PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 4, Pages 987–993
S 0002-9939(04)07779-2
Article electronically published on November 19, 2004

HILBERT-SAMUEL COEFFICIENTS AND POSTULATION NUMBERS OF GRADED COMPONENTS OF CERTAIN LOCAL COHOMOLOGY MODULES

M. BRODMANN AND F. ROHRER

(Communicated by Bernd Ulrich)

ABSTRACT. Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian homogeneous ring with one-dimensional local base ring $(R_0, \mathfrak{m}_0)$. Let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to the irrelevant ideal $R_+ := \bigoplus_{n>0} R_n$ of R. We show that the first Hilbert-Samuel coefficient $e_1\big(\mathfrak{q}_0, H^i_{R_+}(M)_n\big)$ of the n-th graded component of $H^i_{R_+}(M)$ with respect to $\mathfrak{q}_0$ is antipolynomial of degree $< i$ in n. In addition, we prove that the postulation numbers of the components $H^i_{R_+}(M)_n$ with respect to $\mathfrak{q}_0$ have a common upper bound.

1. Introduction

Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian homogeneous ring, so that R is $\mathbb{N}_0$-graded with Noetherian base ring R_0 and of the form $R = R_0[\ell_0, \cdots, \ell_r]$ with finitely many elements $\ell_0, \cdots, \ell_r \in R_1$. Let $R_+ := \bigoplus_{n>0} R_n$ denote the irrelevant ideal of R. Moreover let M denote a finitely generated graded R-module. For $i \in \mathbb{N}_0$ let $H^i_{R_+}(M) = \bigoplus_{n\in\mathbb{Z}} H^i_{R_+}(M)_n$ denote the i-th local cohomology module of M with respect to R_+. Keep in mind that the n-th graded component $H^i_{R_+}(M)_n$ of $H^i_{R_+}(M)$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and vanishes for all $n \gg 0$.

Motivated by the "tameness-problem" for coherent sheaves over projective schemes (cf. [B-H]) one is led to ask for the "asymptotic behaviour" of the graded components $H^i_{R_+}(M)_n$ for $n \to -\infty$. Namely, the mentioned tameness-problem is equivalent to the question of whether for each $i \in \mathbb{N}_0$ either $H^i_{R_+}(M)_n = 0$ for all $n \ll 0$ or $H^i_{R_+}(M)_n \neq 0$ for all $n \ll 0$. We do not know any example in which this tameness property is not satisfied. On the other hand, tameness has been established until now in fairly special cases only (cf. [L], [B-F-L]).

A more specific question is, whether for each $i \in \mathbb{N}_0$, the set $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)_n\right)$ of associated primes of the R_0-module $H^i_{R_+}(M)_n$ is asymptotically stable for $n \to -\infty$. This is true, for example, if R_0 is semilocal and either of dimension one or of dimension two and essentially of finite type over a field (cf. [B-F-L]). On the other

Received by the editors December 1, 2003.
2000 *Mathematics Subject Classification.* Primary 13D45, 13E10.
Key words and phrases. Local cohomology modules, graded components, Hilbert-Samuel polynomials.

hand this "asymptotic stability of associated primes" need not hold in general, as can be deduced from examples of Singh [S] (cf. [B-K-S]) or Katzman [K] (cf. [B-F-T]). It may fail, for example, if R_0 is regular local and of dimension four.

Finally, to be even more specific, one could ask whether certain numerical invariants of the R_0-modules $H^i_{R_+}(M)_n$ "behave well" if i is fixed and n goes to $-\infty$. To be more precise we say that a numerical function $f : \mathbb{Z} \to \mathbb{Z}$ *is antipolynomial (of degree $< i$)* if there is some polynomial $p \in \mathbb{Q}[\mathbf{x}]$ (of degree $< i$) such that $f(n) = p(n)$ for all $n \ll 0$. Moreover, if $(R_0, \mathfrak{m}_0)$ is local and $\mathfrak{q}_0 \subseteq R_0$ is an $\mathfrak{m}_0$-primary ideal, let $e_0(\mathfrak{q}_0, T)$ denote the *Hilbert-Samuel multiplicity* of the finitely generated R_0-module T with respect to $\mathfrak{q}_0$. Then, in this terminology we can say (cf. [B-F-T]):

