ON PEIXOTO’S CONJECTURE FOR FLOWS ON NON-ORIENTABLE 2-MANIFOLDS

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Abstract. Contrary to the case of vector fields on orientable compact 2-manifolds, there is a smooth vector field $X$ on a non-orientable compact 2-manifold with a dense orbit (and therefore without closed orbits) whose phase portrait—up to topological equivalence—remains intact under a one-parameter family of twist perturbations localized in a flow box of $X$.

1. Introduction

The importance of the $C^r$-Closing Lemma Problem lies in the fact that a positive answer to it would lead to very deep positive conclusions in Dynamical Systems, in topics related to the Generic, Stability and Bifurcation Theories. As a consequence of its role, there are several useful $C^r$-Closing Lemmas. As a sample of some of the important results, we wish to mention Peixoto’s $C^r$-Connecting Lemma [24], Pugh’s $C^1$-Closing Lemma [26], Mañé’s $C^1$-Ergodic Closing Lemma [19], Gutierrez’s $C^r$-counterexample [9], Herman’s $C^r$-Closing Lemma [17, 18], Hayashi’s $C^1$-Connecting Lemma [14, 15, 16]. Besides the articles that will be quoted in this paper, in the same way as above, we wish to mention [12, 13, 20, 22, 34]. As a sample of some very recent results that use $C^1$-Closing Lemma results, we wish to mention [3, 5].

Let $\mathcal{X}^r(M)$, $0 \leq r \leq \infty$, denote the space of $C^r$–vector fields (with the $C^r$–topology) on a compact, connected, boundaryless, $C^\infty$, 2-manifold $M$. Our (compact) flow boxes $V \subset M$ of $X$ will be either the standard ones or those such that $X|_V$ is topologically equivalent to the constant vector field $(1, 0)$ on the cylinder $[0, 1] \times \mathbb{R}/\mathbb{Z}$. Notice that in the second case, for all $t \in (0, 1)$, $\{t\} \times \mathbb{R}/\mathbb{Z}$ is a transversal circle to the vector field $(1, 0)$.

To state our main result we shall need the following.

Definition 1.1 (twist perturbation of a vector field). Let $M$ be a 2-manifold, $X \in \mathcal{X}^r(M)$ and $V \subset M$ be a compact flow box of $X$. Given $Y \in \mathcal{X}^r(M)$ with support in $V$, we say that $X + Y$ is a $C^r$–twist perturbation of $X$, localized in $V$ if $X|_V$ is topologically equivalent to $Y|_{V^o}$, where $V^o$ denotes the interior of $V$.

Points of $\mathbb{R}/\mathbb{Z}$ will be denoted as if they were points of $\mathbb{R}$; in this way, we shall use expressions of the form $x + a$ and $x - a$ when referring to points of $\mathbb{R}/\mathbb{Z}$. Also, if...
$x < y$ are real numbers such that $y - x < 1$, the subinterval $(x, y)$ of $\mathbb{R}$ determines a unique subinterval of $\mathbb{R}/\mathbb{Z}$.

**Definition 1.2** (smooth/affine/isometric iet). Let $\Gamma$ be either the the unit interval $[0, 1)$ or the unit circle $\mathbb{R}/\mathbb{Z}$. Given a finite subset $S = \{a_0, a_1, \ldots, a_n\}$ of $\Gamma$, with $0 = a_0 < a_1 < \cdots < a_n = 1$, a smooth interval exchange transformation, shortly smooth iet, will be an injective transformation $T : \Gamma \setminus S \to \Gamma$ such that $T|_{(a_{i-1}, a_i)}$ is a smooth diffeomorphism, $1 \leq i \leq n$, and the range of $T$ is all $\Gamma$ but $n$ points. If, moreover, $T|_{(a_{i-1}, a_i)}$ is an affine (isometric) transformation for all $1 \leq i \leq n$, we call $T$ an affine (isometric) iet. As usual, the term iet will refer to an isometric iet.

**Definition 1.3** (quasi-minimal vector field). A $C^1$ vector field on a compact 2–manifold $M$ is quasi-minimal if its set of singularities $S$ is at most finite and any of its orbits in $M \setminus S$ is dense in $M$.

