-e-\delta) &= 1-e,\\
h(a+b+c) &= a+b+c,\\
h(a+b+e+\delta) &= a+b+e,\\
h(a+2e+2\delta) &= a+2e,\\
h(b+c+e+2\delta) &= b+c+e,\\
h(1) &= 1,\\
h(e-d+\delta) &= e-d.
\end{aligned}$$

We may extend h linearly to the other points so that h becomes a piecewise affine homeomorphism of the unit circle $\mathbb{R}/\mathbb{Z}$. It follows at once that h satisfies (2)–(5). □

4. C^∞–CONJUGACY

In this section we show that the homeomorphism h conjugating T and T_δ and the fixed-point-free homeomorphism H can be substituted by a C^∞–diffeomorphism g and a fixed-point-free C^∞–diffeomorphism G, respectively, in such a way that the relation $g^{-1}\circ T\circ g=G\circ T$ remains true.

Recall $\mathrm{dis}(T)\subset\mathrm{fix}(h)$ and that

$$\begin{aligned}
\mathrm{dom}(T) &= \mathbb{R}/\mathbb{Z}\backslash\{a,a+b,a+b+c,1\},\\
\mathrm{dis}(T) &= \{a,a+b,a+b+c,1\},\\
\mathrm{dom}(T^{-1}) &= \mathbb{R}/\mathbb{Z}\backslash\{e,a+e,a+b+e,e-d\},\\
\mathrm{dis}(T^{-1}) &= \{e,a+e,a+b+e,e-d\}.
\end{aligned}$$

Let $\mathcal{G}$ be the set of C^∞–diffeomorphisms $g:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ such that

(P1) $g|_U=I|_U$, where U is a small neighborhood of $\mathrm{fix}(h)$,

(P2) $g'(x)\geq\frac{1}{2}, \forall x\in\mathbb{R}/\mathbb{Z}$;

Lemma 4.1. *If $g\in\mathcal{G}$, then*

(P3) $T(g(x))-T(x)=T'(x)(g(x)-x), \quad \forall x\in\mathrm{dom}(T)$

(where $T'(x)\in\{-1,1\}$).

Proof. An immediate consequence of (P1) is that for any $x \in \operatorname{dom}(T)$, x and $g(x)$ lie in the same interval of the partition associated to the *iet* T; that is, for each $x \in \operatorname{dom}(T)$, there exists $1 \leq i \leq n$ such that $x,\ g(x) \in (a_{i-1}, a_i)$. This implies the lemma. $\square$

We prove now the main result of this section.

Proposition 4.2. *Let* $g \in \mathcal{G}$.

(1) *The map*

$$G = g^{-1} \circ T \circ g \circ T^{-1}$$

is well defined in $\operatorname{dom}(T^{-1})$ *and extends to a* C^∞*–diffeomorphism* $G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *such that* $\forall x \in \operatorname{dom}(T)$, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$. *In particular, the isometric iet* T *and the smooth iet* $G \circ T$ *are* C^∞*–conjugate.*

(2) *Fix* $\delta > 0$ *small, and let* $h = h_\delta$ *be as in Proposition 3.2. If* $g \in \mathcal{G}$ *is* C^0*-close enough to* h *and* G *is as above, then* $G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ *is a fixed-point-free* C^∞*–diffeomorphism.*

Proof. Extend G to the whole $\mathbb{R}/\mathbb{Z}$ by defining $G(q) = g^{-1}(q), \forall q \in \operatorname{dis}(T^{-1})$. Notice that G is bijective, $G|_{\operatorname{dom}(T^{-1})}$ is smooth, and $\left(G|_{\operatorname{dom}(T^{-1})}\right)^{-1}$ is smooth. If x is in a small neighborhood W_q of $q \in \operatorname{dis}(T^{-1})$, $x \neq q$, then $T^{-1}(x)$ is in a neighborhood U_p of p for some $p \in \operatorname{dis}(T) \subset \operatorname{fix}(h)$ and by (P1) and by definition of G, we get

$$G(x) = g^{-1}(x),\ \forall x \in W_q.$$

Therefore, G is smooth at any point $q \in \operatorname{dis}(T^{-1})$ and as, by (P2), $G'(q) \neq 0$, we obtain that G^{-1} is a C^∞–diffeomorphism. By definition of G, $(g^{-1} \circ T \circ g)(x) = (G \circ T)(x)$, $\forall x \in \operatorname{dom}(T)$. This proves (1). If g is C^0-close to h, then G will also be C^0-close to H; therefore, since H is a fixed-point-free homeomorphism, we will obtain that G is also fixed-point-free. $\square$

5. MAIN RESULTS

In this section we prove Theorems 1.4 and 1.5. We start by proving Theorem 1.4.