(1.1) If $(R_0, \mathfrak{m}_0)$ is local of dimension one, $\mathfrak{q}_0 \subseteq R_0$ is $\mathfrak{m}_0$-primary and $i \in \mathbb{N}_0$, the functions given by

$$n \mapsto \operatorname{length}_{R_0}\left(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)\right), n \mapsto \operatorname{length}_{R_0}(0 \underset{H^i_{R_+}(M)_n}{:} \mathfrak{q}_0),$$

$$n \mapsto \operatorname{length}_{R_0}\left(H^i_{R_+}(M)_n / \mathfrak{q}_0 H^i_{R_+}(M)_n\right) \text{ and } n \mapsto e_0\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right)$$

are all antipolynomial of degree $< i$.

The first aim of this note is to show that under the above hypotheses a further numerical invariant of the R_0-modules $H^i_{R_+}(M)_n$ is antipolynomial of degree $< i$. Namely, assume that $(R_0, \mathfrak{m}_0)$ is local of dimension one, let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal and fix $i \in \mathbb{N}_0$. We then have "asymptotic stability of associated primes" and so $\dim_{R_0}\left(H^i_{R_+}(M)_n\right)$ takes a constant value ≤ 1 if $n \to -\infty$. Assume that this constant value is 1. Then, for all $n \ll 0$ the *Hilbert-Samuel polynomial of $H^i_{R_+}(M)_n$ with respect to* $\mathfrak{q}_0$ can be written as

$$(1.2) \qquad P_{H^i_{R_+}(M)_n, \mathfrak{q}_0}(\mathbf{x}) = e_0\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right)(\mathbf{x}+1) - e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right),$$

where $e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = -P_{H^i_{R_+}(M)_n, \mathfrak{q}_0}(-1) \in \mathbb{Z}$ is the so-called *first Hilbert-Samuel coefficient of $H^i_{R_+}(M)_n$ with respect to* $\mathfrak{q}_0$.

We shall prove that the function given by $n \mapsto e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right)$ is antipolynomial of degree $< i$ (cf. Theorem 3.1).

In addition we show that the *postulation numbers* of the R_0-modules $H^i_{R_+}(M)_n$ with respect to $\mathfrak{q}_0$, hence the numbers

$$(1.3) \quad \mu\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) := \inf\left\{r \in \mathbb{N}_0 \mid \operatorname{length}_{R_0}\left(H^i_{R_+}(M)_n / \mathfrak{q}_0^{t+1} H^i_{R_+}(M)_n\right) = P_{H^i_{R_+}(M)_n, \mathfrak{q}_0}(t), \ \forall t \geq r\right\}$$

have a common upper bound for all $n \in \mathbb{Z}$ (cf. Theorem 3.3).

As for the unexplained terminology we refer to [B-S] and to [E].

2. Hilbert-Samuel coefficients in dimension 1

Throughout this section let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension one, let $\mathfrak{q} \subseteq A$ be an $\mathfrak{m}$-primary ideal and let T be a finitely generated A-module of dimension one.

2.1. *Notation and Remark.* A) Let $P_{T,\mathfrak{q}}(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ denote the *Hilbert-Samuel polynomial* of T with respect to $\mathfrak{q}$, so that

$$P_{T,\mathfrak{q}}(n) = \operatorname{length}_A(T/\mathfrak{q}^{n+1}T) \text{ for all } n \gg 0.$$

As $\dim(T) = 1, P_{T,\mathfrak{q}}(\mathbf{x})$ is of degree 1 and we may write

$$P_{T,\mathfrak{q}}(\mathbf{x}) = e_0(\mathfrak{q}, T)(\mathbf{x}+1) - e_1(\mathfrak{q}, T)$$

where $e_0(\mathfrak{q}, T) \in \mathbb{N}$ is the *(Hilbert-Samuel) multiplicity* of T with respect to $\mathfrak{q}$ and $e_1(\mathfrak{q}, T) \in \mathbb{Z}$ is the *first Hilbert-Samuel coefficient* of T with respect to $\mathfrak{q}$.