In this paper, we prove the following.

**Theorem 1.4** (smooth iet version). There exist a minimal isometric iet $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and a family of diffeomorphisms $\{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon\}$, with $G_0$ being the identity map, depending smoothly on $\mu \in [0, \epsilon)$, such that, for all $(\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z}$,

1. $\frac{d}{\mu_0}G_\mu(p_0) > 0$ (in particular for all $\mu \in (0, \epsilon)$, $G_\mu$ has no fixed points);
2. $G_{\mu_0} \circ T$ is a smooth iet $C^\infty$–conjugate to $T$.

In the theorem above, since $T = T_0$ is minimal and every $T_\mu$ is topologically conjugate to $T$, we obtain that every $T_\mu$ is also minimal; in particular, all orbits of $T_\mu$ are non-trivial recurrent (and so $T_\mu$ has no closed orbits).

Given a smooth iet $T$, defined on $\mathbb{R}/\mathbb{Z}$, there exist a compact 2–manifold $M$, containing $\mathbb{R}/\mathbb{Z}$, and a vector field $X \in \mathcal{X}^\infty(M)$ transversal to $\mathbb{R}/\mathbb{Z}$ such that the first return Poincaré map $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ induced by $X$ is precisely $T$ (see [3]). Using this, the result above can be extended to vector fields as follows:

**Theorem 1.5** (vector field version). Let $M$ be a non-orientable compact 2–manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale vector fields $\{X_\mu \in \mathcal{X}^\infty(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of $X_0$, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,

1. $X_\mu|_V$ is a flow box;
2. If $\mu \neq \nu$, $X_\mu$ is a $C^\infty$–twist perturbation of $X_\nu$ localized in $V$;
3. $X_\mu$ and $X_\nu$ are topologically equivalent.

In the theorem above, since $X = X_0$ is quasi-minimal and every $X_\mu$ is topologically equivalent to $X$, we obtain that every $X_\mu$ is also quasi-minimal; in particular, all regular orbits of $X_\mu$ are non-trivial recurrent (and so $X_\mu$ has no closed orbits).

Now we relate our results to Peixoto’s Conjecture. Let $\Sigma^r$ be the subset of $\mathcal{X}^r(M)$ formed by the Morse-Smale $C^r$–vector fields. M. Peixoto states in [24] the following conjecture.

**PC** Let $M$ be a non-orientable 2–manifold. $X \in \mathcal{X}^r(M)$ is structurally stable if and only if $X \in \Sigma^r$. Moreover, $\Sigma^r$ is open and dense in $\mathcal{X}^r(M)$.

Peixoto [24] proved this conjecture for $M$ orientable. As a consequence of Peixoto’s work and Pugh’s $C^1$–Closing Lemma [26, 30], it follows that $\Sigma^1$ is dense
in $\mathcal{X}^1(M)$. There are some partial results concerning (PC) in class $C^r$, $r \geq 1$: The conjecture is true both for the projective plane $\mathbb{P}^2$ and for the Klein bottle $K^2$ (see the proof in [24] that flows on $K^2$ do not have non-trivial recurrence). Gutierrez [6] showed that (PC) is true for the torus with one cross-cap.

By [24], (PC) is true if, and only if, it is possible to give an affirmative answer for the following $\text{CL}$–Connecting Lemma question:

(CL) Let $X \in \mathcal{X}^r(M)$ have finitely many singularities, all hyperbolic (at least one singularity). Suppose that $X$ has a non-trivial recurrent trajectory. Does there exist an arbitrarily small $C^r$–perturbation of $X$ such that the resulting vector field has one more saddle connection than $X$?

We recall that, when $M$ is orientable, (CL) has a positive answer by arbitrarily small $C^r$–twist perturbations. Also, when $M$ is non-orientable, there are many cases in which (CL) has a positive answer, again by arbitrarily small $C^r$–twist perturbations.

Hence, it is natural to wonder whether, in the non-orientable case, we may make use of an arbitrary $C^r$–twist perturbation in order to give an affirmative answer for (CL). Theorem 1.5 gives a negative answer to this question.