Theorem 1.4 (smooth *iet* version)**.** There exist a minimal isometric *iet* $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a family of diffeomorphisms $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$, with G_0 being the identity map, depending smoothly on $\mu \in [0, \epsilon)$, such that, for all $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$,

(1) $\frac{d}{d\mu}\Big|_{\mu=\mu_0} G_\mu(p_0) > 0$;
(2) $G_{\mu_0} \circ T$ is a smooth *iet* C^∞–conjugate to T.

Proof. By Proposition 4.2, there exist a diffeomorphism g with Properties (P1) to (P3) and a fixed-point-free diffeomorphism G such that $g^{-1} \circ T \circ g = G \circ T$ at every point of $\operatorname{dom}(T)$. Given $\mu \in [0, 1]$, let $g_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by

$$g_\mu(x) = \mu g(x) + (1 - \mu)x. \tag{5.1}$$

Notice that $\{g_\mu : \mu \in [0, 1]\}$ provides a smooth isotopy between the identity map $g_0 = I$ and $g_1 = g$; also by (P2),

$$\frac{dg_\mu}{dx}(x) = \mu g'(x) + 1 - \mu \geq \frac{\mu}{2} + 1 - \mu = 1 - \frac{\mu}{2} \geq \frac{1}{2},\ \forall x \in \mathbb{R}/\mathbb{Z}.$$

Thus, by construction, g_μ is a diffeomorphism of the unit circle satisfying (P1)-(P3), for each $\mu \in [0,1]$, that is, $g_\mu \in \mathcal{G}$. Given $\mu \in [0,1]$, by Proposition 4.2, there exists a diffeomorphism $G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ satisfying

$$(G_\mu \circ T)(x) = \big((g_\mu)^{-1} \circ T \circ g_\mu\big)(x),\ \forall x \in \operatorname{dom}(T). \tag{5.2}$$

The proof of this theorem follows at once from the following lemma.

Lemma 5.1. *Given $x \in \mathbb{R}/\mathbb{Z}$, the map $\mu \in [0,1] \mapsto G_\mu(x) \in \mathbb{R}/\mathbb{Z}$ is differentiable. Moreover, there exist $\epsilon > 0$ and $\sigma > 0$ such that $\forall(\mu_0, x) \in [0,\epsilon) \times \mathbb{R}/\mathbb{Z}$, $\frac{d}{d\mu}\Big|_{\mu=\mu_0} G_\mu(x) > \sigma$.*

Proof. Let $u \in (0,1]$ be a real number. Then, from (5.2),

$$\big(T \circ g_u\big)(x) = \big(g_u \circ G_u \circ T\big)(x),\ \forall x \in \operatorname{dom}(T). \tag{5.3}$$

From (5.1), we reach

$$g_u(y) = y + u(g(y) - y),\ \forall y \in \mathbb{R}/\mathbb{Z}. \tag{5.4}$$

From (5.4) and from property (P3) of g_u, we obtain

$$\frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{T'(x) \cdot \big(g_u(x) - x\big)}{u} = T'(x) \cdot \big(g(x) - x\big). \tag{5.5}$$

From equations (5.3)-(5.5), we get

$$\begin{aligned} T'(x) \cdot \big(g(x) - x\big) &\overset{(5.5)}{=} \{(T \circ g_u)(x) - (T \circ g_0)(x)\}/u \\ &\overset{(5.3)}{=} \{(g_u \circ G_u \circ T)(x) - (g_0 \circ G_0 \circ T)(x)\}/u \\ &\overset{(5.4)}{=} \frac{G_u(T(x)) - G_0(T(x))}{u} + g(G_u(T(x))) - G_u(T(x)). \end{aligned}$$

When $u \to 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T'(x) \cdot \big(g(x) - x\big) - \big(g(T(x)) - T(x)\big), \tag{5.6}$$

and from Property (P3) of g, we reach

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) = T(g(x)) - g(T(x)), \quad \forall x \in \operatorname{dom}(T). \tag{5.7}$$

Since $G = g^{-1} \circ T \circ g \circ T^{-1}$ has no fixed points, we have that $T(g(x)) - g(T(x)) \neq 0$, $\forall x \in \operatorname{dom}(T)$. Therefore,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(T(x)) \neq 0,\ \forall x \in \operatorname{dom}(T).$$

That is,

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) \neq 0, \quad \forall y \in \operatorname{dom}(T^{-1}). \tag{5.8}$$