B) We set

$$\overline{T} := T/\Gamma_{\mathfrak{m}}(T),$$

where $\Gamma_{\mathfrak{m}}(T) = \bigcup_{n\in\mathbb{N}}(0 \underset{T}{:} \mathfrak{m}^n)$ denotes the $\mathfrak{m}$-torsion of T. Keep in mind that $\operatorname{Ass}_A(\overline{T}) = \operatorname{Ass}_A(T)\setminus\{\mathfrak{m}\}$, so that $\dim(\overline{T}) = \operatorname{depth}_A(\overline{T}) = 1$.

C) Keep the above hypotheses and notation. Then, for each $n \in \mathbb{N}_0$, there is a short exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(T)/(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^nT) \to T/\mathfrak{q}^nT \to \overline{T}/\mathfrak{q}^n\overline{T} \to 0.$$

In particular,

$$\operatorname{length}_A(T/\mathfrak{q}^nT) = \operatorname{length}_A(\overline{T}/\mathfrak{q}^n\overline{T}) + \operatorname{length}_A(\Gamma_{\mathfrak{m}}(A)) - \operatorname{length}_A(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^nT).$$

By Artin-Rees there is some $n_0 \in \mathbb{N}$ such that $\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^nT = 0$ for all $n \geq n_0$. This gives the relations

$$\begin{aligned}
\operatorname{length}_A(T/\mathfrak{q}^nT) &= \operatorname{length}_A(\overline{T}/\mathfrak{q}^n\overline{T}) + \operatorname{length}_A(\Gamma_{\mathfrak{m}}(T)) \text{ for all } n \geq n_0;\\
P_{T,\mathfrak{q}}(\mathbf{x}) &= P_{\overline{T},\mathfrak{q}}(\mathbf{x}) + \operatorname{length}_A(\Gamma_{\mathfrak{m}}(T));\\
e_0(\mathfrak{q}, T) &= e_0(\mathfrak{q}, \overline{T});\\
e_1(\mathfrak{q}, T) &= e_1(\mathfrak{q}, \overline{T}) - \operatorname{length}_A(\Gamma_{\mathfrak{m}}(T)).
\end{aligned}$$

2.2. *Notation.* A) Let

$$\mathcal{R}(\mathfrak{q}) := \bigoplus_{n\geq 0} \mathfrak{q}^n, \quad \mathcal{R}(\mathfrak{q}, T) := \bigoplus_{n\geq 0} \mathfrak{q}^nT$$

denote the *Rees ring of* $\mathfrak{q}$ and the *Rees module of* T *with respect to* $\mathfrak{q}$, respectively.

B) In addition, let

$$\begin{aligned}
\operatorname{Gr}(\mathfrak{q}) :=& \mathcal{R}(\mathfrak{q})/\mathfrak{q}\mathcal{R}(\mathfrak{q}) = \bigoplus_{n\geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1},\\
\operatorname{Gr}(\mathfrak{q}, T) :=& \mathcal{R}(\mathfrak{q}, T)/\mathfrak{q}\mathcal{R}(\mathfrak{q}, T) = \bigoplus_{n\geq 0} \mathfrak{q}^nT/\mathfrak{q}^{n+1}T
\end{aligned}$$

denote the *associated graded ring of* $\mathfrak{q}$ and the *associated graded module of* T *with respect to* $\mathfrak{q}$, respectively.