Finally, Theorems 1.4 and 1.5 above are relevant to the $\text{PC}$–Closing Lemma problem, because the positive answer to the $C^r$–Closing Lemma, given in [7], for a large class of flows on orientable two-manifolds, was obtained by means of twist perturbations (see also [11, 2] [4]).

2. A FLOW ON THE TORUS WITH TWO CROSS-CAPS

Let $\varphi : \mathbb{R} \times M \to M$ be a flow on $M$ and $\gamma = \gamma(t)$ a trajectory of $\varphi$. We denote by $\omega(\gamma)(\alpha(\gamma))$ the $\omega$-limit set ($\alpha$-limit set) of $\gamma$. We say that $\gamma$ is $\omega$–recurrent ($\alpha$–recurrent) if $\gamma \subset \omega(\gamma)(\gamma \subset \alpha(\gamma))$. A recurrent trajectory is a trajectory that is $\omega$–recurrent or $\alpha$–recurrent. A fixed point and a closed orbit are called trivial recurrent trajectories.

Given a $C^1$ flow $\varphi : \mathbb{R} \times M \to M$ with a recurrent trajectory $\gamma$ and a point $p \in \gamma$, Peixoto proved that there exists a smooth circle $T$, transversal to $\varphi$, passing through the point $p$ (see [6] [24]). We shall analyse flows in terms of their action on transverse circles. Let us recall now the example of a flow given by Gutierrez in [10].

Gutierrez constructed in [10] an et $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ (cf. Figures 1 and 2) and a suspension of $T$ that is a quasi-minimal $C^\infty$–flow $\varphi : \mathbb{R} \times M \to M$, on the torus with two cross-caps $M$, in such a way that:

1. $\varphi$ has two hyperbolic saddle points as its only singularities;
2. $\mathbb{R}/\mathbb{Z}$ is a subset of $M$, $\varphi$ is transversal to $\mathbb{R}/\mathbb{Z}$, and the et $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is precisely the Poincaré return map induced by $\varphi$ on $\mathbb{R}/\mathbb{Z}$; moreover, for some real positive numbers $a, b, c, d, e$,
   - $\text{dom}(T) = \mathbb{R}/\mathbb{Z} \setminus \{a, a + b, a + b + c, 1\}$, where $\text{dom}(T)$ is the domain of definition of $T$;
   - the numbers $a, a + e - c, a - c - d + 2e, \cdots$, are ordered (modulo 1) according to Figures 1 and 2. In particular, $0 < a < b < e < a + b < 1 - e < a + b + c < a + b + c + d = 1, d < e,$ $a + b + c + e = 1 + e - d$.
Figure 1. The first return map $T$ induced by $\varphi$ on $\mathbb{R}/\mathbb{Z}$

Figure 2. The isometric $iet$ $T$
$T$ operates upon the intervals below, by reversing orientation, in the following way:

\[(a, a + b) \mapsto (a + e, a + b + e),\]

\[(a + b + e, 1) \mapsto (e - d, e);\]

$T$ operates upon the intervals below, by preserving orientation, in the following way:

\[(0, a) \mapsto (e, a + e);\]

\[(a + b, a + b + c) \mapsto (a + b + e, e - d).\]

In the next section, given a small number $\delta > 0$, we will introduce an affine iet $T_\delta$ that is topologically conjugate to $T$ and $2\delta$-close to $T$, in the uniform $C^0$-topology.

3. Topological conjugacy

**Definition 3.1** ($T_\delta$). Let $\delta > 0$ be small and let $T_\delta$ (cf. Figure 3) be the affine iet satisfying

- $T_\delta$ operates upon the intervals below, linearly, diffeomorphically, and reversing orientation, in the following way:
  
  \[(a, a + e - c + \delta) \mapsto [a + b + c, a + b + e + \delta),\]

  \[[a + e - c + \delta, a - c - d + 2e + 2\delta] \mapsto [1 - e - \delta, a + b + c],\]

  \[[a - c - d + 2e + 2\delta, 2a - c + e] \mapsto [b + c + \delta, 1 - e - \delta],\]

  \[[2a - c + e, a + b] \mapsto (a + e + \delta, b + c + \delta),\]

  \[(a + b + c, a + b + e + \delta] \mapsto [c, e + \delta),\]

  \[[a + b + e + \delta, a + 2e + 2\delta] \mapsto [c + b - e, c],\]

  \[[a + 2e + 2\delta, b + c + e + 2\delta] \mapsto [a, c + b - e],\]

  \[[b + c + e + 2\delta, 1) \mapsto (e - d - \delta, a);\]