If $q \in \operatorname{dis}(\mathrm{T}^{-1})$, then $G_\mu(q) = (g_\mu)^{-1}(q)$, $\forall \mu$. Besides, we have by Figure 4 that $q - g(q) > 0$ (the points $(q, g(q))$ with $q \in \operatorname{dis}(T^{-1})$ lie below the diagonal). Observe that

$$\begin{aligned} \big(g_\mu \circ (g_\mu)^{-1}\big)(x) &= x, \quad \forall x \in \mathbb{R}/\mathbb{Z}, \\ \mu \cdot g((g_\mu)^{-1}(x)) + (1-\mu)(g_\mu)^{-1}(x) &= x, \quad \forall x \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Therefore, by differentiating the previous equation with respect to μ at $\mu = 0$, we get

$$\frac{d}{d\mu}\Big|_{\mu=0} G_\mu(q) = \frac{d}{d_\mu}\Big|_{\mu=0} (g_\mu)^{-1}(q) = q - g(q) > 0. \tag{5.9}$$

We remark that equations (5.7) and (5.9) are compatible; that is, for any $q \in \mathrm{dis}(T^{-1})$,

$$\lim_{y\to q} \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(y) = q - g(q).$$

Hence, the map $(\mu, x) \in [0,1] \times \mathbb{R}/\mathbb{Z} \mapsto \frac{d}{d\mu}\Big|_{\mu=0} G_\mu(x)$ is continuous. This implies the lemma. □

Corollary 5.2. *For all $\mu \in (0, \epsilon)$, G_μ has no fixed points.*

Proof. This follows immediately from Theorem 1.4. □

We now prove Theorem 1.5.

Theorem 1.5 (Vector field version)**.** Let M be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale flows $\{X_\mu \in \mathfrak{X}^\infty(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of X_0, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,

1. $X_\mu|_V$ is a flow box;
2. if $\mu \neq \nu$, X_μ is a C^∞-twist perturbation of X_ν localized in V;
3. X_μ and X_ν are topologically equivalent.

Proof. Let T be as in Theorem 1.4. From Theorem 1.4, we know that $\{G_\mu \circ T\}$ is a family of smooth *iet*'s conjugate to T. Each *iet* $\{G_\mu \circ T\}$ may be suspended to obtain a smooth vector field X_μ on a non-orientable compact manifold M of genus 4 (see [8]). We may assume that M contains $\mathbb{R}/\mathbb{Z}$ and does not depend on μ. By definition of suspension, each X_μ is tranversal to $\mathbb{R}/\mathbb{Z}$ and $G_\mu \circ T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the forward Poincaré map induced by X_μ. This family $\{X_\mu\}$ can be constructed to satisfy the condition of the theorem in the case in which V is a cylinder. The other case is similar. In both cases, the fact that T is minimal and every $G_\mu \circ T$ is conjugate to T ensures that the family $\{X_\mu\}$ has the required properties of quasi-minimality and topological equivalence. □

ACKNOWLEDGMENTS

We wish to thank Carlos Gustavo Moreira from IMPA and the referee for very helpful comments to previous versions of this article.

REFERENCES

1. A. Pierre, M. Malkin and E. Zhuzhoma. *On the C^r–closing lemma for surface flows and expansions of points of the circle at infinity.* Preprint IML-2001.
2. S. Aranson and E. Zhuzhoma. *On the C^r-closing lemma and the Koebe-Morse coding of geodesics on surfaces.* Journ. Dyn. Contr. Syst., **7** (2001), No. 1, 15–48. MR1817328 (2001m:37042)
3. F. Abdenur, A. Avila, J. Bochi. *Robust transitivity and topological mixing for C^1-flows.* Proc. Amer. Math. Soc. **132** (2004), 699–705. MR2019946
4. C. R. Carrol. *Rokhlin towers and C^r closing for flows on* $\mathbf{C}^2$. Ergod. Th. & Dynam. Sys. (1992), **12**, 683–706. MR1200337 (94c:58177)