2.3. *Notation.* For a finitely generated graded module M over the Noetherian homogeneous ring $R = \bigoplus_{n\geq 0} R_n$ let $\operatorname{reg}(M)$ denote the *(Castelnuovo-Mumford) regularity* of M, thus

$$\operatorname{reg}(M) := \inf\{r \in \mathbb{Z} \mid H^i_{R_+}(M)_{n-i+1} = 0, \ \forall i \in \mathbb{N}_0, \forall n \geq r\}.$$

2.4. **Lemma.** *Let* $\operatorname{depth}_A(T) \neq 0$. *Then* $\operatorname{reg}(\operatorname{Gr}(\mathfrak{q}, T)) \leq \operatorname{reg}(\operatorname{Gr}(\mathfrak{q}))$.

Proof. Let $\mathbf{x}$ be an indeterminate. Then $B := A[\mathbf{x}]_{\mathfrak{m}A[\mathbf{x}]}$ is a Noetherian local flat one-dimensional extension ring of A with maximal ideal $\mathfrak{n} := \mathfrak{m}B$ and infinite residue field $B/\mathfrak{n}$. Moreover $\mathfrak{q}B \subseteq B$ is an $\mathfrak{n}$-primary ideal and $T \otimes_A B$ is a finitely generated B-module of dimension 1 and depth $\neq 0$.

In addition, there is an isomorphism of graded rings $\mathrm{Gr}(\mathfrak{q}B) \cong \mathrm{Gr}(\mathfrak{q}) \otimes_A B$ and an isomorphism of graded $\mathrm{Gr}(\mathfrak{q}B)$-modules $\mathrm{Gr}(\mathfrak{q}B, T \otimes_A B) \cong \mathrm{Gr}(\mathfrak{q}, T) \otimes_A B$. In view of the graded flat base-change property of local cohomology (cf. [B-S]), these latter isomorphisms induce

$$\mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}B)) = \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$$

and

$$\mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}B, T \otimes_A B)) = \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}, T))\,.$$

Altogether, we thus may replace $(A, \mathfrak{m}), \mathfrak{q}$ and T respectively by $(B, \mathfrak{n}), \mathfrak{q}B$ and $T \otimes_A B$ and hence assume that $A/\mathfrak{m}$ is infinite.

As $\mathfrak{q}$ is $\mathfrak{m}$-primary, its analytic spread equals $\dim(A) = 1$. So there is some $x \in \mathfrak{q}$ such that Ax is a minimal reduction of $\mathfrak{q}$. As $A/\mathfrak{m}$ is infinite the reduction number of $\mathfrak{q}$ with respect to Ax is $\leq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$ (cf. [T, Proposition 3.2] or [B-S, 18.3.12]) so that $x\mathfrak{q}^n = \mathfrak{q}^{n+1}$ for all $n \geq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$. Let $x^* = (0, x, 0, \cdots) \in \mathcal{R}(\mathfrak{q})_1$ be the element $x \in \mathfrak{q}$ considered as a one-form in $\mathcal{R}(\mathfrak{q})$. It follows that $\sqrt{\mathcal{R}(\mathfrak{q})_+} = \sqrt{x^*\mathcal{R}(\mathfrak{q})}$ so that $H^i_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = H^i_{(x^*)}(\mathcal{R}(\mathfrak{q}, T))$ for all $i \geq 0$. Therefore $H^i_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = 0$ for all $i > 1$. As $x \in A$ is a parameter, $\mathrm{depth}_A(T) = 1 = \dim(A)$ implies that x is T-regular. So x^* is $\mathcal{R}(\mathfrak{q}, T)$-regular and hence $H^0_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T)) = 0$. Moreover we have the following short exact sequence of graded $\mathcal{R}(\mathfrak{q})$-modules (cf. [B-S, 2.2.17]):

$$0 \to \mathcal{R}(\mathfrak{q}, T) \xrightarrow{\eta} \mathcal{R}(\mathfrak{q}, T)_{x^*} \to H^1_{(x^*)}(\mathcal{R}(\mathfrak{q}, T)) \to 0.$$