- $T_\delta$ operates upon the intervals below, linearly, diffeomorphically and preserving orientation, in the following way:
  
  \[(a + b, a + e + \delta] \mapsto (a + b + e + \delta, a + 2e + 2\delta],\]

  \[[a + e + \delta, b + c + \delta] \mapsto [a + 2e + 2\delta, b + c + e + 2\delta],\]

  \[[b + c + \delta, 1 - e - \delta) \mapsto [b + c + e + 2\delta, 1),\]

  \[(1 - e - \delta, a + b + c) \mapsto (0, e - d + \delta),\]

  \[(0, e - d + \delta] \mapsto (e + \delta, 2e - d + 2\delta],\]

  \[[e - d + \delta, a) \mapsto [2e - d + 2\delta, a + e + \delta).\]

**Proposition 3.2** (topological conjugacy). Given $\delta > 0$, there exist a fixed-point-free homeomorphism

\[H = H_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\]

and a piecewise affine homeomorphism (cf. Figure 4)

\[h = h_\delta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\]

such that

\[1) \ T_\delta = H \circ T;\]
Figure 3. The affine iet $T_\delta$

Figure 4. The homeomorphism $h$ in local coordinates

(2) $\text{fix}(h) \supset \text{dis}(T_\delta) = \text{dis}(T) = \mathbb{R}/\mathbb{Z} \setminus \text{dom}(T_\delta) = \{a, a + b, a + b + c, 1\},$

where $\text{fix}(h)$ (resp. $\text{dis}(T)$) denotes the set of fixed points of $h$ (resp. the discontinuity point set of $T$);
(3) \((h^{-1} \circ T \circ h)(x) = T_{\delta}(x) = (H \circ T)(x), \forall x \in \text{dom}(T_{\delta}); \) in particular, \(T\) and \(T_{\delta}\) are topologically conjugate;

(4) \(h = h_{\delta} \to I, \) in the uniform \(C^0\)-topology, as \(\delta\) goes to 0, where \(I : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\) denotes the identity transformation; and

(5) the derivative \((h_{\delta})'\) of \(h_{\delta}\) is a piecewise constant map that converges uniformly to the constant map 1 as \(\delta \to 0.\)

Proof. The existence of \(h\) follows from the definition of \(T_{\delta}.\) Put

\[
\begin{align*}
h(a) & = a, \\
h(a + e - c + \delta) & = a + e, \\
h(a - c - d + 2e + 2\delta) & = a - c - d + 2e, \\
h(2a - c + e) & = 2a - c + e, \\
h(a + b) & = a + b, \\
h(a + e + \delta) & = a + e, \\
h(b + c + \delta) & = b + c, \\
h(1 - e - \delta) & = 1 - e, \\
h(a + b + c) & = a + b + c, \\
h(a + b + e + \delta) & = a + b + e, \\
h(a + 2e + 2\delta) & = a + 2e, \\
h(b + c + e + 2\delta) & = b + c + e, \\
h(1) & = 1, \\
h(e - d + \delta) & = e - d.
\end{align*}
\]

We may extend \(h\) linearly to the other points so that \(h\) becomes a piecewise affine homeomorphism of the unit circle \(\mathbb{R}/\mathbb{Z}.\) It follows at once that \(h\) satisfies (2)–(5). \(\square\)

4. \(C^\infty\)-Conjugacy

In this section we show that the homeomorphism \(h\) conjugating \(T\) and \(T_{\delta}\) and the fixed-point-free homeomorphism \(H\) can be substituted by a \(C^\infty\)-diffeomorphism \(g\) and a fixed-point-free \(C^\infty\)-diffeomorphism \(G,\) respectively, in such a way that the relation \(g^{-1} \circ T \circ g = G \circ T\) remains true.