5. J. Franks. *Rotation numbers and instability sets.* Bull. Amer. Math. Soc. **40** (2003), 263-279. MR1978565 (2004h:37063)
6. C. Gutierrez. *Structural stability for flows on the torus with a cross-cap.* Transactions of the American Mathematical Society, Volume 241 (1978). MR0492303 (80k:58065)
7. C. Gutierrez. *On the C^r-Closing for Flows on 2-Manifolds.* Nonlinearity **13** (2000), No. 6, 1883–1888. MR1794837 (2001m:37043)
8. C. Gutierrez. *Smoothing continuous flows on two-manifolds and recurrences.* Erg. Th. and Dyn. Sys. (1986), **6**, 17–44. MR0837974 (87k:58222)
9. C. Gutierrez. *A counter-example to a C^2- closing lemma.* Erg. Th. and Dyn. Sys. (1987), **7**, 509–530. MR0922363 (89k:58240)
10. C. Gutierrez. *Smooth Nonorientable Nontrivial Recurrence on Two-Manifolds.* Journal of Dif. Equations, vol. 29, N^0 3, pp. 338-395, 1978. MR0507486 (80d:58059)
11. C. Gutierrez and B. F. Pires. *On a C^r–Connecting Lemma for Flows on a Non-Orientable 2-Manifold.* To be written.
12. C. Gutierrez and J. Sotomayor *An Approximation Theorem for Immersions with Stable Configurations of Lines of Principal Curvature.* Bifurcation, Théorie Ergodique et Applications. Dijon, Juin 1981: Astérisque, 98-99, pp. 195-215, 1982. MR0730276 (85b:53002)
13. C. Gutierrez and J. Sotomayor. *Lines of Curvature and Umbilical Points on Surfaces.* 18^0 Colóquio Brasileiro de Matemática, IMPA, 1991.
14. S. Hayashi. *Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjecture for flows.* Annals of Mathematics, **145** (1997), 81–137. MR1432037 (98b:58096)
15. S. Hayashi. *Correction to Connecting invariant manifolds and the solution of the C^1 stability and Ω-stability conjecture for flows.* Annals of Mathematics, **150** (1999), 353–356. MR1715329 (2000h:37029)
16. S. Hayashi. *A C^1 make or break lemma.* Bol. Soc. Brasil. Mat. (N.S.) **31** (2000), no.3, 337–350. MR1817092 (2002c:37042)
17. M. Herman. *Exemples de flots hamiltoniens dont aucune perturbation en topologie C^∞ n'a d'orbites périodiques sur un ouvert de surfaces d'énergies.* C. R. Acad. Sci. Paris Sér. I Math. **t. 312** (1991), No. 13, 989–994. MR1113091 (92j:58034)
18. M. Herman. *Différentiabilité optimale et contre-exemples à la fermeture en topologie C^∞ des orbites récurrentes de flots hamiltoniens.* C. R. Acad. Sci. Paris Sér. I Math. **t. 313** (1991), No. 1, 49–51. MR1115947 (92m:58046)
19. R. Mañé. *An ergodic closing lemma.* Ann. of Math. (2) **116** (1982), No. 3, 503–540. MR0678479 (84f:58070)
20. R. Mañé. *On the creation of homoclinic points.* Inst. Hautes Études Sci. Publ. Math. No. 66, (1988), 139–159. MR0932137 (89e:58089)
21. N. Markley.*The Poincaré-Bendixson Theorem for the Klein Bottle.* Trans. Amer. Math. Soc. **135** (1969). MR0234442 (38:2759)
22. F. Oliveira. *On the generic existence of homoclinic points.* Ergodic Theory and Dynamical Systems **7** (1987) No. 4, 567–595. MR0922366 (89j:58104)
23. F. Oliveira. *On the C^∞ genericity of homoclinic orbits.* Nonlinearity **13** No. 3 (2000), 653–662. MR1758993 (2001h:37106)
24. M. Peixoto. *Structural stability on two-dimensional manifolds.* Topology, 1 (1962), pp. 101–120. MR0142859 (26:426)
25. D. Pixton. *Planar homoclinic points.* Jr. Differential Equations **44** (1982), No. 3, 365–382. MR0661158 (83h:58077)
26. C. Pugh. *An improved closing lemma and a general density theorem.* Amer. J. Math. **89** (1967), 1010–1021. MR0226670 (37:2257)
27. C. Pugh. *Against the C^2-closing lemma.* J. Differential Equations 17 (1975), 435–443. MR0368079 (51:4321)
28. C. Pugh. *A special C^r-closing lemma.* Geometric Dynamics (Rio de Janeiro, 1981), 636–650, Lecture Notes in Math., **1007**, Springer, Berlin, 1983. MR0730291 (85e:58120)
29. C. Pugh. *The C^1 connecting lemma.* J. Dynam. Differential Equations 4 (1992), no. 4, 545–553. MR1187222 (93i:58091)
30. C. Pugh and C. Robinson. *The C^1-closing lemma, including Hamiltonians.* Ergodic Th. and Dynamical Systems **3** (1983), No. 2, 261–313. MR0742228 (85m:58106)
31. C. Bonatti and S. Crovisier. *Récurrence et généricité.* Preprint (2003), no. 324, Institute de Mathématiques de Bourgone. MR1990025 (2004d:37028)

32. C. Robinson *Closing stable and unstable manifolds on the two sphere.* Proc. Amer. Math. Soc. 41 (1973), 299–303. MR0321141 (47:9674)
33. F. Takens. *Homoclinic points in conservative systems.* Invent. Math. **18** (1972), 267–292. MR0331435 (48:9768)
34. L. Wen and Z. Xia. C^1 *connecting lemmas.* Trans. Amer. Math. Soc. **352** (2000), no. 11, 5213–5230. MR1694382 (2001b:37024)

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