As $x^*\mathcal{R}(\mathfrak{q}, T)_n = x\mathfrak{q}^n T = \mathfrak{q}^{n+1}T = \mathcal{R}(\mathfrak{q}, T)_{n+1}$ for all $n \geq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$, the natural map η becomes an isomorphism in all degrees $\geq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$ so that

$$H^1_{\mathcal{R}(\mathfrak{q})_+}(\mathcal{R}(\mathfrak{q}, T))_n = H^1_{(x^*)}(\mathcal{R}(\mathfrak{q}, T))_n = 0$$

for all $n \geq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$. Thus, finally we get $\mathrm{reg}\,(\mathcal{R}(\mathfrak{q}, T)) \leq \mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}))$.

Now, in view of the well-known behaviour of regularities in short exact sequences (cf. [B-S, 15.2.15]) and as $\mathrm{reg}\,(\mathcal{R}(\mathfrak{q}, T)) \geq 0$ (cf. [B-S, 15.3.1]), we get our claim by the graded exact sequences

$$0 \to \mathfrak{q}\mathcal{R}(\mathfrak{q}, T) \to \mathcal{R}(\mathfrak{q}, T)(1) \to [T]_{-1} \to 0,$$
$$0 \to \mathfrak{q}\mathcal{R}(\mathfrak{q}, T) \to \mathcal{R}(\mathfrak{q}, T) \to \mathrm{Gr}(\mathfrak{q}, T) \to 0.$$

□

2.5. **Lemma.** *Let* $r \geq \mathrm{reg}(\mathrm{Gr}(\mathfrak{q}))$*, let* $n \geq r$ *and assume that* $\mathrm{depth}_A(T) = 1$*. Then*

$$\mathrm{length}_A(T/\mathfrak{q}^{n+1}T) = e_0(\mathfrak{q}, T)n - re_0(\mathfrak{q}, T) + \mathrm{length}_A(T/\mathfrak{q}^{r+1}T).$$

Proof. For all $n \in \mathbb{N}_0$ we have

$$\mathrm{length}_{A/\mathfrak{q}}\,(\mathrm{Gr}(\mathfrak{q}, T)_n) = \mathrm{length}_A(T/\mathfrak{q}^{n+1}T) - \mathrm{length}_A(T/\mathfrak{q}^n T).$$

For all $n \gg 0$, the right-hand side of this equality takes the value $P_{T,\mathfrak{q}}(n) - P_{T,\mathfrak{q}}(n-1) = e_0(\mathfrak{q}, T)$, so that the characteristic function of the graded $\mathrm{Gr}(\mathfrak{q})$-module $\mathrm{Gr}(\mathfrak{q}, T)$ takes the constant value $e_0(\mathfrak{q}, T)$. As $\mathrm{reg}\,(\mathrm{Gr}(\mathfrak{q}, T)) \leq r$ it follows that $\text{length}_{A/\mathfrak{q}}(\text{Gr}(\mathfrak{q},T)_n) = e_0(\mathfrak{q},T)$ for all $n > r$ and hence $\text{length}_A(T/\mathfrak{q}^{n+1}T) - \text{length}_A(T/\mathfrak{q}^nT) = e_0(\mathfrak{q},T)$ for all $n \geq r+1$. So, for all $n \geq r$ we get

$$\begin{aligned}\text{length}_A(T/\mathfrak{q}^{n+1}T) &= \sum_{m=r+1}^{n} \big(\text{length}_A(T/\mathfrak{q}^{m+1}T) - \text{length}_A(T/\mathfrak{q}^mT)\big) \\ &\quad + \text{length}_A(T/\mathfrak{q}^{r+1}T) = e_0(\mathfrak{q},T)(n-r) + \text{length}_A(T/\mathfrak{q}^{r+1}T) \\ &= e_0(\mathfrak{q},T)n - re_0(\mathfrak{q},T) + \text{length}_A(T/\mathfrak{q}^{r+1}T).\end{aligned}$$