Recall \(\text{dis}(T) \subset \text{fix}(h)\) and that

\[
\begin{align*}
\text{dom}(T) & = \mathbb{R}/\mathbb{Z} \setminus \{a, a + b, a + b + c, 1\}, \\
\text{dis}(T) & = \{a, a + b, a + b + c, 1\}, \\
\text{dom}(T^{-1}) & = \mathbb{R}/\mathbb{Z} \setminus \{e, a + e, a + b + e, e - d\}, \\
\text{dis}(T^{-1}) & = \{e, a + e, a + b + e, e - d\}.
\end{align*}
\]

Let \(\mathcal{G}\) be the set of \(C^\infty\)-diffeomorphisms \(g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}\) such that

(P1) \(g|_U = I|_U,\) where \(U\) is a small neighborhood of \(\text{fix}(h),\)

(P2) \(g'(x) \geq \frac{1}{2}, \forall x \in \mathbb{R}/\mathbb{Z};\)

Lemma 4.1. If \(g \in \mathcal{G},\) then

(P3) \(T(g(x)) - T(x) = T'(x)(g(x) - x), \forall x \in \text{dom}(T)\)

(\text{where} \(T'(x) \in \{-1, 1\}).\)
Proof. An immediate consequence of (P1) is that for any \( x \in \text{dom}(T) \), \( x \) and \( g(x) \) lie in the same interval of the partition associated to the iet \( T \); that is, for each \( x \in \text{dom}(T) \), there exists \( 1 \leq i \leq n \) such that \( x, g(x) \in (a_{i-1}, a_i) \). This implies the lemma.

We prove now the main result of this section.

**Proposition 4.2.** Let \( g \in \mathcal{G} \).

1. The map \( G = g^{-1} \circ T \circ g^{-1} \) is well defined in \( \text{dom}(T^{-1}) \) and extends to a \( C^\infty \)-diffeomorphism \( G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) such that \( \forall x \in \text{dom}(T) \), \((g^{-1} \circ T \circ g)(x) = (G \circ T)(x) \). In particular, the isometric iet \( T \) and the smooth iet \( G \circ T \) are \( C^\infty \)-conjugate.

2. Fix \( \delta > 0 \) small, and let \( h = h_\delta \) be as in Proposition 3.2. If \( g \in \mathcal{G} \) is \( C^0 \)-close enough to \( h \) and \( G \) is as above, then \( G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) is a fixed-point-free \( C^\infty \)-diffeomorphism.

Proof. Extend \( G \) to the whole \( \mathbb{R}/\mathbb{Z} \) by defining \( G(q) = g^{-1}(q), \forall q \in \text{dis}(T^{-1}) \). Notice that \( G \) is bijective, \( G|_{\text{dom}(T^{-1})} \) is smooth, and \( (G|_{\text{dom}(T^{-1})})^{-1} \) is smooth. If \( x \) is in a small neighborhood \( W_q \) of \( q \in \text{dis}(T^{-1}) \), \( x \neq q \), then \( T^{-1}(x) \) is in a neighborhood \( U_p \) of \( p \) for some \( p \in \text{dis}(T) \subset \text{dis}(h) \) and by (P1) and by definition of \( G \), we get

\[
G(x) = g^{-1}(x), \forall x \in W_q.
\]

Therefore, \( G \) is smooth at any point \( q \in \text{dis}(T^{-1}) \) and as, by (P2), \( G'(q) \neq 0 \), we obtain that \( G^{-1} \) is a \( C^\infty \)-diffeomorphism. By definition of \( G \), \((g^{-1} \circ T \circ g)(x) = (G \circ T)(x), \forall x \in \text{dom}(T) \). This proves (1). If \( g \) is \( C^0 \)-close to \( h \), then \( G \) will also be \( C^0 \)-close to \( H \); therefore, since \( H \) is a fixed-point-free homeomorphism, we will obtain that \( G \) is also fixed-point-free.

\[ Hence, \text{dis}(G) \subset \text{dis}(H) \;
\]

5. MAIN RESULTS

In this section we prove Theorems 1.4 and 1.5. We start by proving Theorem 1.4 (smooth iet version). There exist a minimal isometric iet \( T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) and a family of diffeomorphisms \( \{G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, 0 \leq \mu < \epsilon \} \), with \( G_0 \) being the identity map, depending smoothly on \( \mu \in [0, \epsilon) \), such that, for all \((\mu_0, p_0) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z} \),

1. \( \frac{d}{d\mu}|_{\mu=\mu_0} G_\mu(p_0) > 0 \);

2. \( G_{\mu_0} \circ T \) is a smooth iet \( C^\infty \)-conjugate to \( T \).

Proof. By Proposition 4.2 there exist a diffeomorphism \( g \) with Properties (P1) to (P3) and a fixed-point-free diffeomorphism \( G \) such that \( g^{-1} \circ T \circ g = G \circ T \) at every point of \( \text{dom}(T) \). Given \( \mu \in [0, 1] \), let \( g_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) be defined by

\[
g_\mu(x) = \mu g(x) + (1 - \mu)x.
\]

Notice that \( \{g_\mu : \mu \in [0, 1]\} \) provides a smooth isotopy between the identity map \( g_0 = I \) and \( g_1 = g \); also by (P2),

\[
\frac{dg_\mu}{dx}(x) = \mu g'(x) + 1 - \mu \geq \frac{\mu}{2} + 1 - \mu = 1 - \frac{\mu}{2} \geq \frac{1}{2}, \forall x \in \mathbb{R}/\mathbb{Z}.
\]
Thus, by construction, \( g_\mu \) is a diffeomorphism of the unit circle satisfying (P1)-(P3), for each \( \mu \in [0, 1] \), that is, \( g_\mu \in G \). Given \( \mu \in [0, 1] \), by Proposition 4.2 there exists a diffeomorphism \( G_\mu : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) satisfying

\[
(G_\mu \circ T)(x) = ((g_\mu)^{-1} \circ T \circ g_\mu)(x), \quad \forall x \in \text{dom}(T).
\]

The proof of this theorem follows at once from the following lemma.

**Lemma 5.1.** Given \( x \in \mathbb{R}/\mathbb{Z} \), the map \( \mu \in [0, 1] \mapsto G_\mu(x) \in \mathbb{R}/\mathbb{Z} \) is differentiable. Moreover, there exist \( \epsilon > 0 \) and \( \sigma > 0 \) such that \( \forall (\mu_0, x) \in [0, \epsilon) \times \mathbb{R}/\mathbb{Z} \),

\[
\frac{d}{d\mu}
\bigg|_{\mu=\mu_0} G_\mu(x) > \sigma.
\]

**Proof.** Let \( u \in (0, 1] \) be a real number. Then, from (4.2),

\[
(T \circ g_u)(x) = (g_u \circ G_u \circ T)(x), \quad \forall x \in \text{dom}(T).
\]

From (5.1), we reach

\[
g_u(y) = y + u(g(y) - y), \quad \forall y \in \mathbb{R}/\mathbb{Z}.
\]

From (5.4) and from property (P3) of \( g_u \), we obtain

\[
\frac{(T \circ g_u)(x) - (T \circ g_0)(x)}{u} = \frac{T'(x) \cdot (g_u(x) - x)}{u} = T'(x) \cdot (g(x) - x).
\]

From equations (5.3) - (5.5), we get

\[
T'(x) \cdot (g(x) - x) = \left\{(T \circ g_u)(x) - (T \circ g_0)(x)\right\}/u = \left\{(g_u \circ G_u \circ T)(x) - (g_0 \circ G_0 \circ T)(x)\right\}/u = \frac{G_u(T(x)) - G_0(T(x))}{u} + g(G_u(T(x))) - g(G_0(T(x))).
\]

When \( u \to 0 \), we get

\[
\frac{d}{d\mu}
\bigg|_{\mu=0} G_\mu(T(x)) = T'(x) \cdot (g(x) - x) - (g(T(x)) - T(x)),
\]

and from Property (P3) of \( g \), we reach

\[
\frac{d}{d\mu}
\bigg|_{\mu=0} G_\mu(T(x)) = T(g(x)) - g(T(x)), \quad \forall x \in \text{dom}(T).
\]