□

2.6. **Proposition.** *Let* $r \geq \text{reg}(\text{Gr}(\mathfrak{q}))$. *Then*

$$e_1(\mathfrak{q},T) = (r+1)e_0(\mathfrak{q},T) - \text{length}_A(T/\mathfrak{q}^{r+1}T) - \text{length}_A\left(\mathfrak{q}^{r+1}T \cap \Gamma_{\mathfrak{m}}(T)\right).$$

Proof. As $\overline{T}$ has dimension and depth 1, Lemma 2.5 gives $e_1(\mathfrak{q},\overline{T}) = (r+1)e_0(\mathfrak{q},\overline{T}) - \text{length}_A(\overline{T}/\mathfrak{q}^{r+1}\overline{T})$. As

$$e_0(\mathfrak{q},T) = e_0(\mathfrak{q},\overline{T}),\ \ e_1(\mathfrak{q},T) = e_1(\mathfrak{q},\overline{T}) - \text{length}_A\left(\Gamma_{\mathfrak{m}}(T)\right)$$

and

$$\begin{aligned}\text{length}_A(\overline{T}/\mathfrak{q}^{r+1}\overline{T}) &= \text{length}_A(T/\mathfrak{q}^{r+1}T) - \text{length}_A\left(\Gamma_{\mathfrak{m}}(T)\right) \\ &\quad + \text{length}_A(\Gamma_{\mathfrak{m}}(T) \cap \mathfrak{q}^{r+1}T),\end{aligned}$$

(cf. 2.1 C)), we get our claim. □

3. Antipolynomial growth of first Hilbert-Samuel coefficients and boundedness of postulation numbers

Now, we are ready to formulate and to prove our first main result.

3.1. **Theorem.** *Let* $R = \bigoplus_{n\geq 0} R_n$ *be a Noetherian homogeneous ring such that* $(R_0,\mathfrak{m}_0)$ *is local and of dimension* 1. *Let* $\mathfrak{q}_0 \subseteq R_0$ *be an* $\mathfrak{m}_0$*-primary ideal, let* M *be a finitely generated graded* R*-module and let* $i \in \mathbb{N}_0$. *Assume that* $\dim_{R_0}\left(H^i_{R_+}(M)_n\right) = 1$ *for all* $n \ll 0$. *Then there is a polynomial* $S(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$ *of degree* $< i$ *such that*

$$e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = S(n) \text{ for all } n \ll 0.$$

Proof. By our hypotheses there is some $r \geq \text{reg}(\text{Gr}(\mathfrak{q}_0))$ such that $\dim_{R_0}\left(H^i_{R_+}(M)_n\right) = 1$ for all $n \leq -r$. So by Proposition 2.6 we obtain for all $n \leq -r$:

$$\begin{aligned}&e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = (r+1)e_0\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) \\ &- \text{length}_{R_0}\left(H^i_{R_+}(M)_n/\mathfrak{q}_0^{r+1}H^i_{R_+}(M)_n\right) \\ &- \text{length}_{R_0}\left(\mathfrak{q}_0^{r+1}H^i_{R_+}(M)_n \cap \Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right)\right).\end{aligned}$$

According to (1.1) there are polynomials $P, Q \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that

$$e_0\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = Q(n),\ \ \text{length}_{R_0}\left(H^i_{R_+}(M)_n/\mathfrak{q}_0^{r+1}H^i_{R_+}(M)_n\right) = P(n)$$

for all $n \ll 0$.

By [B-F-T, Theorem 2.5], the graded R-module $\Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)\right)$ is Artinian, and hence so is its graded submodule

$$U := \mathfrak{q}_0^{r+1} H^i_{R_+}(M) \cap \Gamma_{\mathfrak{m}_0 R}\left(H^i_{R_+}(M)\right).$$

So, there is a polynomial $F \in \mathbb{Q}[\mathbf{x}]$ such that $\text{length}_{R_0}(U_n) = F(n)$ for all $n \ll 0$ (cf. [Ki]). According to (1.1) there is a polynomial $\overline{Q} \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that $\text{length}_{R_0}\left(\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n)\right) = \overline{Q}(n)$ for all $n \ll 0$. As $U_n \subseteq \Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right)$ for all n, it follows that $\deg(F) \leq \deg(\overline{Q}) < i$.