Since \( G = g^{-1} \circ T \circ g \circ T^{-1} \) has no fixed points, we have that \( T(g(x)) - g(T(x)) \neq 0, \forall x \in \text{dom}(T) \). Therefore,

\[
\frac{d}{d\mu}
\bigg|_{\mu=0} G_\mu(T(x)) \neq 0, \forall x \in \text{dom}(T).
\]

That is,

\[
\frac{d}{d\mu}
\bigg|_{\mu=0} G_\mu(y) \neq 0, \quad \forall y \in \text{dom}(T^{-1}).
\]

If \( q \in \text{dis}(T^{-1}) \), then \( G_\mu(q) = (g_\mu)^{-1}(q) \), \( \forall \mu \). Besides, we have by Figure 4 that \( q - g(q) > 0 \) (the points \( (q, g(q)) \) with \( q \in \text{dis}(T^{-1}) \) lie below the diagonal). Observe that

\[
(g_\mu \circ (g_\mu)^{-1})(x) = x, \quad \forall x \in \mathbb{R}/\mathbb{Z},
\]

\[
\mu \cdot g((g_\mu)^{-1}(x)) + (1 - \mu)(g_\mu)^{-1}(x) = x, \quad \forall x \in \mathbb{R}/\mathbb{Z}.
\]
Therefore, by differentiating the previous equation with respect to $\mu$ at $\mu = 0$, we get

\begin{equation}
\left. \frac{d}{d\mu} \right|_{\mu=0} G_\mu(q) = \left. \frac{d}{d\mu} \right|_{\mu=0} (g_\mu)^{-1}(q) = q - g(q) > 0.
\end{equation}

We remark that equations (5.7) and (5.9) are compatible; that is, for any $q \in \text{dis}(T^{-1})$,

$$\lim_{y \to q} \left. \frac{d}{d\mu} \right|_{\mu=0} G_\mu(y) = q - g(q).$$

Hence, the map $(\mu, x) \in [0,1] \times \mathbb{R}/\mathbb{Z} \mapsto \left. \frac{d}{d\mu} \right|_{\mu=0} G_\mu(x)$ is continuous. This implies the lemma.

\begin{corollary}
For all $\mu \in (0, \epsilon)$, $G_\mu$ has no fixed points.
\end{corollary}
\begin{proof}
This follows immediately from Theorem 1.4.
\end{proof}

We now prove Theorem 1.5 (Vector field version). Let $M$ be a non-orientable compact 2-manifold of genus 4. Then there exists a family of quasi-minimal, Kupka-Smale flows $\{X_\mu \in \mathcal{X}(M)\}$, depending smoothly on $\mu \in [0, \epsilon)$, such that, for some flow box $V \subset M$ of $X_0$, which can be taken to be homeomorphic to either a rectangle or a cylinder, and for all $\mu, \nu \in [0, \epsilon)$,

1. $X_\mu|_V$ is a flow box;
2. if $\mu \neq \nu$, $X_\mu$ is a $C^\infty$-twist perturbation of $X_\nu$ localized in $V$;
3. $X_\mu$ and $X_\nu$ are topologically equivalent.

\begin{proof}
Let $T$ be as in Theorem 1.4. From Theorem 1.4, we know that $\{G_\mu \circ T\}$ is a family of smooth iter's conjugate to $T$. Each iter $\{G_\mu \circ T\}$ may be suspended to obtain a smooth vector field $X_\mu$ on a non-orientable compact manifold $M$ of genus 4 (see [8]). We may assume that $M$ contains $\mathbb{R}/\mathbb{Z}$ and does not depend on $\mu$. By definition of suspension, each $X_\mu$ is transversal to $\mathbb{R}/\mathbb{Z}$ and $G_\mu \circ T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the forward Poincaré map induced by $X_\mu$. This family $\{X_\mu\}$ can be constructed to satisfy the condition of the theorem in the case in which $V$ is a cylinder. The other case is similar. In both cases, the fact that $T$ is minimal and every $G_\mu \circ T$ is conjugate to $T$ ensures that the family $\{X_\mu\}$ has the required properties of quasi-minimality and topological equivalence.
\end{proof}

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References


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