As $U_n = \mathfrak{q}_0^{r+1} H^i_{R_+}(M)_n \cap \Gamma_{\mathfrak{m}_0}\left(H^i_{R_+}(M)_n\right)$, the equality given at the beginning of this proof yields

$$e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = (r+1)Q(n) - P(n) - F(n) \quad \text{for all } n \ll 0.$$

This proves our claim. □

Now we prove the announced boundedness of postulation numbers (cf. (1.3)). We begin with a preliminary remark.

3.2. *Remark.* Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring, let $W = \bigoplus_{n\in\mathbb{Z}} W_n$ be a graded and Artinian R-module and let $(U^{(m)})_{m\in\mathbb{N}_0}$ be a descending sequence of graded submodules of W. Assume that for each $n \in \mathbb{Z}$ there is some $m_n \in \mathbb{N}_0$ such that $U_n^{(m_n)} = 0$.

As W is Artinian, there is some $t \in \mathbb{N}_0$ such that $U^{(m)} = U^{(t)}$ for all $m \geq t$. So, for each $n \in \mathbb{Z}$ and for each $m \geq \max\{t, m_n\}$, we obtain $U_n^{(t)} = U_n^{(m)} \subseteq U_n^{(m_n)} = 0$. Therefore $U^{(t)} = 0$.

3.3. **Theorem.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian homogeneous ring such that $(R_0, \mathfrak{m}_0)$ is local and of dimension ≤ 1. Let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then, there is some $c \in \mathbb{N}_0$ such that $\mu(\mathfrak{q}_0, H^i_{R_+}(M)_n) \leq c$ for all $n \in \mathbb{Z}$.*

Proof. By [B-F-T, Theorem 3.5 e)] there is some $\delta \in \{0, \pm 1\}$ such that

$$\dim_{R_0}(H^i_{R_+}(M)_n) = \delta \quad \text{for all } n \ll 0.$$

If $\delta = -1$ we have $H^i_{R_+}(M)_n = 0$ for all but finitely many values of n and our claim is clear.

If $\delta \geq 0$ we shall apply Remark 3.2 to the graded Artinian R-module

$$W := \Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M))$$

(cf. [B-F-T, Theorem 2.5 b)]). First, let $\delta = 0$, so that $W_n = H^i_{R_+}(M)_n$ for all $n \ll 0$. If we apply Remark 3.2 with $U^{(m)} := \mathfrak{q}_0^m W$, we find some $t \in \mathbb{N}_0$ with $\mathfrak{q}_0^t W = 0$ so that $\mathfrak{q}_0^t H^i_{R_+}(M)_n = 0$ for all $n \ll 0$. As $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$ we get our claim if $\delta = 0$.

So, let $\delta = 1$. If we apply Remark 3.2 with $U^{(m)} := W \cap \mathfrak{q}_0^m H^i_{R_+}(M)$ and use the lemma of Artin-Rees we find some $t \in \mathbb{N}$ with $\Gamma_{\mathfrak{m}_0 R}(H^i_{R_+}(M)) \cap \mathfrak{q}_0^t H^i_{R_+}(M) = 0$, so that $\Gamma_{\mathfrak{m}_0}(H^i_{R_+}(M)_n) \cap \mathfrak{q}_0^t H^i_{R_+}(M)_n = 0$ for all $n \in \mathbb{Z}$. Now, let n_0 be such that $\dim_{R_0}(H^i_{R_+}(M)_n) = 1$ for all $n \leq n_0$. Then, for each $r \geq \max\{\text{reg}\,(\text{Gr}(\mathfrak{q}_0))\,, t-1\}$

and each $n \leq n_0$, Proposition 2.6 — applied with $A = R_0, \mathfrak{q} = \mathfrak{q}_0$ and $T = H^i_{R_+}(M)_n$ — gives

$$\begin{aligned} &\operatorname{length}_{R_0}\left(H^i_{R_+}(M)_n/\mathfrak{q}_0^{r+1}H^i_{R_+}(M)_n\right) \\ &\qquad = (r+1)e_0\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) - e_1\left(\mathfrak{q}_0, H^i_{R_+}(M)_n\right) = P_{H^i_{R_+}(M)_n,\mathfrak{q}_0}(r). \end{aligned}$$

As $H^i_{R_+}(M)_n = 0$ for all $n \gg 0$, this proves our claim. □

3.4. **Corollary.** *Let $R = \bigoplus_{n\geq 0} R_n$ be a Noetherian homogeneous ring such that $(R_0, \mathfrak{m}_0)$ is local and of dimension ≤ 1. Let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$-primary ideal, let M be a finitely generated graded R-module and let $i \in \mathbb{N}_0$. Then, there are integers $n_0 \in \mathbb{Z}, c \in \mathbb{N}_0$ and polynomials $S, Q \in \mathbb{Q}[\mathbf{x}]$ of degree $< i$ such that for each $n \leq n_0$ and each $r \geq c$ we have*

$$\operatorname{length}_{R_0}\left(H^i_{R_+}(M)_n/\mathfrak{q}_0^{r+1}H^i_{R_+}(M)_n\right) = Q(n)(r+1) - S(n).$$

Proof. Immediate by Theorem 3.1, Theorem 3.3 and [B-F-T, Theorem 3.5]. □

References

[B-F-L] M. BRODMANN, S. FUMASOLI and C.S. LIM: *Low-codimensional associated primes of graded components of local cohomology modules*, Journal of Algebra 275 (2004), 867–882. MR2052643.

[B-F-T] M. BRODMANN, S. FUMASOLI and R. TAJAROD: *Local cohomology over homogeneous rings with one-dimensional local base ring*, Proceedings of the AMS 131, No 10 (2003), 2977–2985. MR1993202 (2004f:13021)

[B-H] M. BRODMANN and M. HELLUS: *Cohomological patterns of coherent sheaves over projective schemes*, Journal of Pure and Applied Algebra 172 (2002), 165–182. MR1906872 (2003f:13017)

[B-K-S] M. BRODMANN, M. KATZMAN and R.Y. SHARP: *Associated primes of graded components of local cohomology modules*, Transactions of the AMS 354, No 11 (2002), 4261–4283. MR1926875 (2003h:13020)

[B-S] M. BRODMANN and R.Y. SHARP: *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press (1998). MR1613627 (99h:13020)

[E] D. EISENBUD: *Commutative algebra with a view towards algebraic geometry*, Springer, New York (1996). MR1322960 (97a:13001)

[K] M. KATZMAN: *An example of an infinite set of associated primes of a local cohomology module*, Journal of Algebra 252 (2002), 161–166. MR1922391 (2003h:13021)

[Ki] D. KIRBY: *Artinian modules and Hilbert polynomials*, Quarterly Journal of Mathematics, Oxford (2) 24 (1973), 47–57. MR0316446 (47:4993)

[L] C.S. LIM: *Graded local cohomology modules and their associated primes*, Communications in Algebra 32, No 2 (2004), 727–745.

[S] A.K. SINGH: *p-torsion elements in local cohomology modules*, Mathematics Research Letters 7 (2000), 165–176. MR1764314 (2001g:13039)

[T] N.V. TRUNG: *Reduction exponent and degree bound for the defining equations of graded rings*, Proceedings of the AMS 101 (1987), 229 -236. MR0902533 (89i:13031)

Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: brodmann@math.unizh.ch

Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: fred@math.unizh.ch